Hyperplane Arrangements & Diagonal Harmonics

Drew Armstrong

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Coinvariants

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Theorems (Newton-Chevalley-etc):

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- Let $\mathfrak{S}_n$ act on $S = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables.
- Then we have

$$S^{\mathfrak{S}_n} \cong \mathbb{C}[p_1, \ldots, p_n]$$

where $p_k = \sum_{i=1}^{n} x_i^k$ are the

power sum symmetric polynomials.

Coinvariants

Theorems (Newton-Chevalley-etc):

- The **coinvariant ring** $R := S/(p_1, \ldots, p_n)$ is isomorphic to the regular representation:

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- The **coinvariant ring** $R := S/(p_1, \ldots, p_n)$ is isomorphic to the regular representation:

$$R \cong_{\mathfrak{S}_n} \mathbb{C}\mathfrak{S}_n$$

- And it's **graded**, with **Hilbert series**

$$\sum_i \dim R_i \, q^i = \prod_{j=1}^{n} (1 + q + \cdots + q^{j-1}) = [n]_q!$$

"the q–factorial"

Coinvariants

What's next?

- Let $\mathfrak{S}_n$ act on $DS = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ **diagonally**.

Coinvariants

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- Let $\mathfrak{S}_n$ act on $DS = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ **diagonally**.
- (Weyl) Then the ring of **diagonal invariants** $DS^{\mathfrak{S}_n}$ is generated by the "polarized" power sums

$$p_{k,\ell} = \sum_{i=1}^{n} x_i^k y_i^\ell \quad \text{for} \quad k + \ell > 0$$

NOT algebraically independent

Coinvariants

Hard Theorem (Haiman, 2001):

- The **diagonal coinvariant ring**

$$DR := DS/(p_{k,\ell} : k + \ell > 0)$$

has dimension $(n+1)^{n-1}$

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Hard Theorem (Haiman, 2001):

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has dimension $(n+1)^{n-1}$

Ongoing Project:

- Describe the (bigraded) Hilbert/Frobenius series!
- New science of "parking functions"

Affine Permutations

Affine Permutations

- **Bijections:** $\pi : \mathbb{Z} \to \mathbb{Z}$
- **“Periodic”:** $\forall k \in \mathbb{Z},\ \pi(k+n) = \pi(k) + n$
- **Frame of Reference:** $\pi(1) + \pi(2) + \cdots + \pi(n) = \binom{n+1}{2}$

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example

$$\begin{array}{c|ccc|ccc|ccc|c} k & \cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \pi(k) & \cdots & -3 & -1 & 1 & 0 & 2 & 4 & 3 & 5 & 7 & \cdots \end{array}$$

The "window notation": $\pi = [0, 2, 4]$

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The "window notation": $\pi = [0, 2, 4]$

Also observe: $\pi = \cdots(-3,-2)(0,1)(3,4)(6,7)\cdots$

Affine Permutations

Define affine transpositions:

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Then we have:

$$\tilde{\mathfrak{S}}_n = \langle ((1,2)), ((2,3)), \ldots, ((n,n+1)) \rangle$$

"**affine symmetric group**" generated by "**affine adjacent transpositions**"

Affine Permutations

(Lusztig, 1983) says it's a Weyl group

"transposition"		"reflection in hyperplane"
$((1,2))$	$\longrightarrow$	$x_1 - x_2 = 0$
$((2,3))$	$\longrightarrow$	$x_2 - x_3 = 0$
	$\vdots$	
$((n-1,n))$	$\longrightarrow$	$x_{n-1} - x_n = 0$
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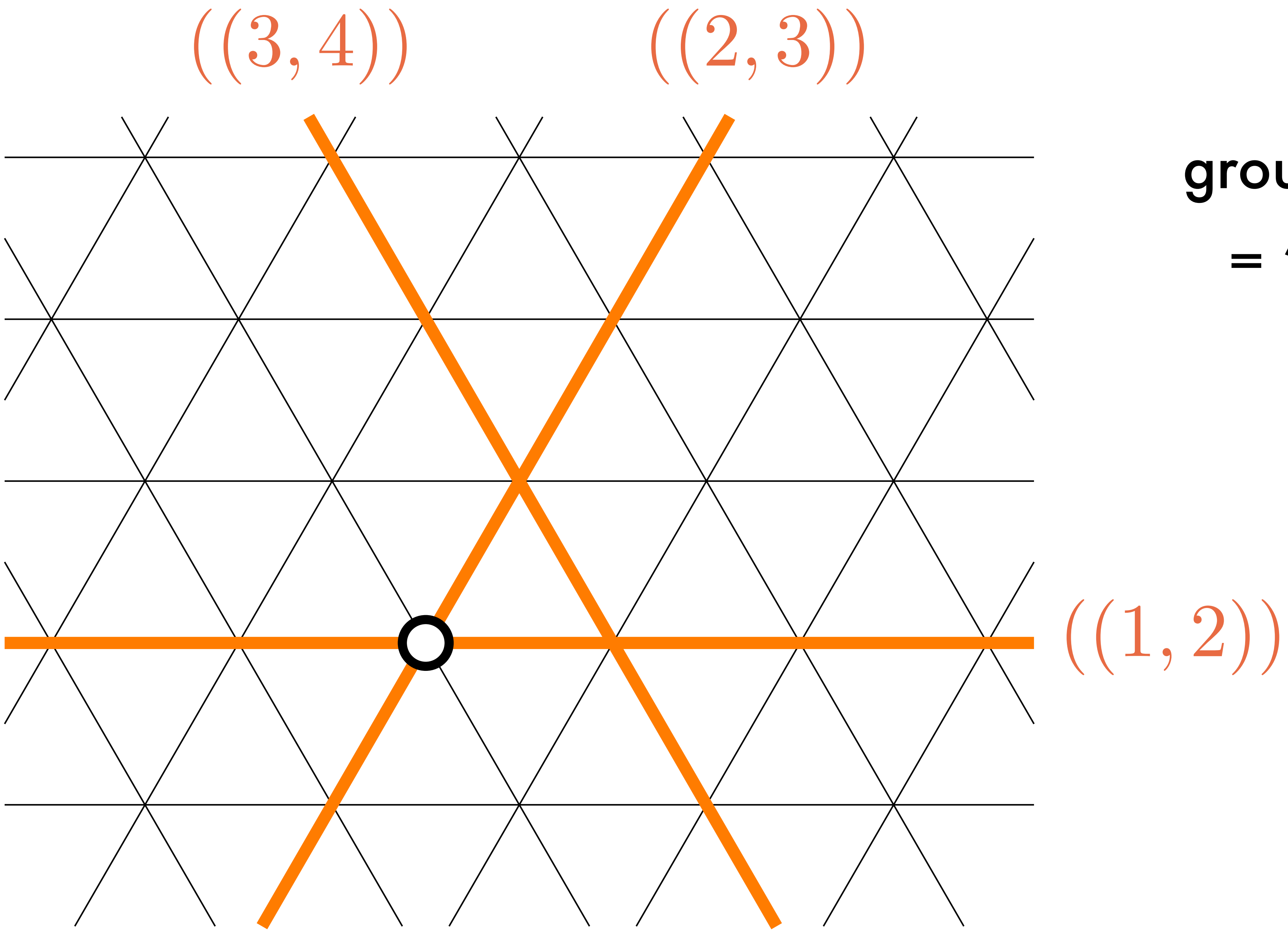
Abuse of notation:

$$\mathfrak{S}_n = \langle ((1,2)), ((2,3)), \ldots, ((n-1,n)) \rangle$$

"**finite** symmetric group"

Picture of Affine S3



group elements

= “alcoves”

Two ways to think

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Way 1.

$$\tilde{\mathfrak{S}}_n = \mathfrak{S}_n \times \mathfrak{S}^n$$

$=$ (finite symmetric group) X (minimal coset reps)

$=$ (which cone are you in?) X (where in the cone?)

$=$ (permute window notation) X (into increasing order)

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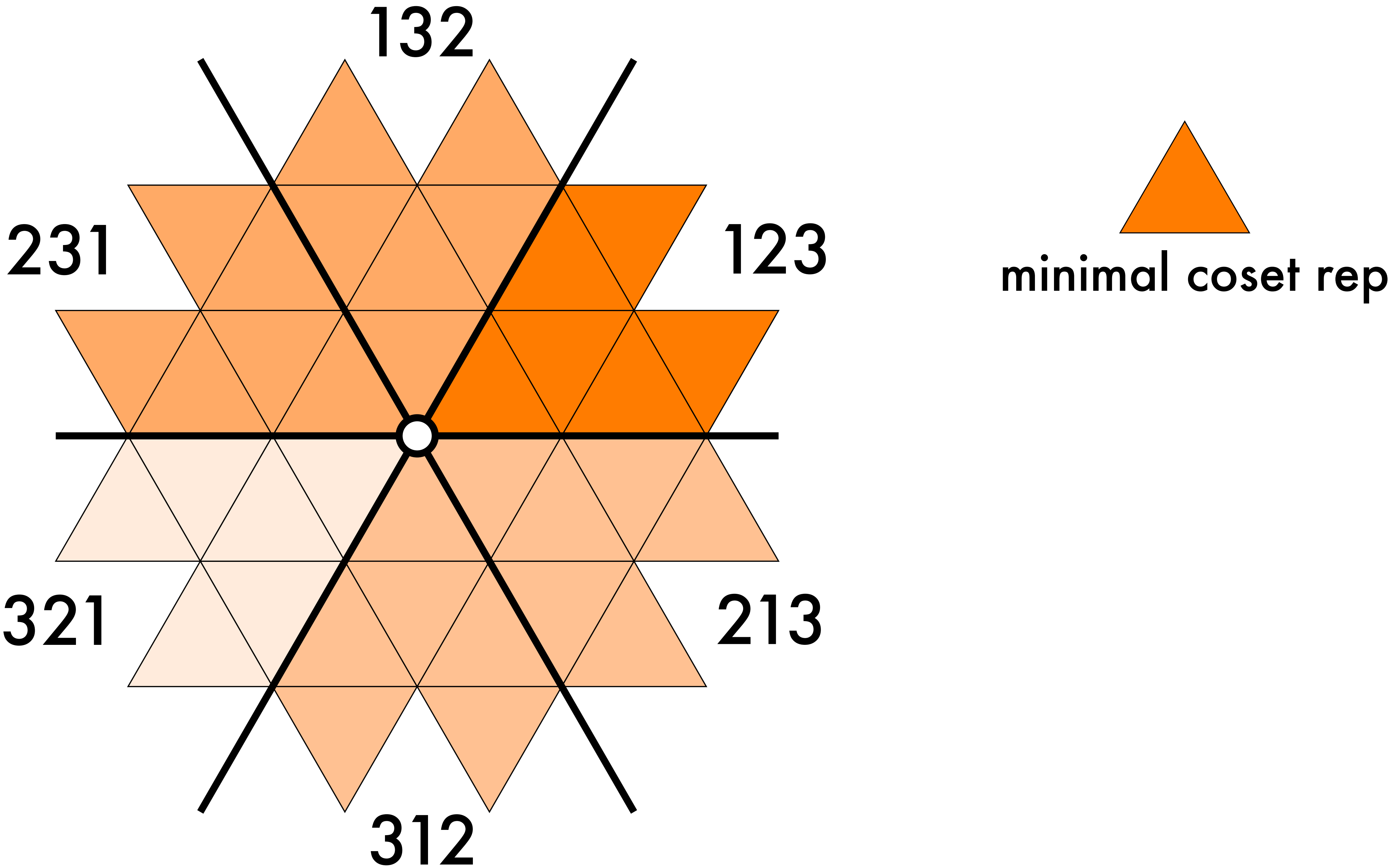
example

