Parking Modules

Armstrong, Reiner, Rhoades

PF

PF

A **parking function** is a vector $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ whose increasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies:

$$\boxed{\forall i,\ b_i \leq i}$$

"fits under a staircase"

PF

A **parking function** is a vector $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ whose increasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies:

$$\boxed{\forall i,\ b_i \leq i}$$

"fits under a staircase"

Think:

- One way street with n parking spaces
- Car i **wants** to park in space a_i
- If space a_i is full, she parks in next available space
- Car 1 parks first, then Car 2, etc.
- " $\vec{a}$ is a **parking function**" means "everyone **is able** to park"

PF

Example: 3 cars

$$
\begin{array}{llllll}
(1,1,1) \\
(1,1,2) & (1,2,1) & (2,1,1) \\
(1,1,3) & (1,3,1) & (3,1,1) \\
(1,2,2) & (2,1,2) & (2,2,1) \\
(1,2,3) & (1,3,2) & (2,1,3) & (2,3,1) & (3,1,2) & (3,2,1)
\end{array}
$$

PF

Example: 3 cars

$$
\begin{array}{llllll}
(1,1,1) & & & & & \\
(1,1,2) & (1,2,1) & (2,1,1) & & & \\
(1,1,3) & (1,3,1) & (3,1,1) & & & \\
(1,2,2) & (2,1,2) & (2,2,1) & & & \\
(1,2,3) & (1,3,2) & (2,1,3) & (2,3,1) & (3,1,2) & (3,2,1)
\end{array}
$$

16 parking functions & 5 orbits

PF

Example: 3 cars

$$
\begin{array}{llllll}
(1,1,1) \\
(1,1,2) & (1,2,1) & (2,1,1) \\
(1,1,3) & (1,3,1) & (3,1,1) \\
(1,2,2) & (2,1,2) & (2,2,1) \\
(1,2,3) & (1,3,2) & (2,1,3) & (2,3,1) & (3,1,2) & (3,2,1)
\end{array}
$$

16 parking functions & 5 orbits

$$(n+1)^{n-1} \qquad \frac{1}{n+1}\binom{2n}{n}$$

"Catalan"

PF to algebra

Idea (Pollack, 1974): a **circular** street with $n + 1$ spaces

- choice functions $= (\mathbb{Z}/(n+1)\mathbb{Z})^n$
- Everyone can park. One empty spot remains.
- is parking function $\iff$ space $n + 1$ is empty
- one parking function per rotation class

PF to algebra

Idea (Pollack, 1974): a **circular** street with $n+1$ spaces

- choice functions $= (\mathbb{Z}/(n+1)\mathbb{Z})^n$
- Everyone can park. One empty spot remains.
- is parking function $\iff$ space $n+1$ is empty
- one parking function per rotation class

Conclusion:

"modulo rotation"

Parking Functions $=$ cosets $(\mathbb{Z}/(n+1)\mathbb{Z})^n / (1, 1, \ldots, 1)$

$$(n+1)^{n-1} = \frac{(n+1)^n}{n+1}$$

PF to algebra

Idea (Haiman, 1996): generalize to **Weyl groups**

- type A root lattice $Q = \mathbb{Z}^n/(1,1,\ldots,1)$
- ...so Parking Functions are

$$(\mathbb{Z}/(n+1)\mathbb{Z})^n/(1,1,\ldots,1) \cong_{S_n} Q/(n+1)Q$$

the "finite torus"

PF to algebra

Idea (Haiman, 1996): generalize to **Weyl groups**

- type A root lattice $Q = \mathbb{Z}^n/(1,1,\ldots,1)$
- ...so Parking Functions are

$$(\mathbb{Z}/(n+1)\mathbb{Z})^n/(1,1,\ldots,1) \cong_{S_n} Q/(n+1)Q$$

the "finite torus"

Now consider:

- **Weyl group** $W \subseteq GL(V)$
- **root lattice** Q
- **Coxeter number** h order of a "Coxeter element"

PF to algebra

The Original Parking Module (Haiman)

$$\mathsf{Park}(W) = Q/(h+1)Q$$

PF to algebra

The Original Parking Module (Haiman)

$$\mathsf{Park}(W) = Q/(h+1)Q$$

The character is...

$$\chi_{\mathsf{Park}}(w) = (h+1)^{\dim(V^w)}$$

dimension of the “fixed space”

PF to algebra

The Original Parking Module (Haiman)

$$\mathsf{Park}(W) = Q/(h+1)Q$$

The number of orbits is...