$$[6, -3, 8, -1] = [3, 1, 4, 2] \times [-3, -1, 8, 6]$$

Picture of Way 1



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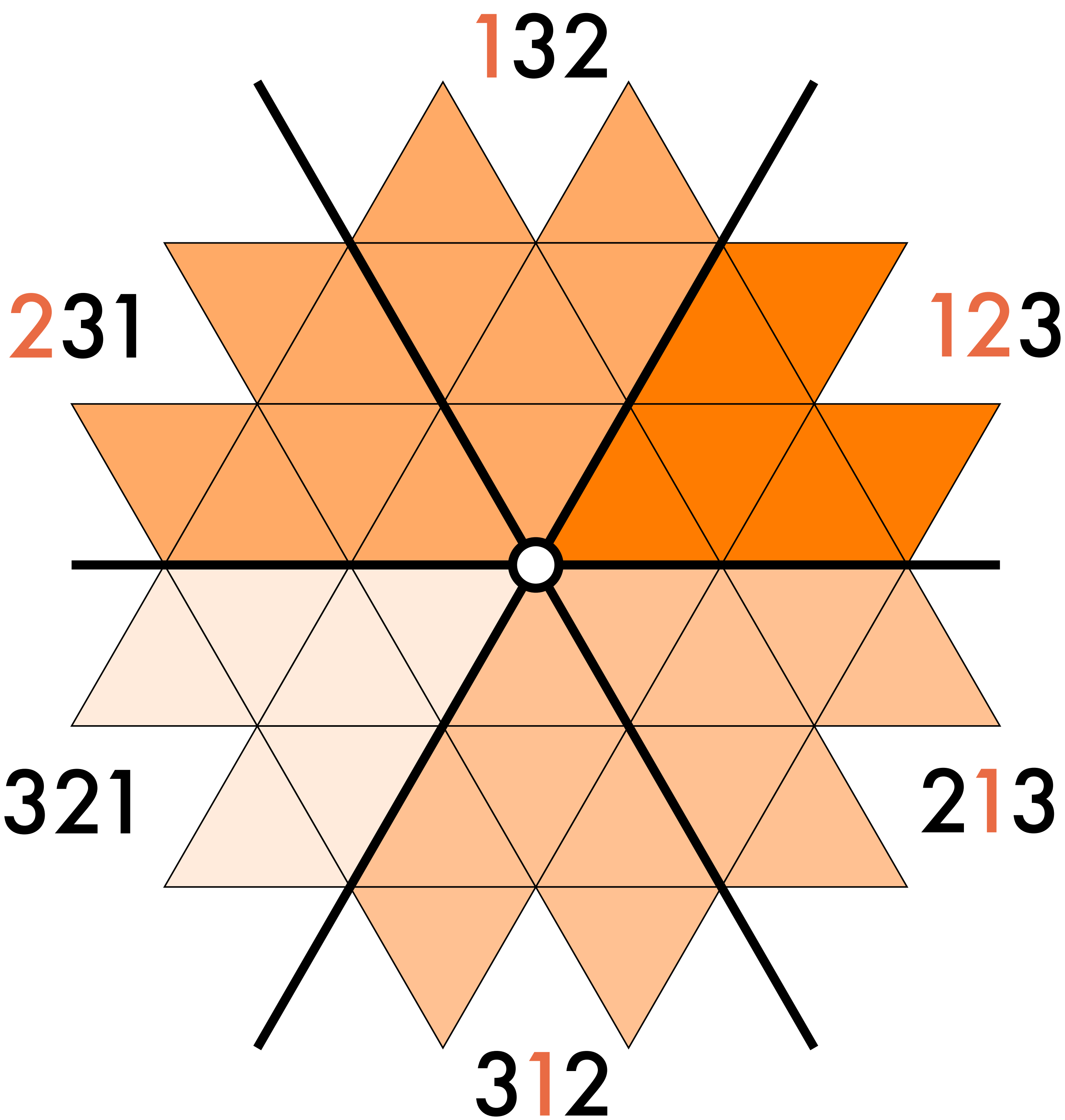
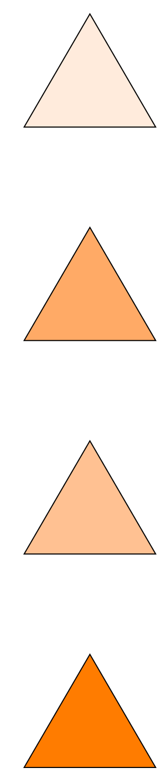


For Posterity:

Note (finite) **ascent sets** in window notation



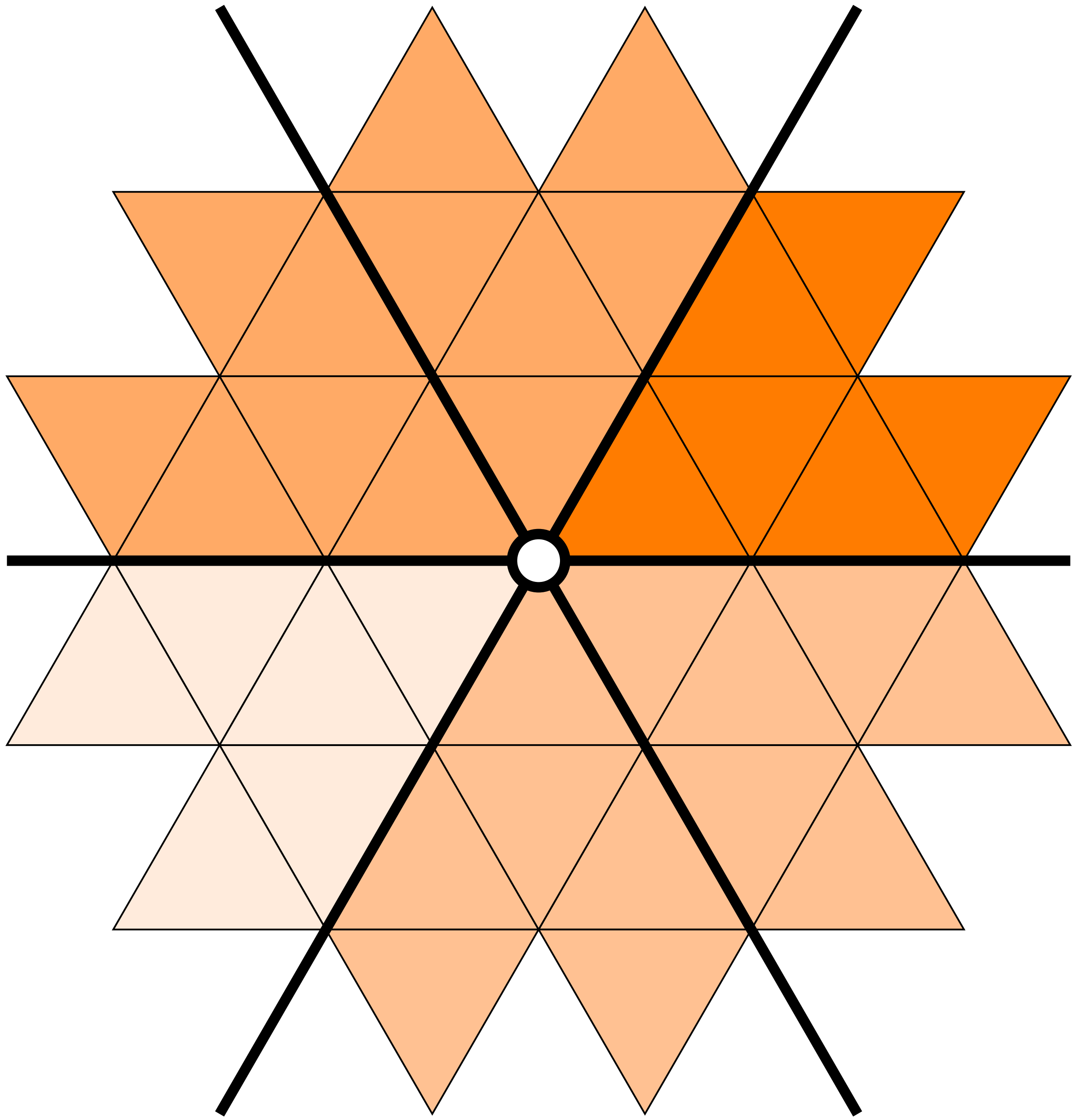
$= \emptyset$

$= \{1\}$

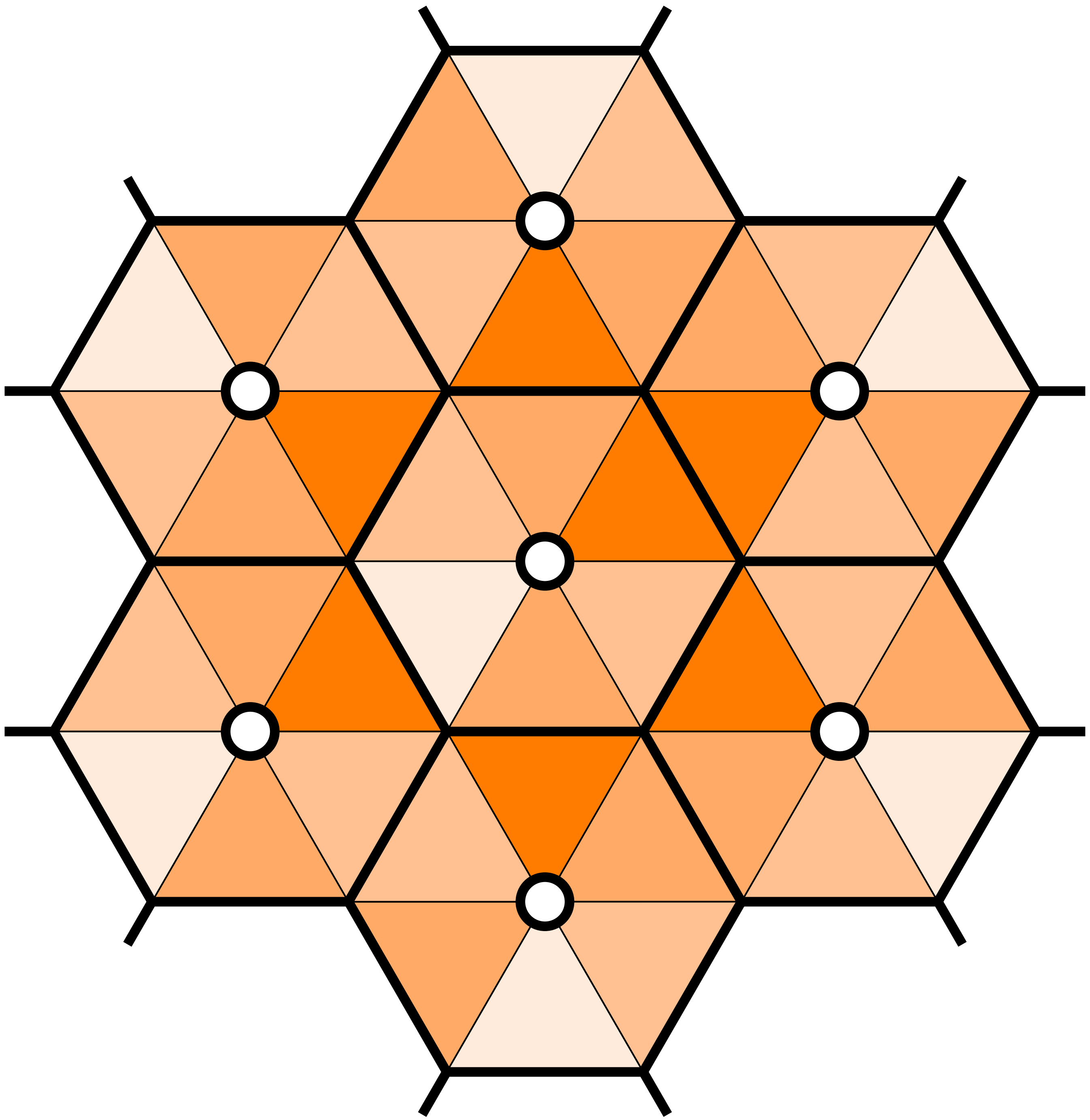
$= \{2\}$

$= \{1, 2\}$

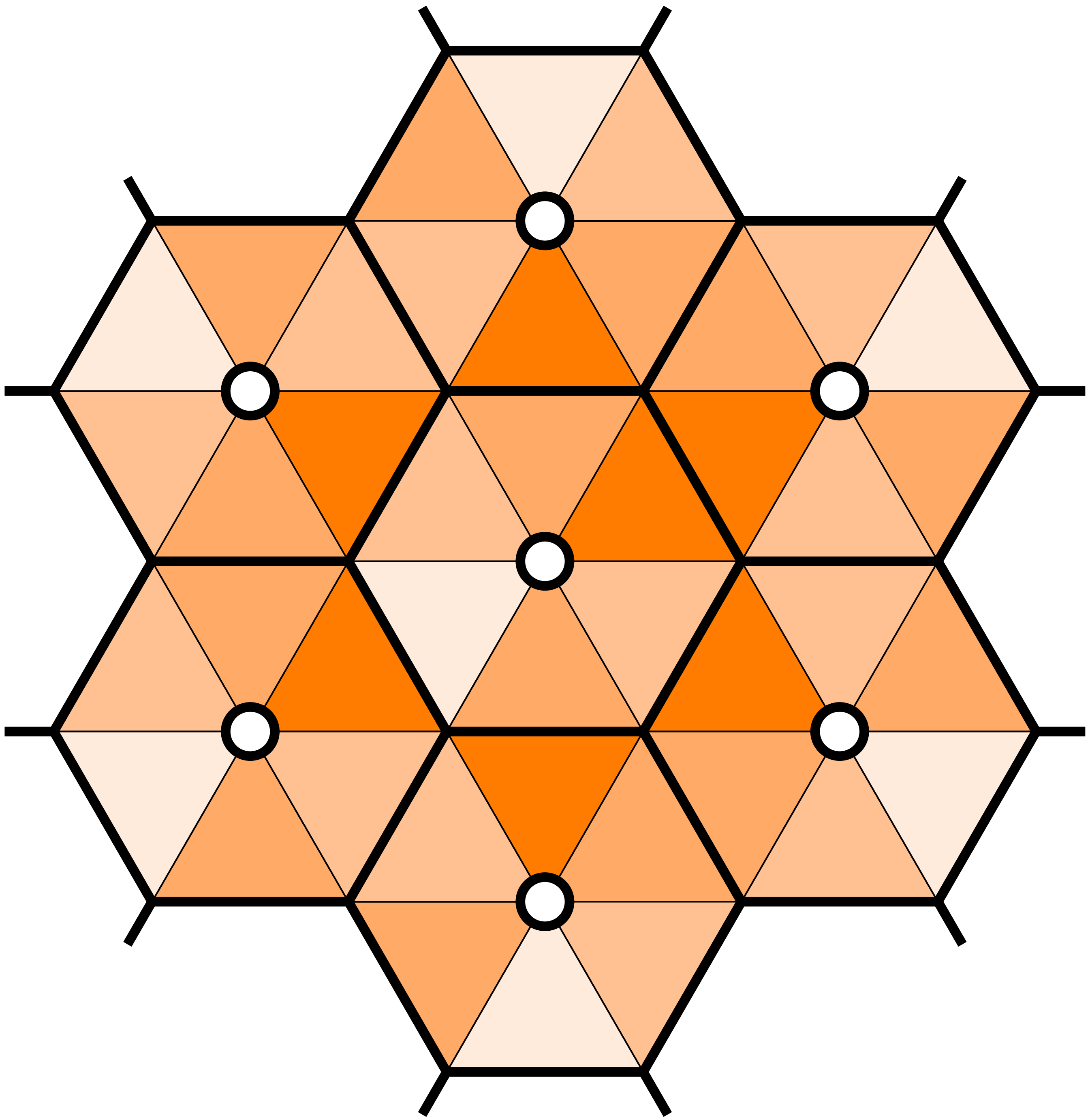
What if we invert?



Invert!



This is Way 2 to think.



This is Way 2 to think.

$$\tilde{\mathfrak{S}}_n = \mathfrak{S}_n \ltimes Q_n$$

$= \mathfrak{S}_n$ semi-direct product with the **root lattice**

$$Q_n = \left\{ (r_1, \ldots, r_n) \in \mathbb{Z}^n : \sum_i r_i = 0 \right\}$$

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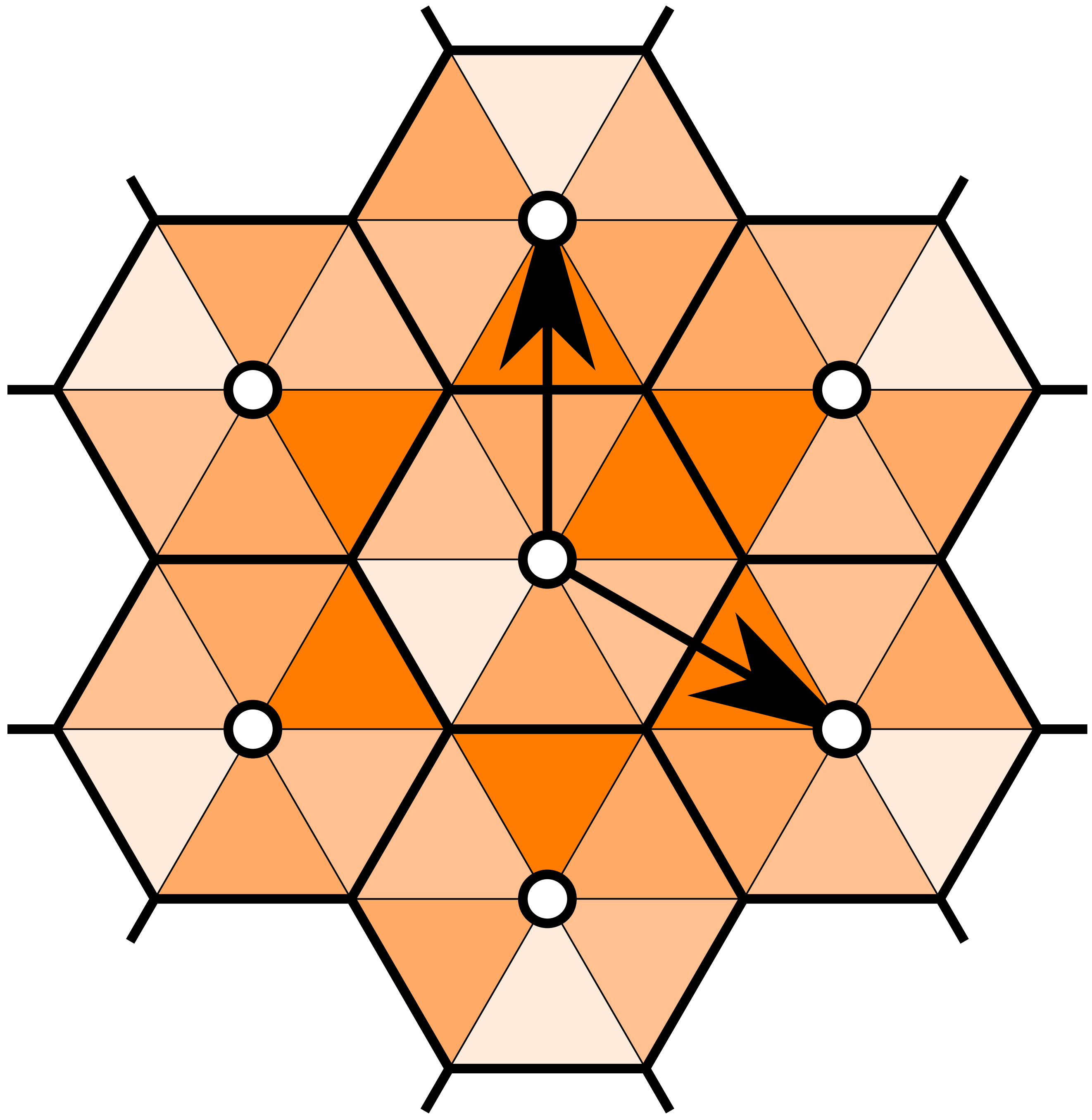
In terms of window notation:

$$[6, -3, 8, -1] = (2, 1, 4, 3) + 4 \cdot (1, -1, 1, -1)$$

"finite permutation + *n* times a root"

"division with remainder"

This is Way 2 to think.