$$\mathsf{Cat}(W) = \prod_i \frac{h+d_i}{d_i}$$

the "degrees" of the group

PF to Shi

PF to Shi

The **Shi arrangement** of hyperplanes is:

$$\mathsf{Shi}(W) := \{H_{\alpha,k} : \alpha \in \Phi^+, k \in \{0,1\}\}$$

where $H_{\alpha,k} := \{x \in V : (x,\alpha) = k\}$

PF to Shi

The **Shi arrangement** of hyperplanes is:

$$\mathrm{Shi}(W) := \{H_{\alpha,k} : \alpha \in \Phi^+, k \in \{0,1\}\}$$

where $H_{\alpha,k} := \{x \in V : (x,\alpha) = k\}$

There exists a uniform **BIJECTION** (Cellini-Papi, Shi):

$$Q/(h+1)Q \longrightarrow \text{regions of } \mathrm{Shi}(W)$$

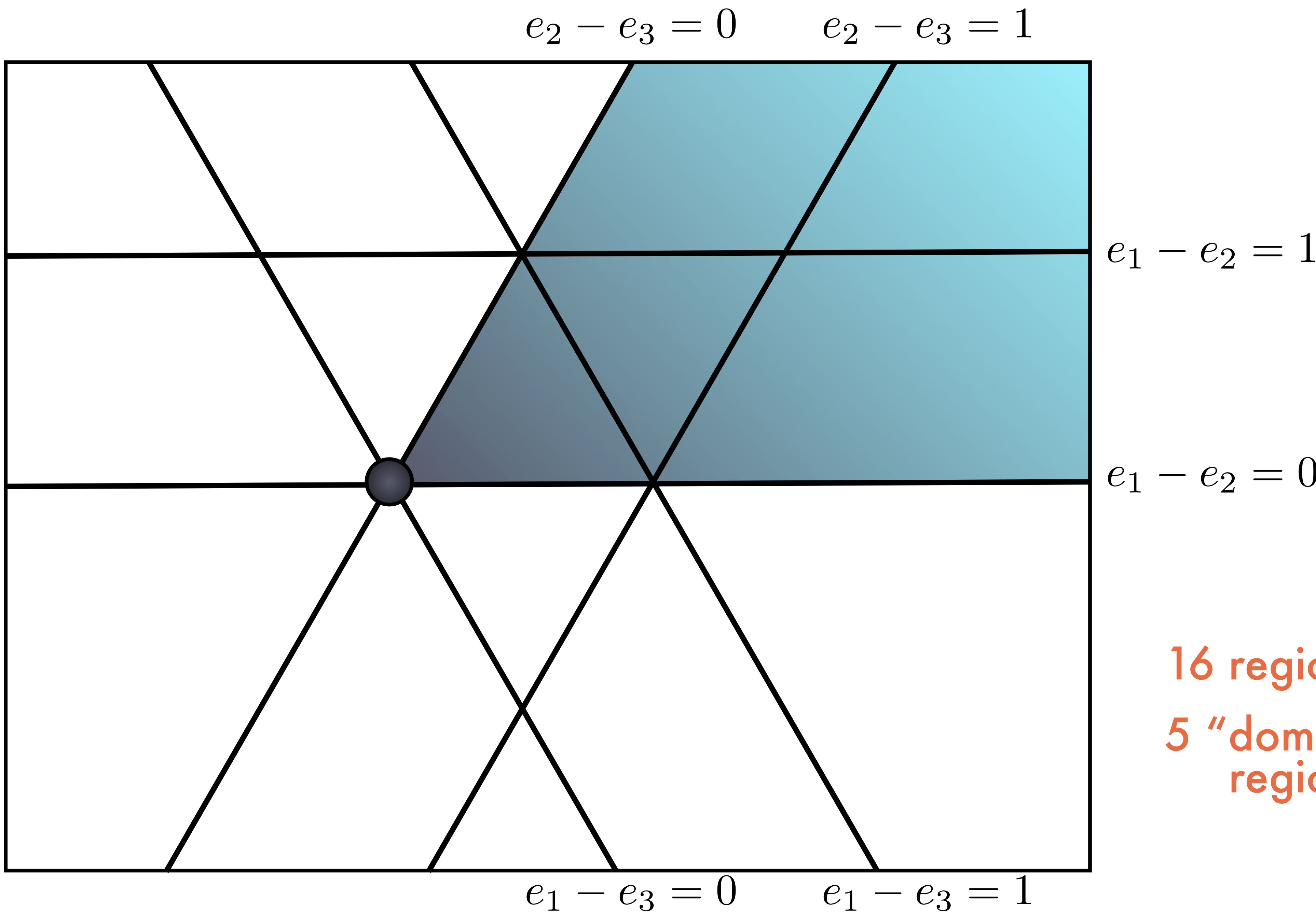
Cellini-Papi

Shi

$$(h+1)^{\dim(V)}$$

PF to Shi

Picture for $W = S_3$



16 regions

5 "dominant" regions

Shi to NN

Type A: Label region R by pair (w, π) where

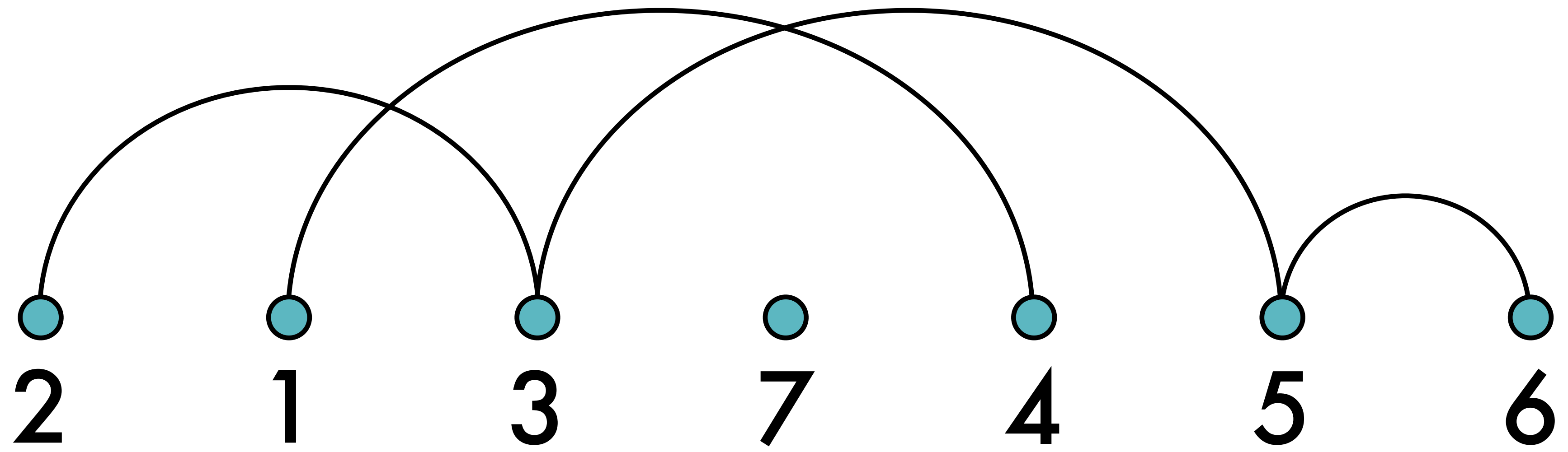
permutation, partition

- R is in the **cone** corresponding to $w \in S_n$
- π is a **partition** of $\{1, 2, \ldots, n\}$ defined by