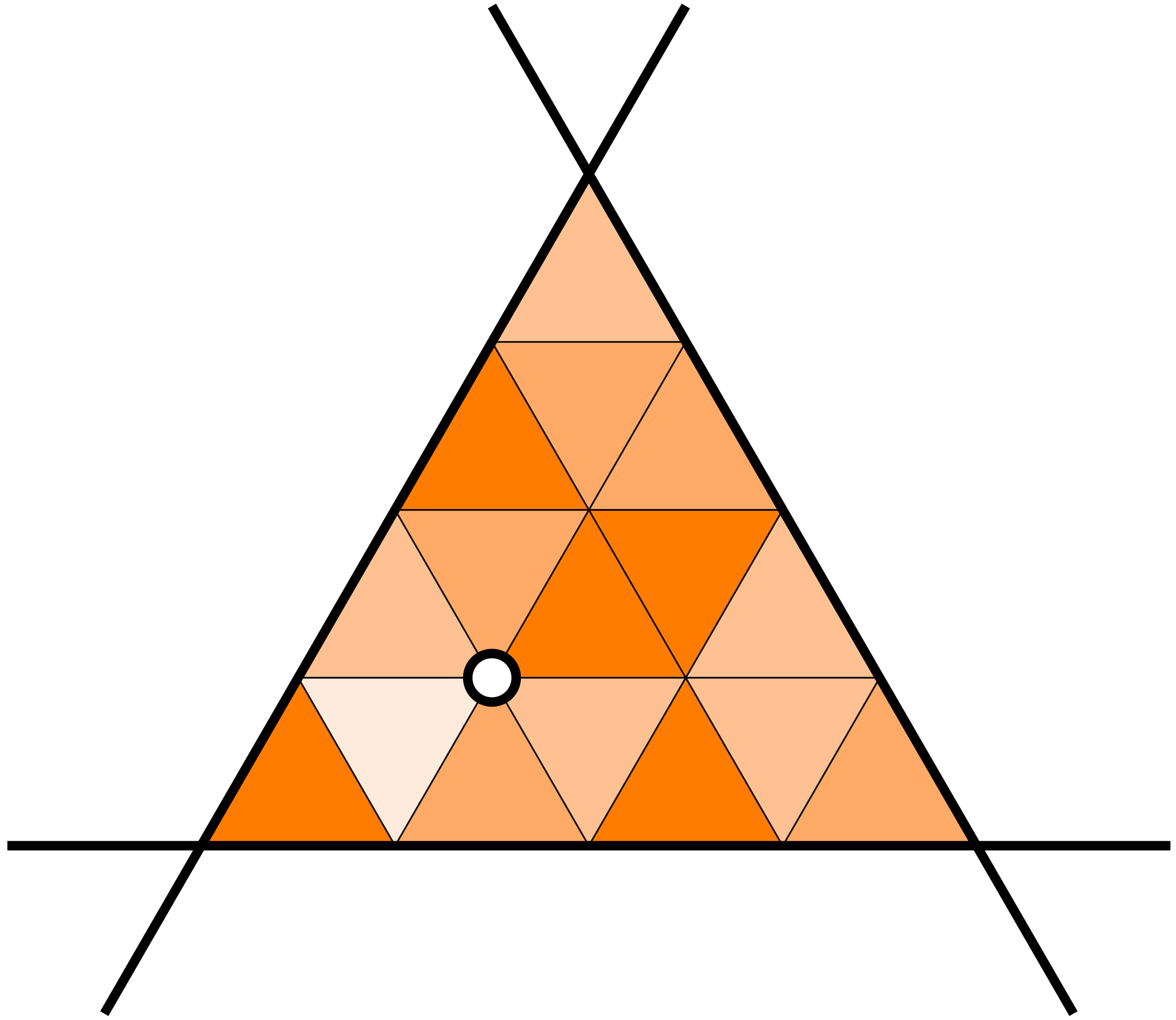


a copy of $\mathfrak{S}_3$ around each root vector $\circ \in Q_3$

Now for Shi and Ish

Now for Shi and Ish

Consider a special simplex

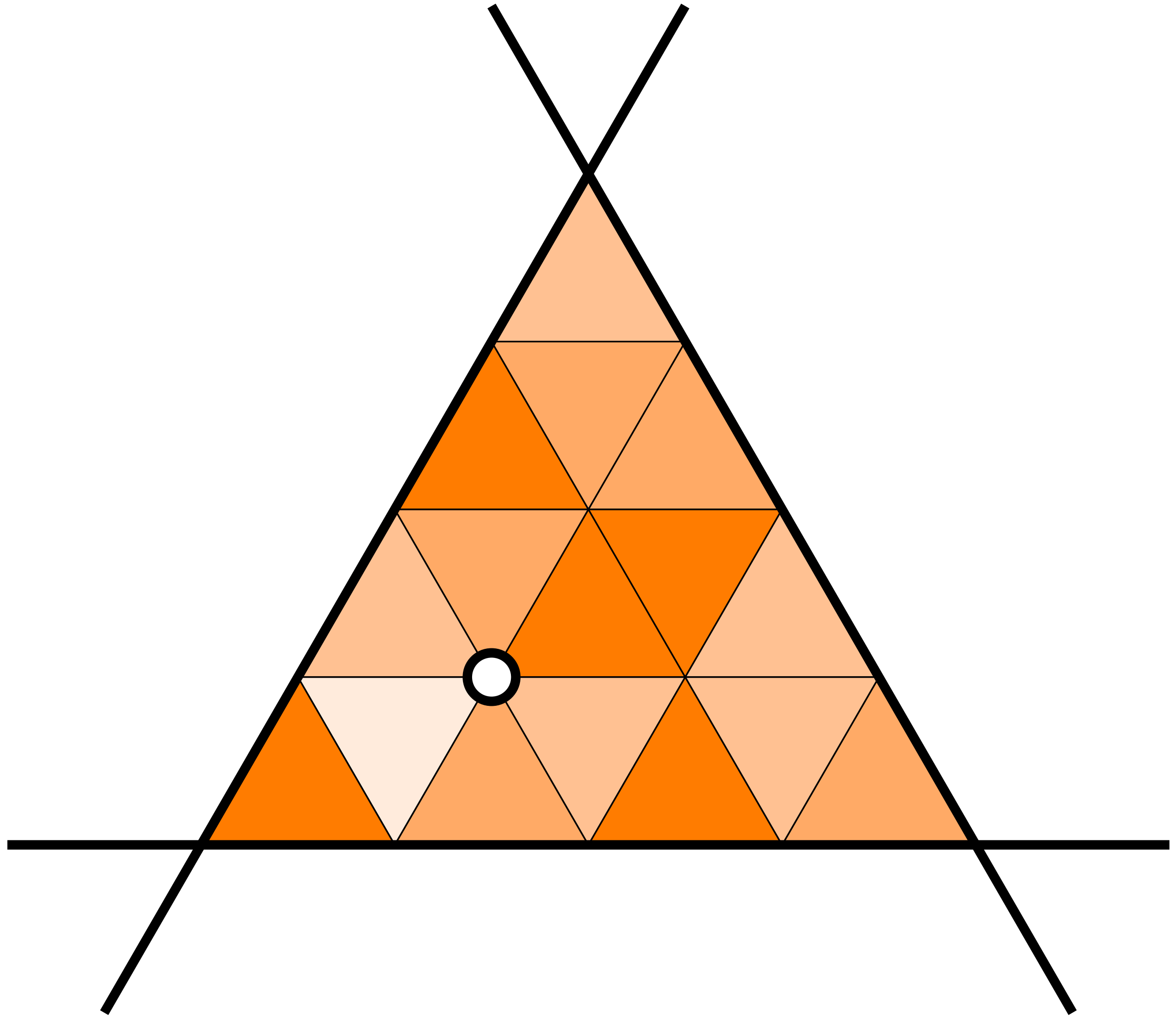


Bounded by:

$$\begin{aligned} x_1 - x_2 &= -1 \\ x_2 - x_3 &= -1 \\ &\vdots \\ x_{n-1} - x_n &= -1 \\ x_1 - x_n &= 2 \end{aligned}$$

Now for Shi and Ish

Consider a special simplex

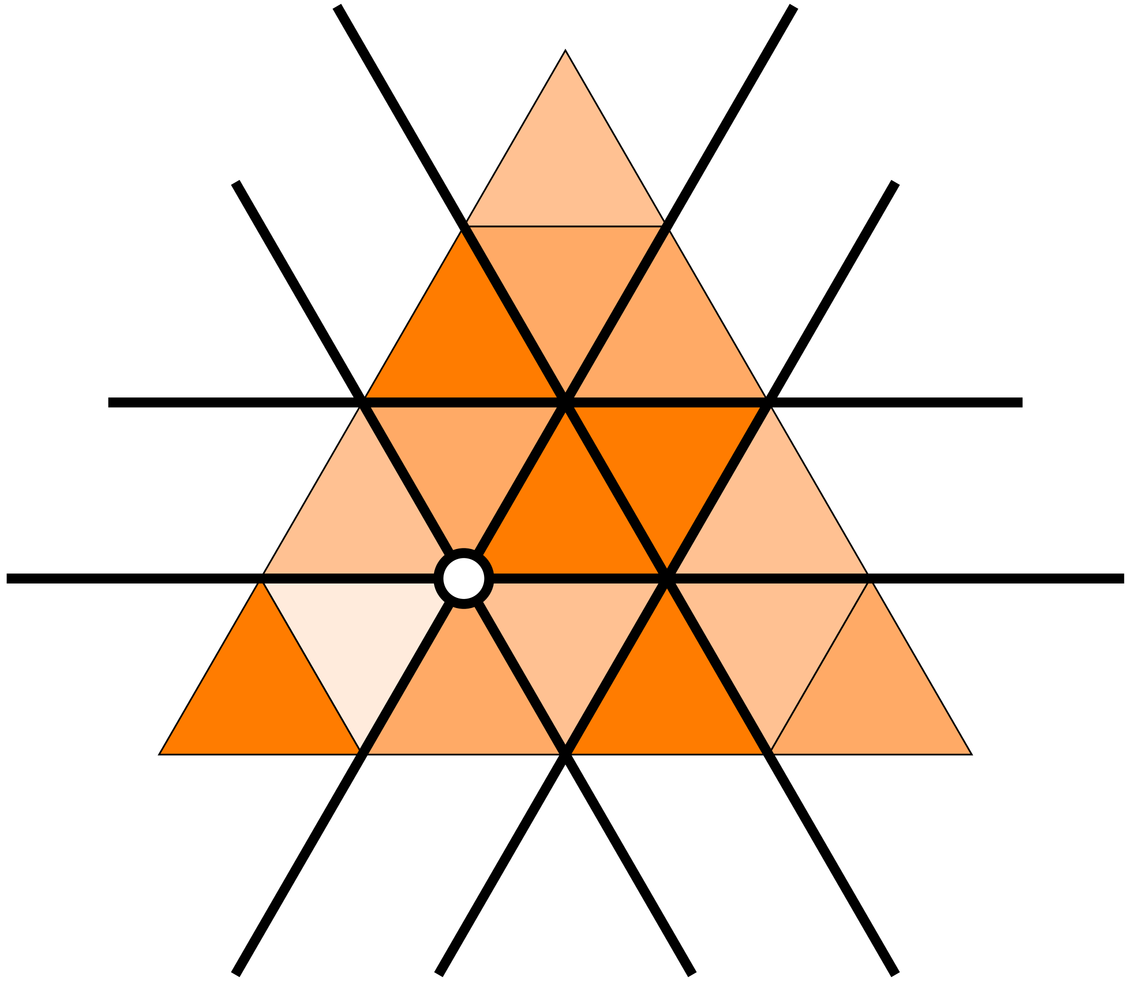


It's a dilation of the **fundamental alcove** by a factor of $n+1$

Hence it contains $(n+1)^{n-1}$ alcoves!

Now for Shi and Ish

Cut it with the **Shi arrangement**



Shi arrangement:

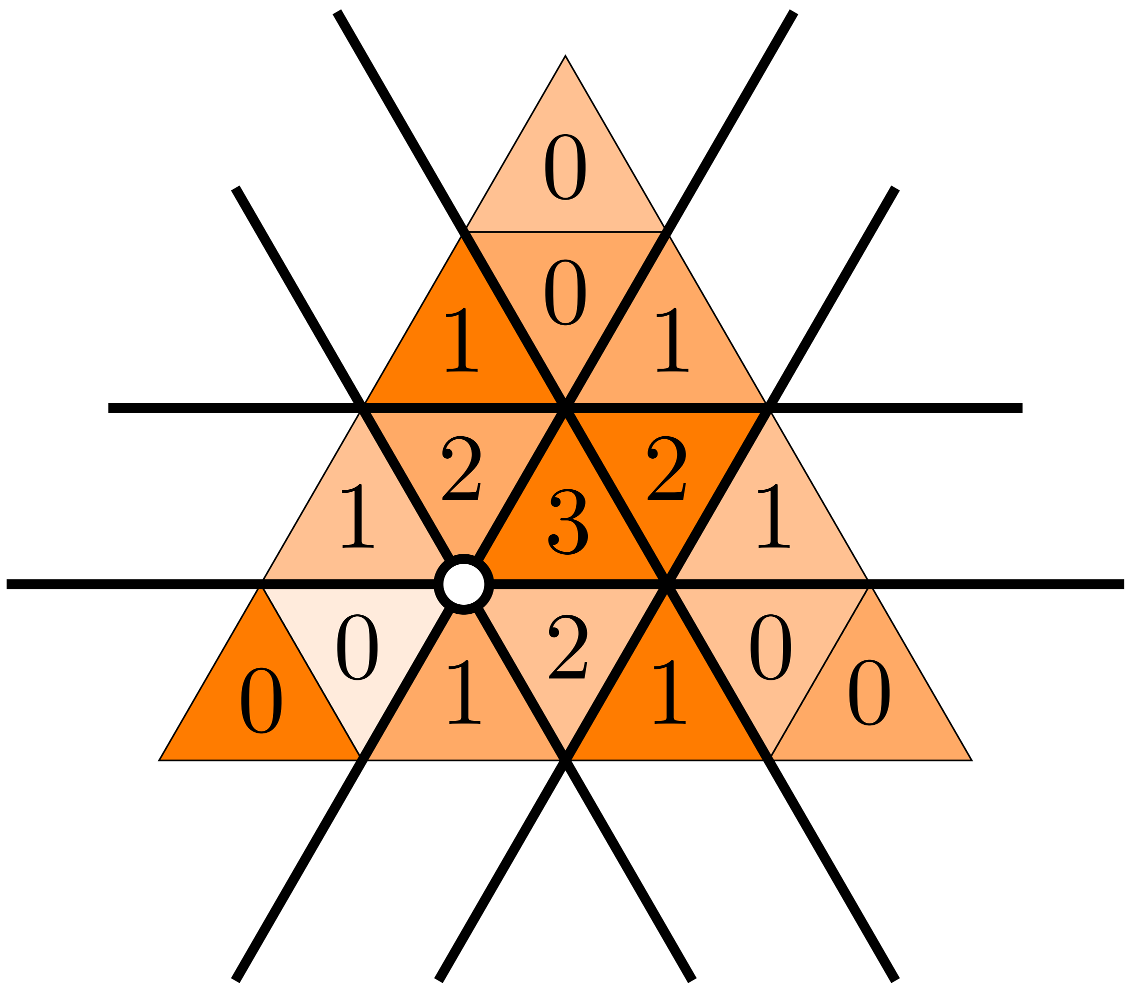
$$x_i - x_j = 0, 1$$

for all

$$1 \leq i < j \leq n$$

Now for Shi and Ish

And consider the "distance enumerator"
(call it **"shi"**)



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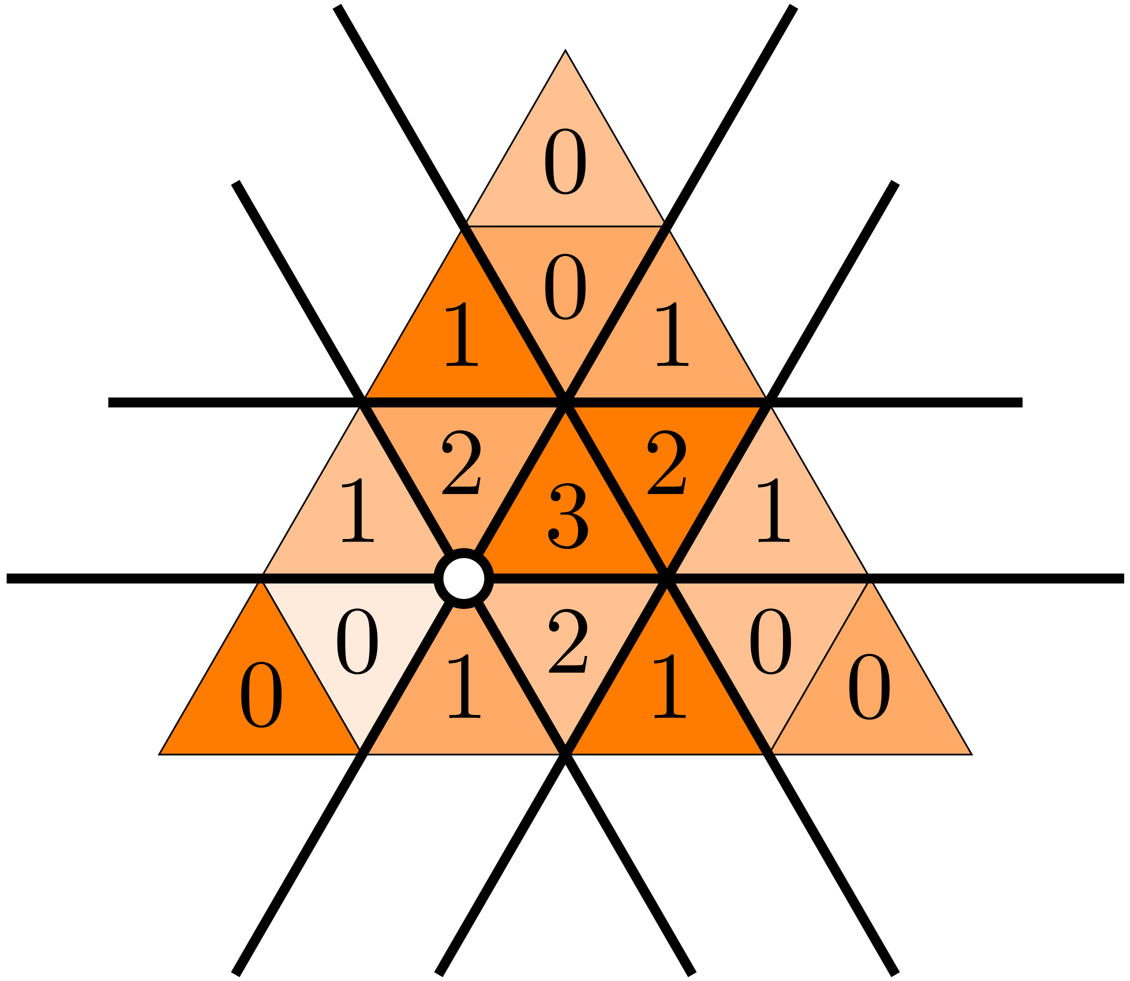

example

$$\sum q^{\mathsf{shi}} = 6 + 6q + 3q^2 + q^3$$

Now for Shi and Ish

Next define a statistic on the root lattice:

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let j be **maximal** such that r_j is **minimal**.

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$$\text{ish}(\mathbf{r}) := j - n(r_j + 1)$$

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let j be **maximal** such that r_j is **minimal**.

Then

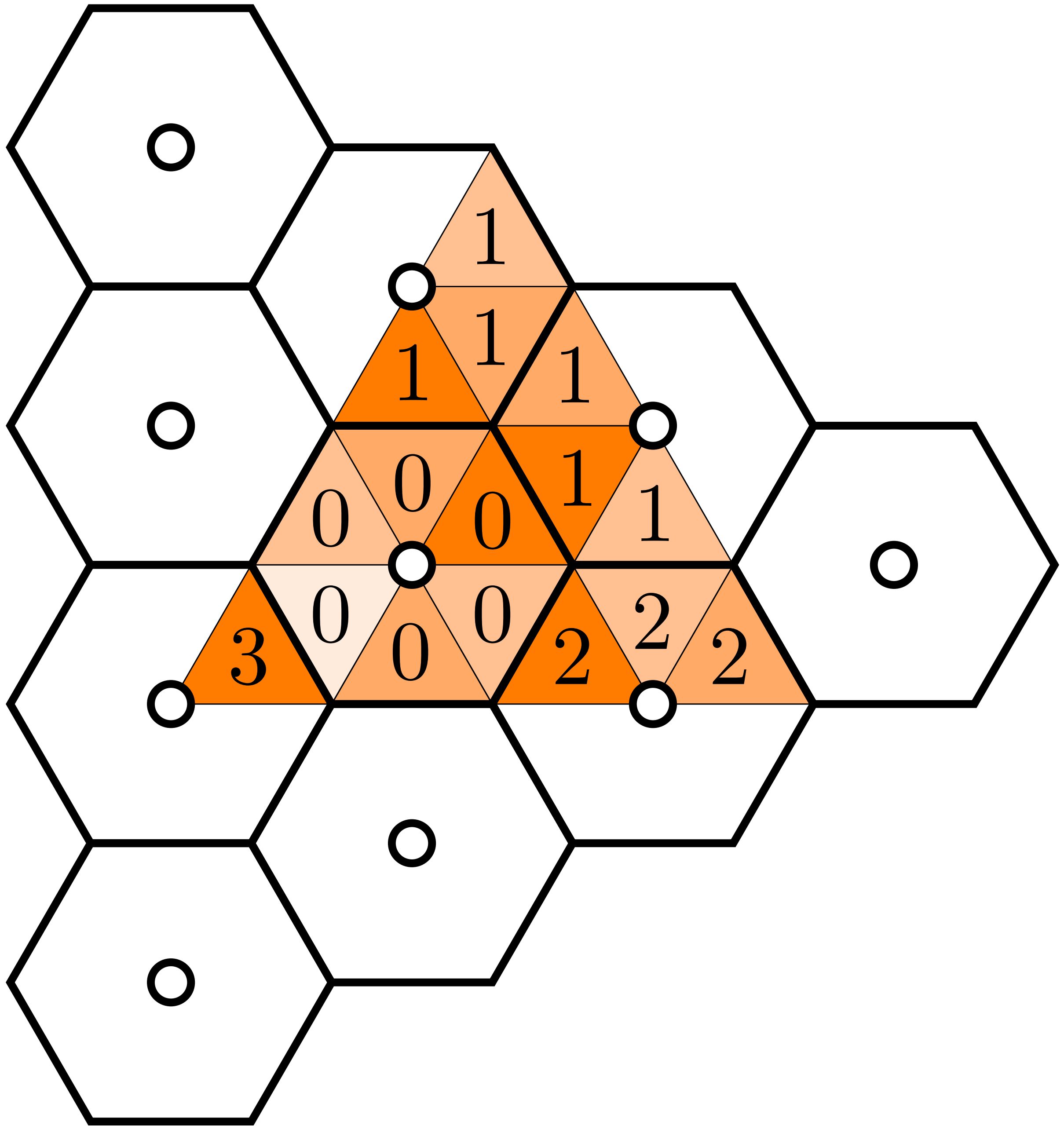
$$\text{ish}(\mathbf{r}) := j - n(r_j + 1)$$

$$\text{ish}(2, -2, 2, -2, 0) = 4 - 5 \cdot (-2 + 1) = 9$$

here $n = 5, \quad j = 4, \quad r_j = -2$

Now for Shi and Ish

ish spirals.



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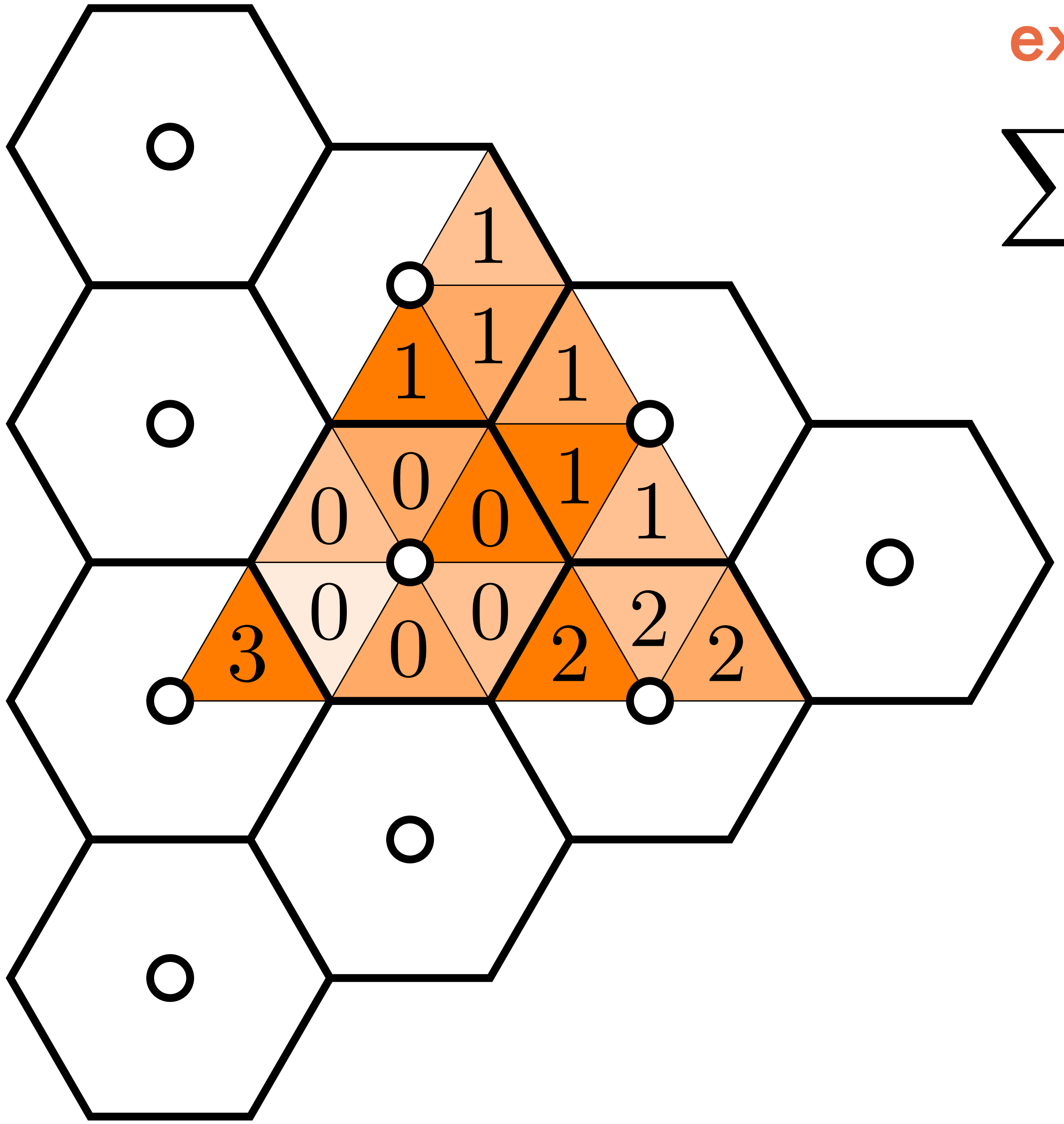

example

$$\sum t^{\mathsf{ish}} = 6 + 6t + 3t^2 + t^3$$

Now for Shi and Ish

The joint distribution:

ish

shi		0	1	2	3
	0	1	2	2	1
	1	2	3	1	
	2	2	1		
	3	1			

symmetry??