$i \sim_\pi j$ $i \leq j$	$\Longleftrightarrow$	$e_i - e_j = 1$ is **above** R

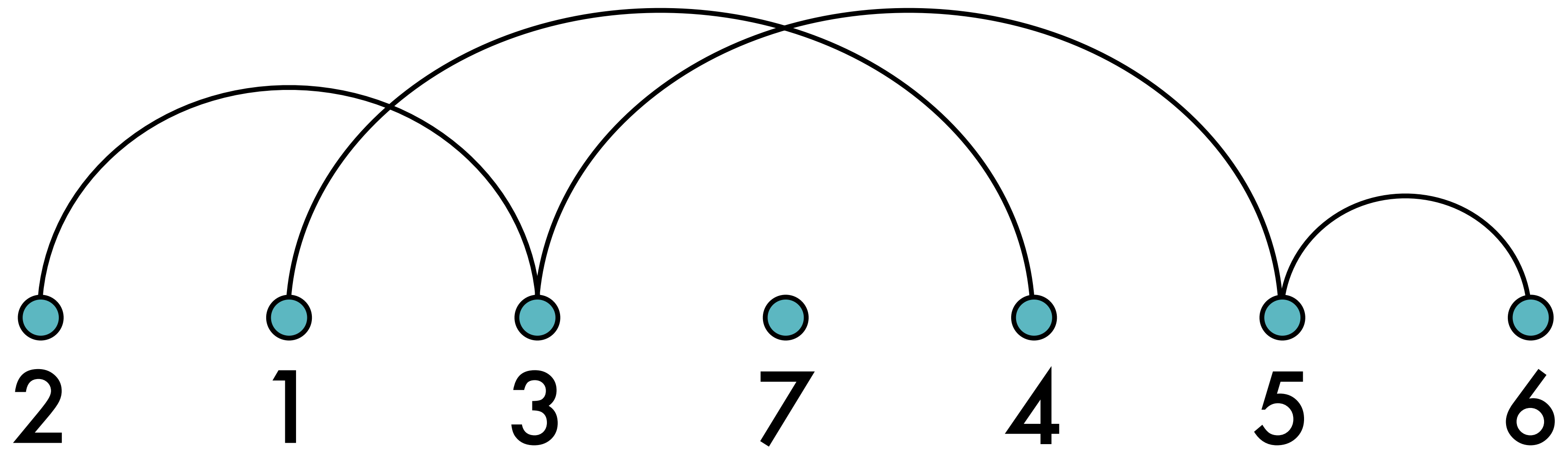
Shi to NN

For example:



Shi to NN

For example:



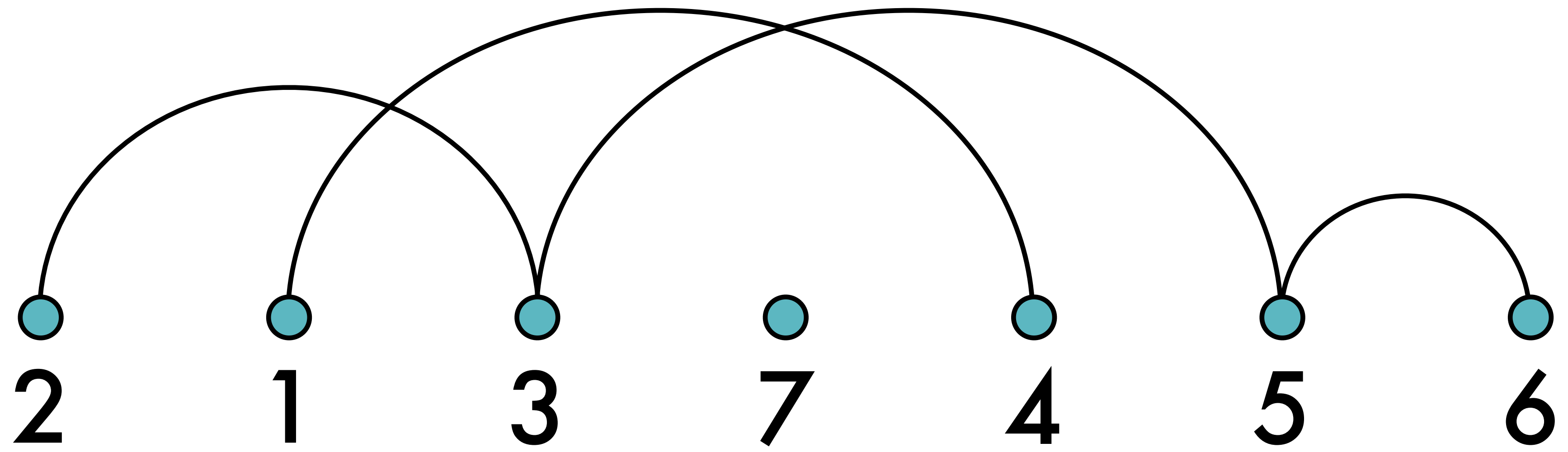
Theorem (Pak-Stanley, Athanasiadis-Linusson):

Shi regions $=$ labels (w, π) where

π is NN and blocks are **increasing** wrt w

Shi to NN

For example:

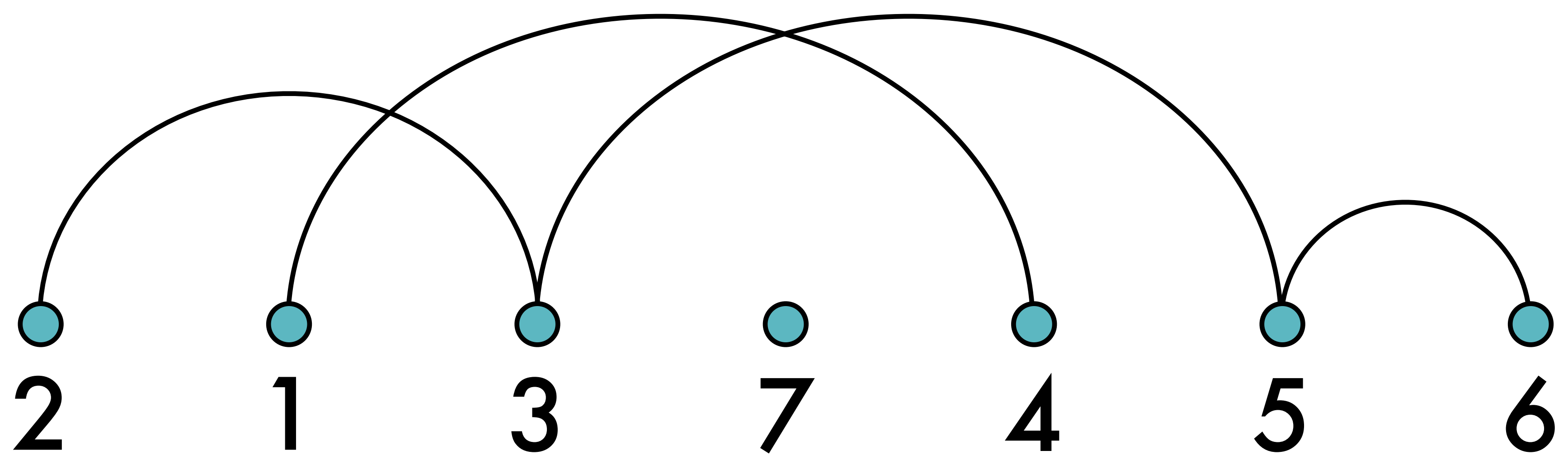


Definition: The "**ceiling partition**" of the label (w, π) is...

$w \cdot \pi$ i.e. w acting on π

Shi to NN

For example:



Definition: The "**ceiling partition**" of the label (w, π) is...

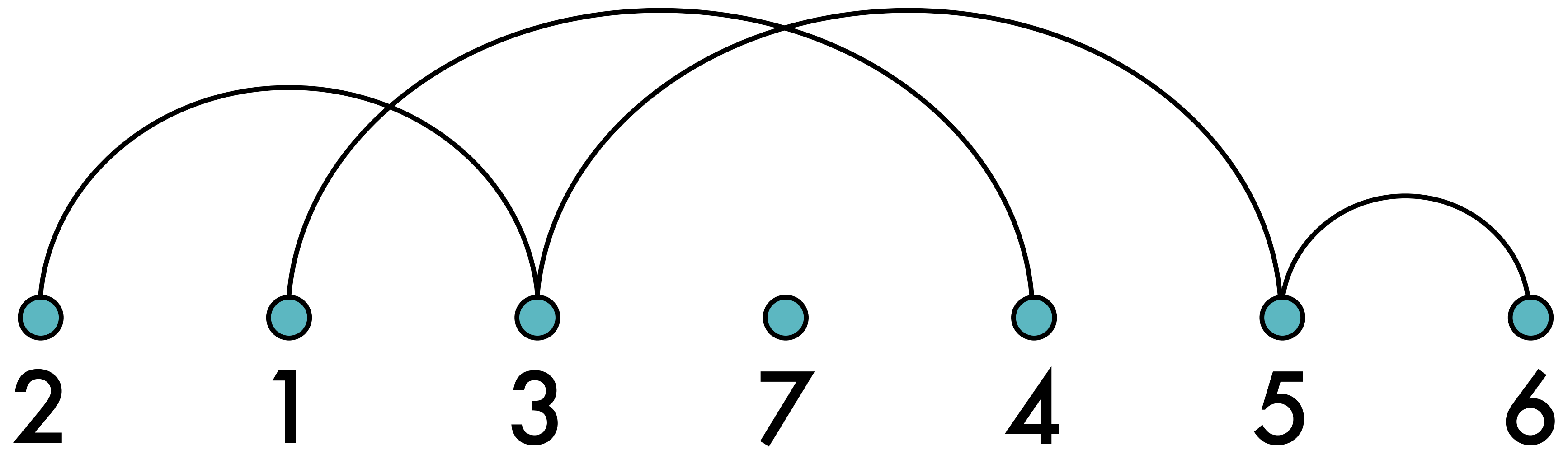
$w \cdot \pi$ i.e. w acting on π

e.g.

$$
\begin{aligned}
w &= 2137456 \\
\pi &= \{1,3,6,7\},\{2,5\},\{4\} \\
w \cdot \pi &= \{1,4\},\{2,3,5,6\},\{7\}
\end{aligned}
$$

Shi to NN

For example:



Theorem (Armstrong-Rhoades):

Let μ be a partition of $\{1, 2, \ldots, n\}$ with k blocks. Then...