Now for Shi and Ish

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Conjectures:

- Joint Symmetry: $\sum q^{\mathsf{shi}} t^{\mathsf{ish}} = \sum t^{\mathsf{shi}} q^{\mathsf{ish}}$

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Conjectures:

- Joint Symmetry: $\sum q^{\text{shi}} t^{\text{ish}} = \sum t^{\text{shi}} q^{\text{ish}}$
- In fact, we have $\sum q^{\text{shi}} t^{\text{ish}} =$ Hilbert series of DR

diagonal coinvariants

Now for Shi and Ish

Finally...

to each alcove ascent set Asc we associate the

(Gessel) Fundamental Quasisymmetric Function

$$F_{\mathsf{Asc}} = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in \mathsf{Asc} \Rightarrow i_j < i_{j+1}}} z_{i_1} z_{i_2} \cdots z_{i_n}$$

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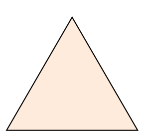
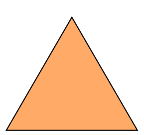
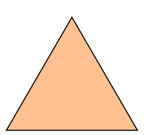
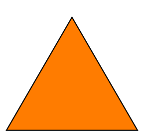
 $= F_{\emptyset}$  $= F_{\{1\}}$  $= F_{\{2\}}$  $= F_{\{1,2\}}$

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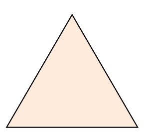
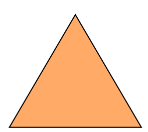
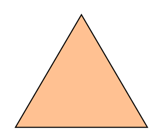
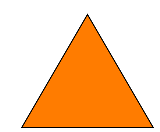
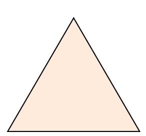
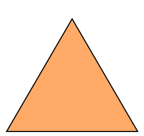
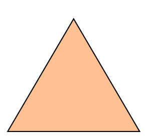
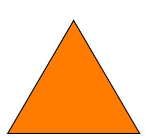
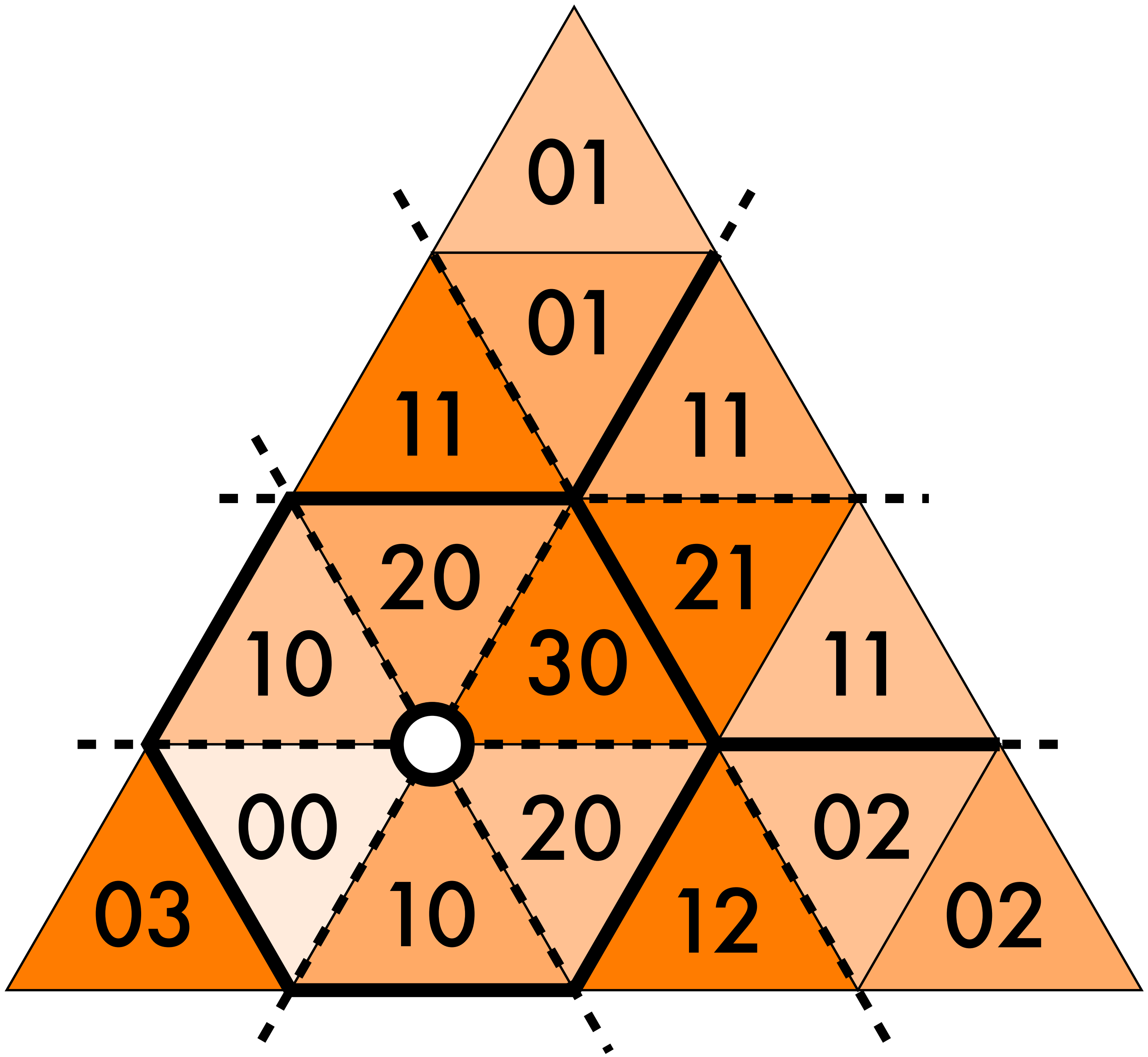
 $= F_{\emptyset}$  $= F_{\{1\}}$  $= F_{\{2\}}$  $= F_{\{1,2\}}$

 $=$ schur(3) (trivial representation)

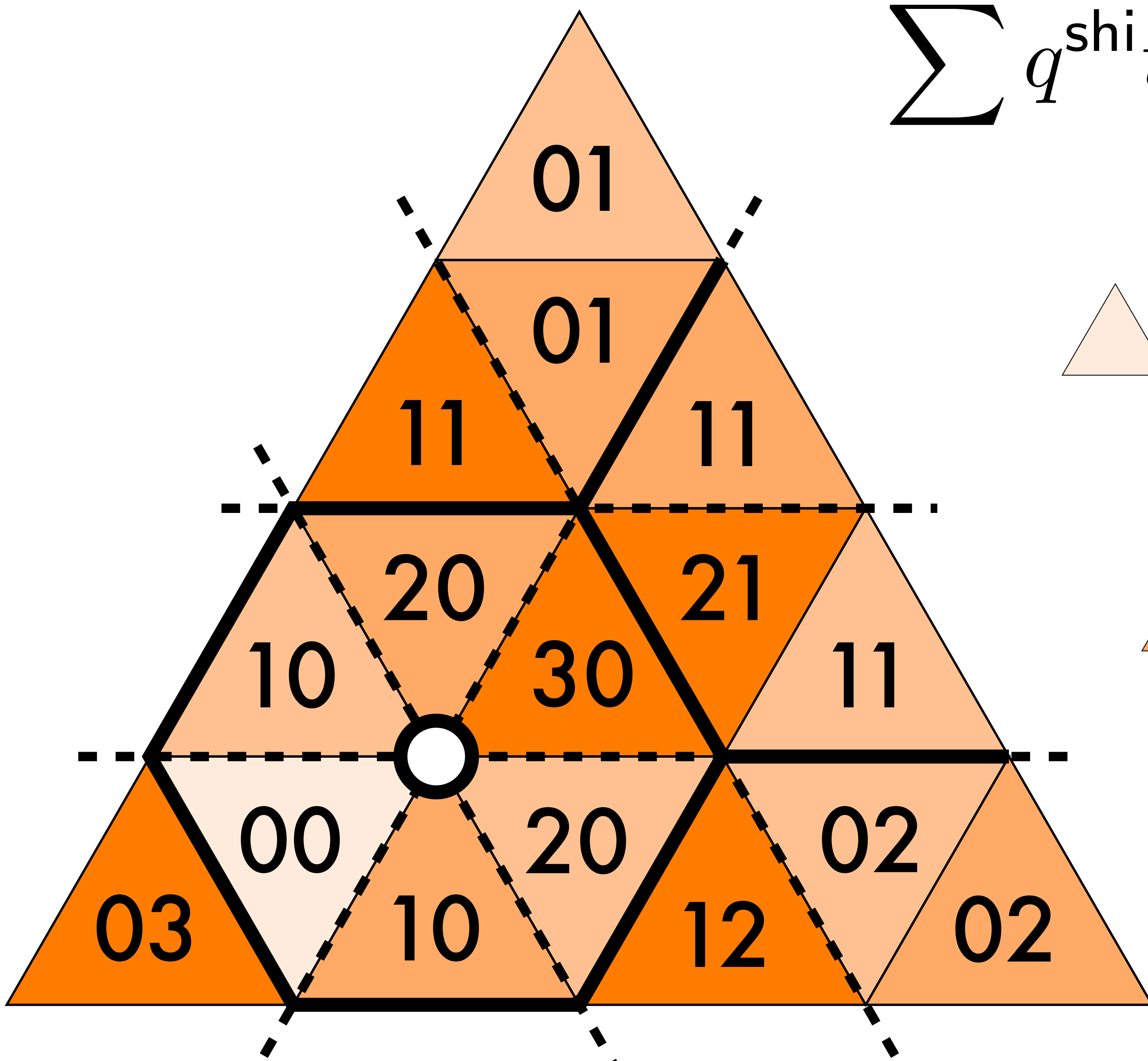
 $+$  $=$ schur$(2,1)$ (the other one)

 $=$ schur$(1,1,1)$ (sign representation)

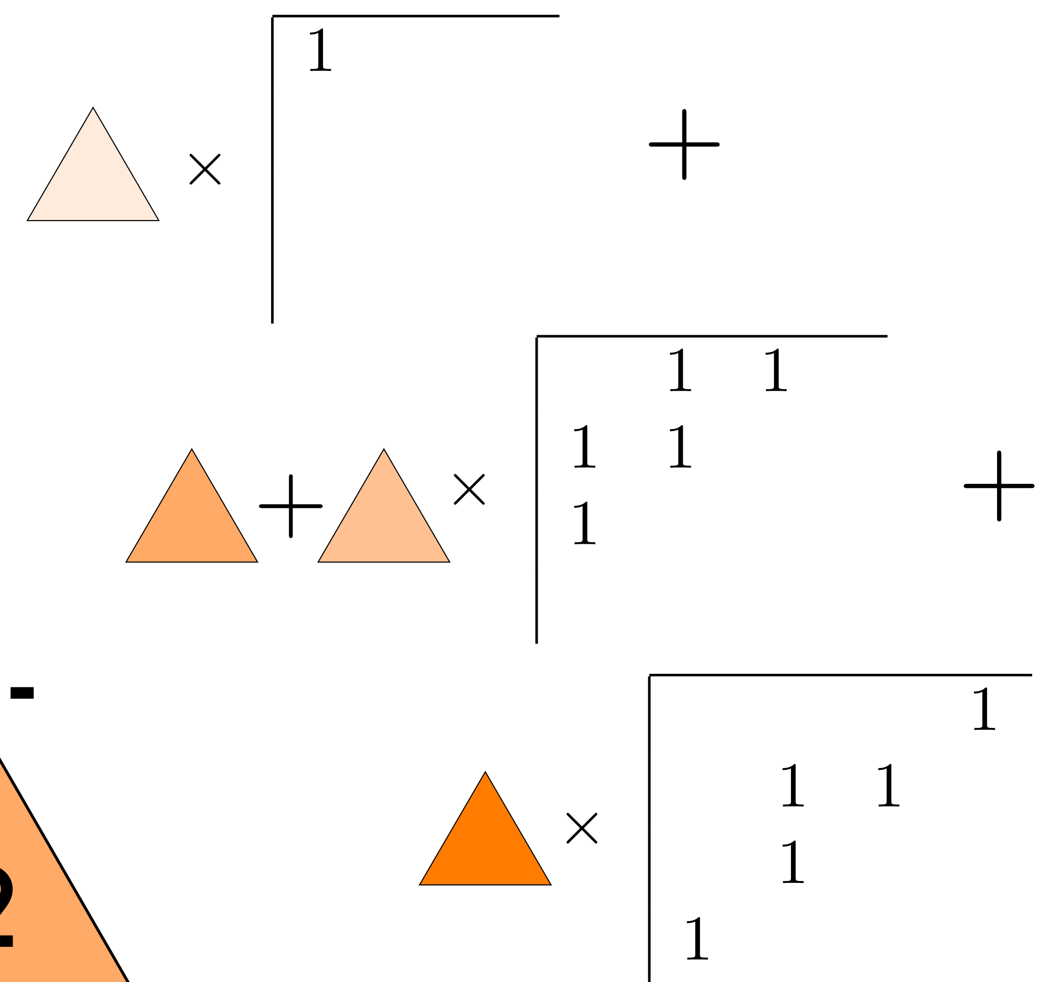
Now for Shi and Ish



Now for Shi and Ish



$$\sum q^{\mathsf{shi}} t^{\mathsf{ish}} F_{\mathsf{Asc}} =$$



the "(q,t)–Catalan"