Number of Shi regions with ceiling partition μ is $\dfrac{n!}{(n-k+1)!}$

Shi to NN

Corollary:

Total number of Shi chambers

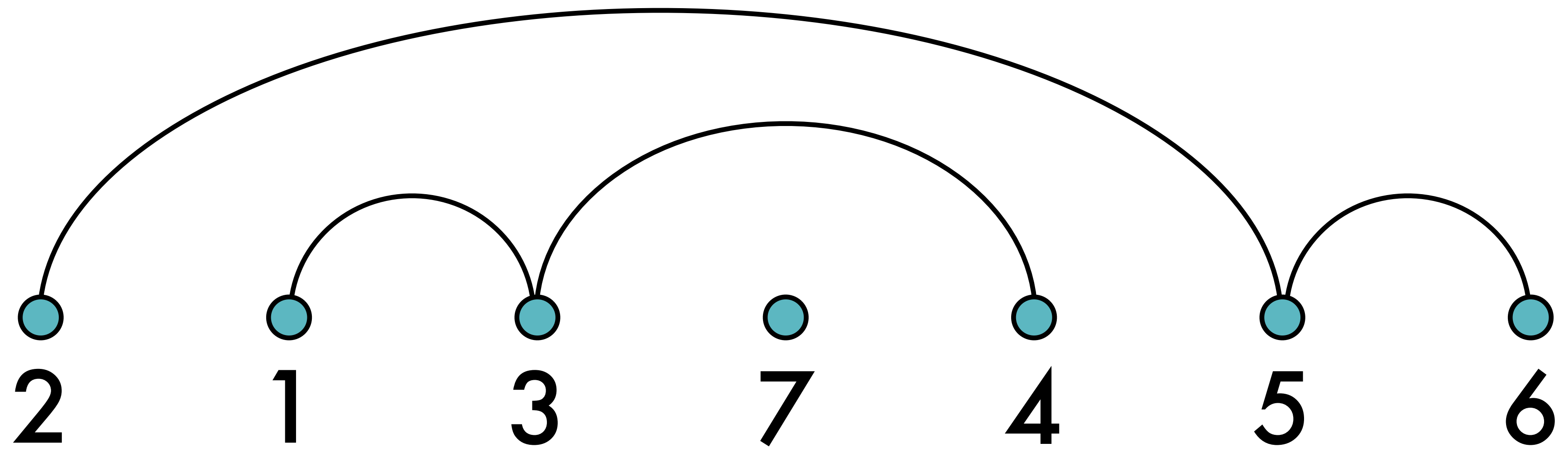
$$= \sum_k \text{Stir}(n,k) \frac{n!}{(n-k+1)!}$$

$$= (n+1)^{n-1}$$

new proof

NN to NC

Q: Why not do the same for non**Crossing** partitions?