Now for Shi and Ish

(Shuffle) Conjecture:

$\sum q^{\mathsf{shi}} t^{\mathsf{ish}} F_{\mathsf{Asc}}$ is the **Frobenius series** of DR

Now for Shi and Ish

Theorem (me):

- My “shuffle conjecture” = The Shuffle Conjecture (HHLRU05).

Now for Shi and Ish

Theorem (me):

- My “shuffle conjecture” = The Shuffle Conjecture (HHLRU05).
- That is, $\exists$ (at least) two natural maps to parking functions:

$$\begin{matrix} \textbf{dinv} & & \textbf{shi} & & \textbf{area}' \\ & \longleftarrow & & \longrightarrow & \\ \textbf{area} & & \textbf{ish} & & \textbf{bounce} \end{matrix}$$

“Haglund-Haiman-Loehr statistics”

Where does it come from?

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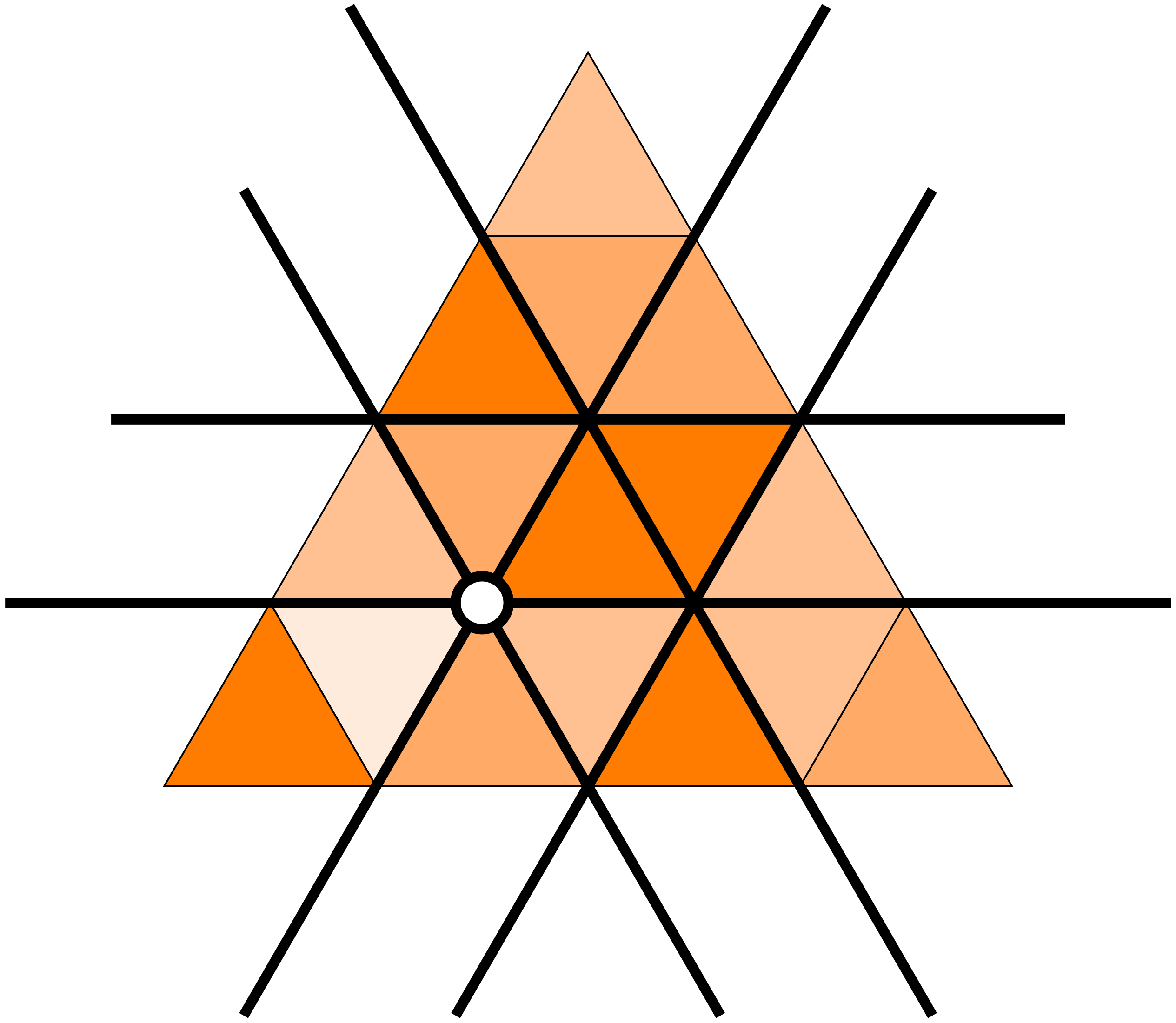
If you invert this picture...



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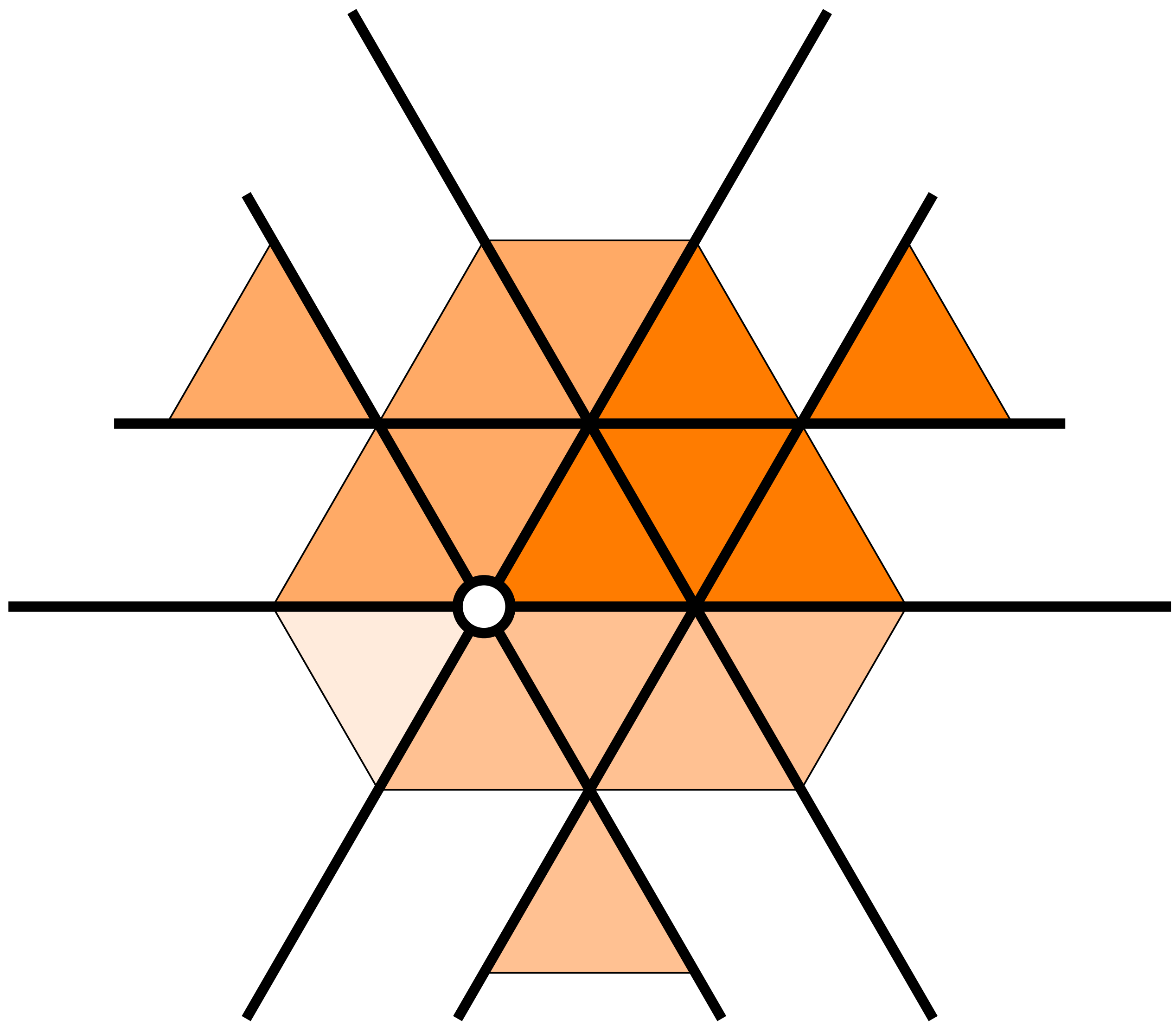

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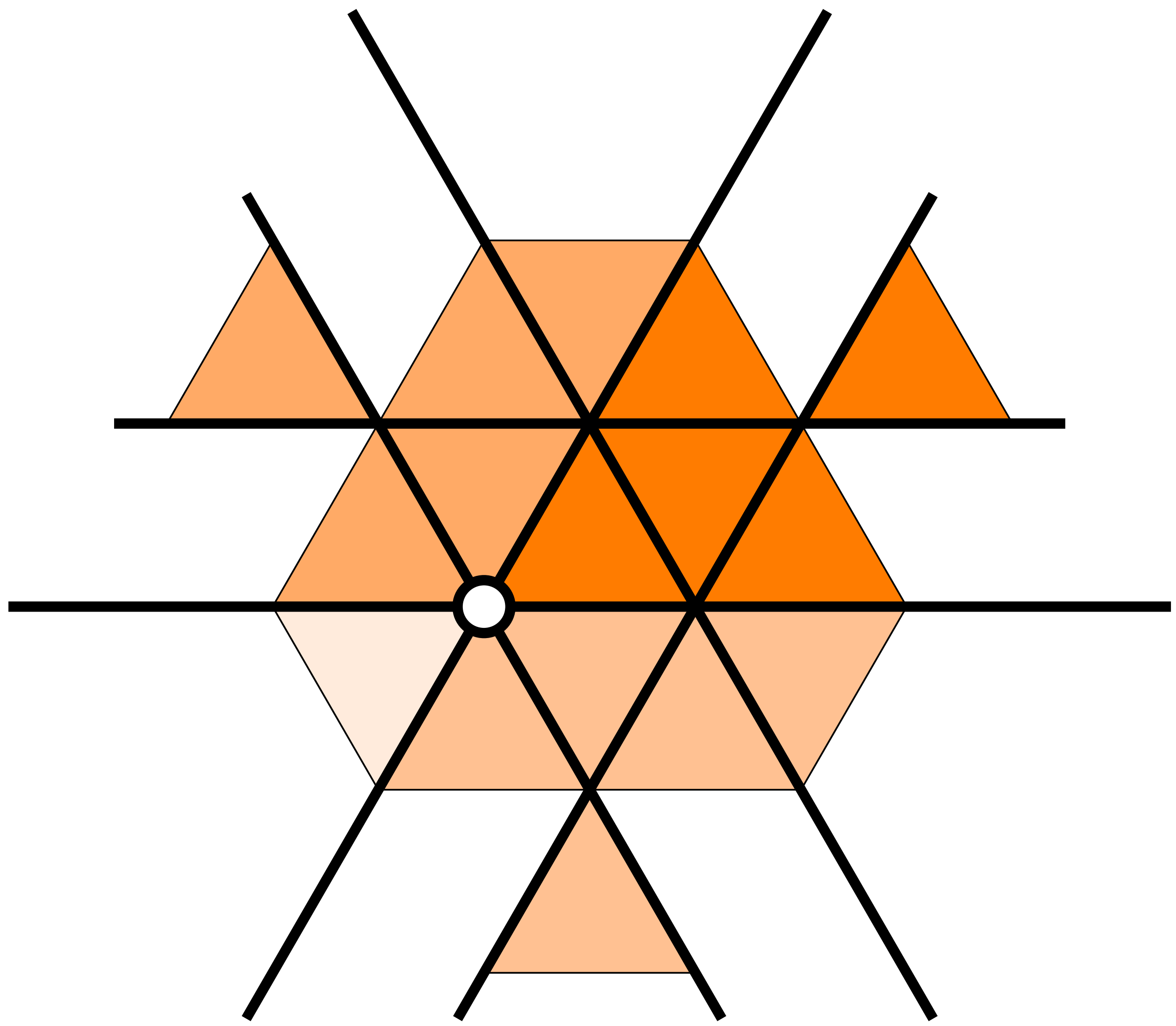

The Shi Arrangement

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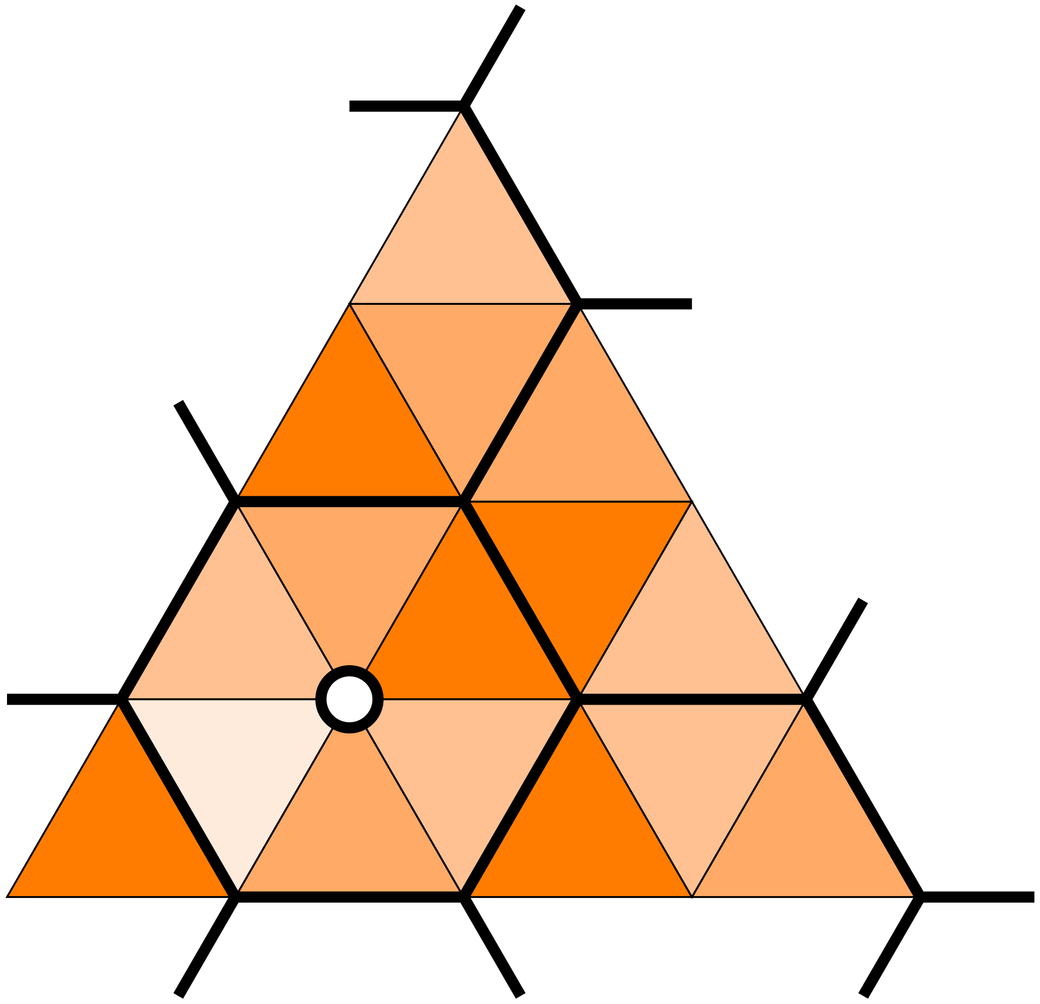

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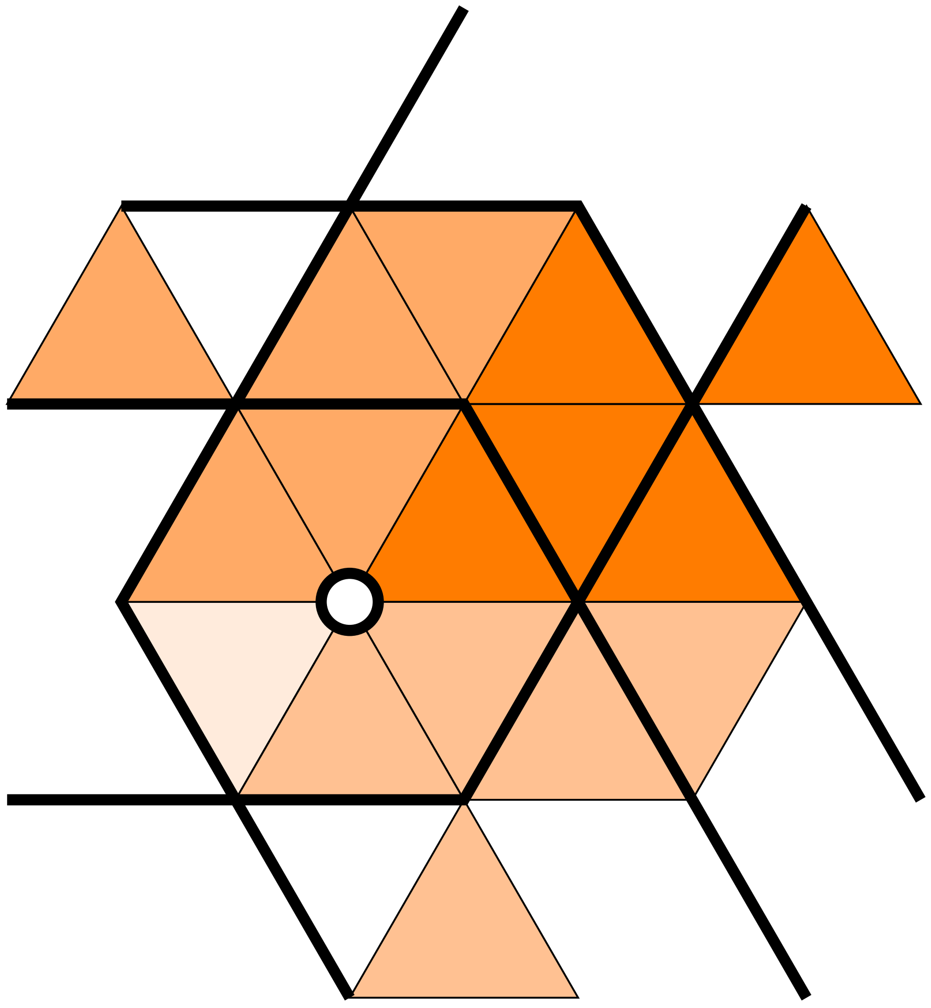

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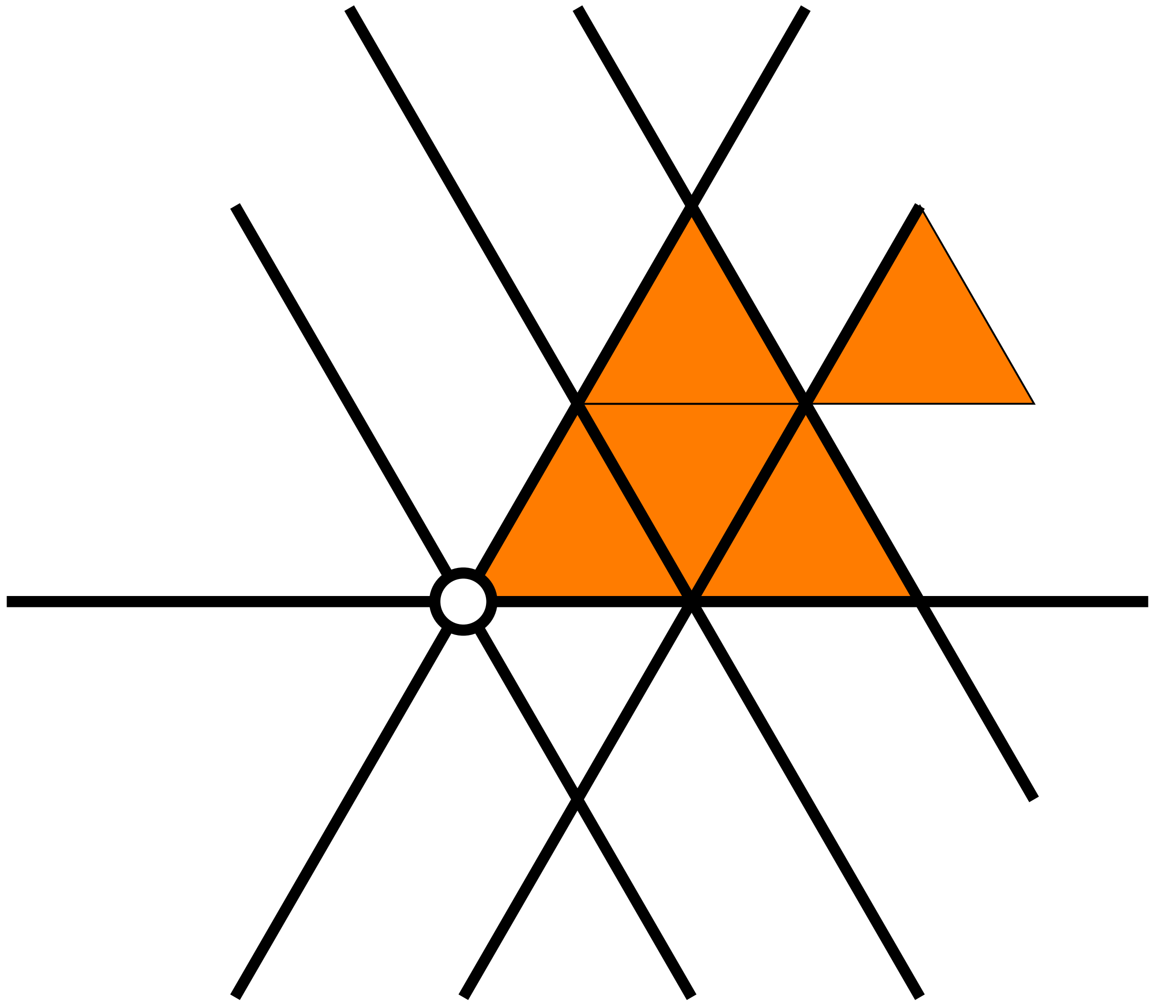

The Ish Arrangement

(please see Brendon's talk)

Thanks!

Þakka þér fyrir