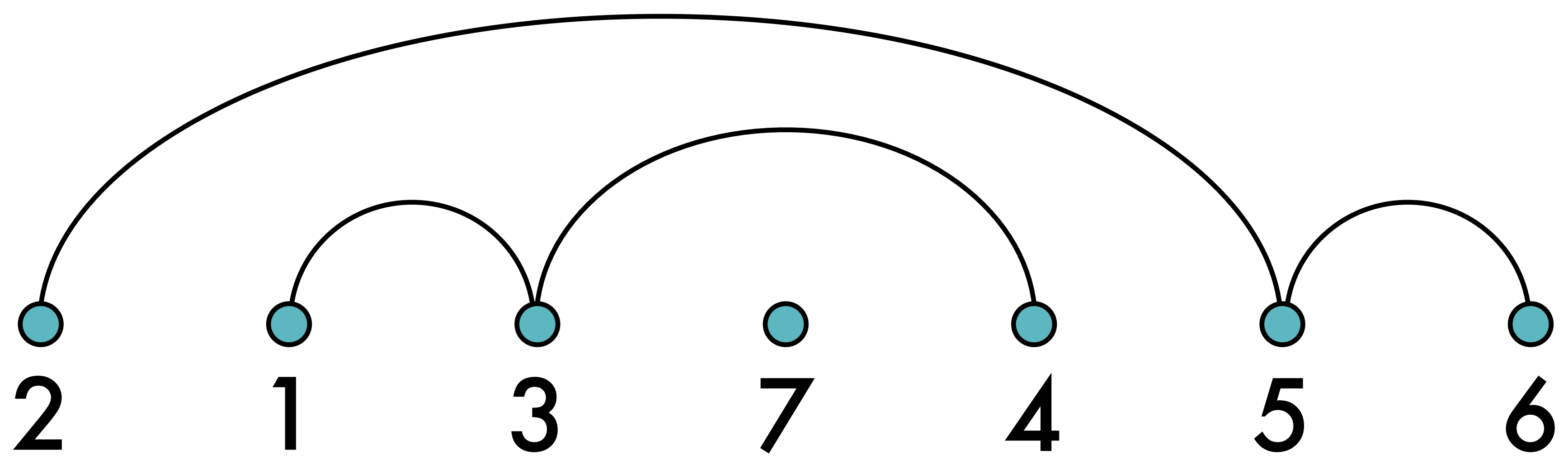


i.e. Consider pairs (w, π) where

- π is a NC partition
- w is a permutation **increasing** on blocks of π

NN to NC

Q: Why not do the same for non**Crossing** partitions?



i.e. Consider pairs (w, π) where

- π is a NC partition
- w is a permutation **increasing** on blocks of π

A: Indeed, why not?

Now Some Algebra

Again consider Weyl group $W \subseteq GL(V)$

A "Galois Correspondence":

lattice of parabolic subgroups

lattice of "flats" ...or "partitions"

$$\mathscr{P}(W) \quad \rightarrow \quad \mathscr{L}(W)$$

$$U \quad \mapsto \quad V^U$$

parabolic subgroup

the "fixed space"

...some flats/partitions are NN, some are NC...

Now Some Algebra

A General Construction: For $\mathscr{F} \subseteq \mathscr{L}(W)$ define

$$\text{Park}^{\mathscr{F}} := \{(w, X) : w \in W, X \in \mathscr{L}(W), w \cdot X \in \mathscr{F}\}/\sim$$

where

$$(w, X) \sim (w', X') \iff \begin{array}{c} X = X' \\ \text{and} \\ wW_X = w'W_{X'} \end{array}$$

Note: $\text{Park}^{\mathscr{F}}$ is a W-module.

$$u \cdot [w, X] := [wu^{-1}, uX]$$

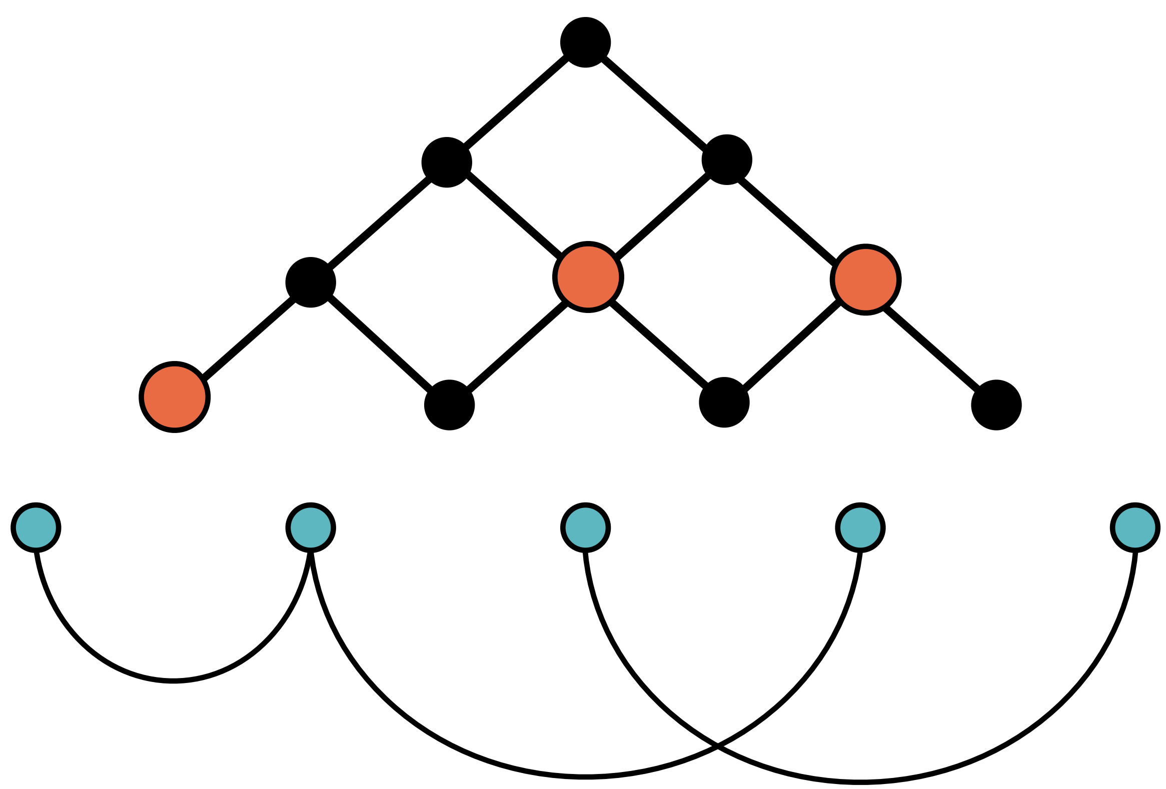
Now Some Algebra

Special Kinds of Flats/Partitions:

1. $\mathsf{NN}(W) \subseteq \mathscr{L}(W)$

- A “**nonnesting partition**” is an antichain in $(\Phi^+, \leq)$

e.g.



Now Some Algebra

Special Kinds of Flats/Partitions:

1. $\text{NN}(W) \subseteq \mathscr{L}(W)$

- A **flat/partition** $X \in \mathscr{L}(W)$ is called "**nonnesting**" if

$$\boxed{X = \bigcap_{a \in A} a^{\perp}}$$

for some **antichain** $A \in \Phi^{+}$

Now Some Algebra

Special Kinds of Flats/Partitions:

2. $\mathrm{NC}(W) \subseteq \mathscr{L}(W)$

- Let $c \in W$ be a "**Coxeter element**" (e.g. an n-cycle)
- A "**noncrossing partition**" is a group element

$$w \in [\mathrm{id}, c]_T$$

where $T \subseteq W$ are the **reflections** in W

and $[\mathrm{id}, c]_T$ is an interval in the **Cayley graph** wrt T

Now Some Algebra

Special Kinds of Flats/Partitions:

2. $\mathsf{NC}(W) \subseteq \mathscr{L}(W)$

- A **flat/partition** $X \in \mathscr{L}(W)$ is called "**noncrossing**" if

$$\boxed{X = V^w}$$

for some noncrossing $w \in W$

Now Some Algebra

Special Kinds of Flats/Partitions:

2. $\mathrm{NC}(W) \subseteq \mathscr{L}(W)$

- A **flat/partition** $X \in \mathscr{L}(W)$ is called "**noncrossing**" if

$$\boxed{X = V^w}$$

for some noncrossing $w \in W$

Now Some Theorems??

Now Some Theorems

Some Theorems (ARR):

1. $\mathrm{Park}^{\mathrm{NN}(W)}$ is a **parking module**.

   i.e. $\mathrm{Park}^{\mathrm{NN}(W)} \cong_W Q/(h+1)Q$

Proof is **Uniform** for Weyl Groups. (smile)

Now Some Theorems

Some Theorems (ARR):

2. $\text{Park}^{\text{NC}(W)}$ is a **parking module**.

i.e. $\text{Park}^{\text{NC}(W)} \cong_W Q/(h+1)Q$

Proof is **Case-By-Case**. (frown)

Now Some Theorems

Some Theorems (ARR):

But wait... $\mathsf{Park}^{\mathsf{NC}(W)}$ has **more structure**.

cyclic group of order "h" generated by Coxeter element

Let $C = \langle c \rangle$

Then $\mathsf{Park}^{\mathsf{NC}(W)}$ is a $W \times C$-module:

$$(u, c^d) \cdot [w, X] := [c^d w u^{-1}, uX]$$

Now Some Theorems

Some Theorems (ARR):

$2'$. $\mathrm{Park}^{\mathrm{NC}(W)}$ **has $W \times C$ -character**
(not yet checked for exceptional types)

$$\chi(w, c^d) = \lim_{q \to \zeta^d} \frac{\det(1 - q^{h+1} w)}{\det(1 - qw)}$$

$$\zeta = e^{2\pi i/h}$$

$$= (h+1)^{\mathrm{mult}_w(\zeta^d)}$$

eigenvalue multiplicity

Second Last Slide

Some Remarks:

1. $\text{Park}^{\mathscr{F}}$ for other kinds of flats/partitions?

2. $\text{Park}^{\text{NC}(W)}$ works for more groups W.

3. Where's the cyclic action on $\text{Park}^{\text{NN}(W)}$?
   Panyushev?

Last Slide

Wanted: Equivariant Bijection

$$\mathrm{Park}^{\mathrm{NN}(W)} \longleftrightarrow \mathrm{Park}^{\mathrm{NC}(W)}$$

$\mathrm{Park}^{\mathrm{NN}(W)}$		$\mathrm{Park}^{\mathrm{NC}(W)}$	
enumeration	✓	enumeration	?
cyclic action	?	cyclic action	✓

Thanks!