What is ADE?

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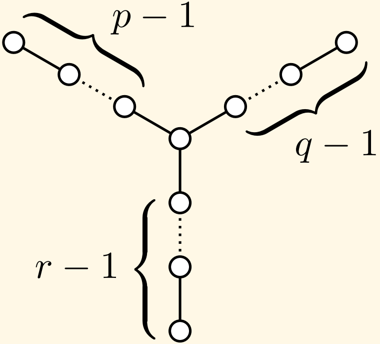
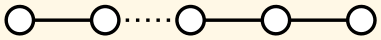
Preamble

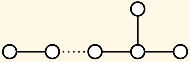
Consider the following graph with $p + q + r - 2$ vertices. Let's call it Y_{pqr}.

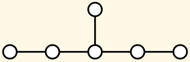


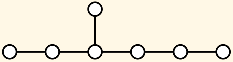
Preamble

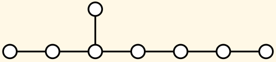
The graphs Y_{pqr} satisfying $\boxed{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1}$ have special names.

A_n  $\{p,q,r\}=\{1,k,n-k+1\}$

D_n  $\{p,q,r\}=\{2,2,n-2\}$

E_6  $\{p,q,r\}=\{2,3,3\}$

E_7  $\{p,q,r\}=\{2,3,4\}$

E_8  $\{p,q,r\}=\{2,3,5\}$

Preamble

You may have noticed that these “ADE diagrams” show up everywhere in mathematics.

- **Where** do these diagrams come from?
- **What** do they mean?
- **Why** do they show up everywhere?

Terry Gannon (in *Moonshine Beyond the Monster*) calls ADE a “meta-pattern” in mathematics, i.e., a structure that shows up more often than we would expect. Vladimir Arnold (in *Symplectization, Complexification and Mathematical Trinities*) describes ADE as “a kind of religion rather than mathematics”. You should not expect me to be able to explain it, nor *am* I able to. But I will try.

Preamble

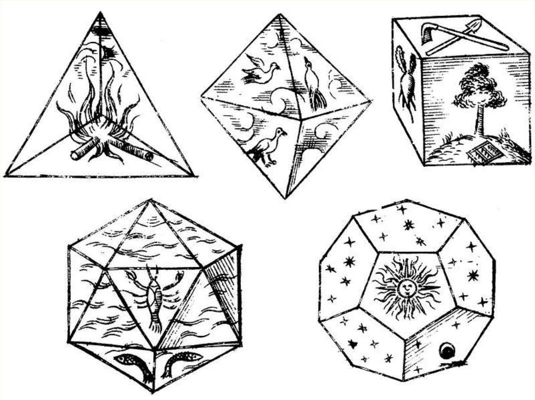
In this talk I will approach the problem from two points of view:

1. Historical: I will describe the earliest examples of ADE classification and how they entered mainstream mathematics.

2. Ahistorical: I will describe the most basic mathematical problem (that I know of) to which ADE is the answer.

History

The earliest example of ADE classification is the Platonic solids, as described in Plato's *Timaeus* (the figure is from Kepler's *Mysterium Cosmographicum*, 1596).



History

In modern terms, we associate the Platonic solids with their groups of rotational symmetries. The finite subgroups of SO(3) are classified as:

Type	Group	Symmetries of $\cdots$	p, q, r
A_n	cyclic	1-sided n-gon	$1, 1, n$
D_n	dihedral	2-sided $(n-2)$-gon	$2, 2, n-2$
E_6	T	tetrahedron	$2, 3, 3$
E_7	O	cube/octahedron	$2, 3, 4$
E_8	I	dodecahedron/icosahedron	$2, 3, 5$

The numbers p,q,r describe the amount of rotational symmetry around vertices, edges, faces of the polyhedron.

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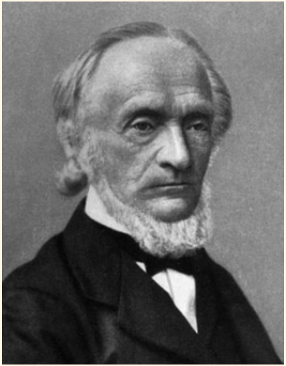
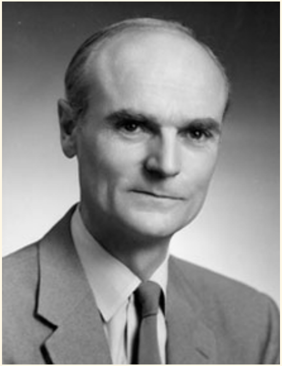
The meaning of the actual ADE **diagrams** in this case is called the *McKay Correspondence*. It is quite modern (post 1980) so I won't talk about it today.

History

Quiz: Who are these people?





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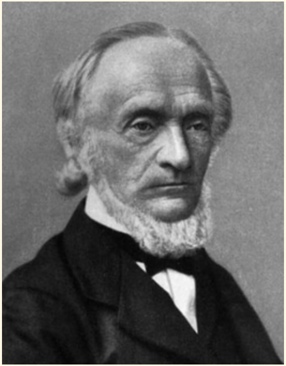
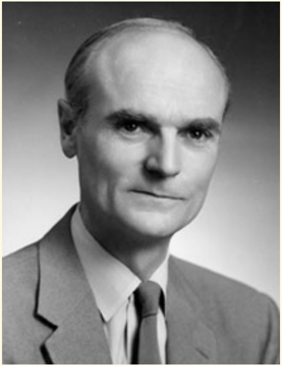


Ludwig Schläfli
1814-1895

Wilhem Killing
1847-1923



HSM Coxeter
1907-2003

History

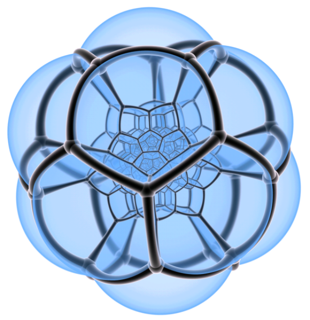
Some time between 1827 (August Möbius' *Der barycentrische Calcul*) and 1844 (Hermann Grassmann's *Ausdehnungslehre*) the concept of higher dimensional space became thinkable. At this point it was natural to look for "Platonic solids" in higher dimensions. The following theorem was proved by Ludwig Schläfli prior to 1952.

Classification of Regular Polytopes: There are three infinite families consisting of (1) regular polygons in dimension two, (2) a regular hypersimplex and (3) a regular hypercube/hyperoctahedron in each dimension. In addition to these infinite families there are exactly three "exceptional types":

- dodecahedron/icosahedron in 3 dimensions
- 24-cell in 4 dimensions
- 120-cell/600-cell in 4 dimensions

History

Here is a picture of the 120-cell stereographically projected onto three dimensional space:



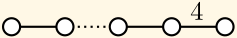
It is built from 120 regular dodecahedra glued together along faces. The 120-cell played an important part in the history of topology (see John Stillwell's *Story of the* 120-*cell*).

History

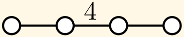
Ludwig Schläfli's work of 1952 was not very influential. Harold Scott Macdonald Coxeter mentions (in his *Regular Polytopes*, 1948) that the classification was independently rediscovered **at least eight times** between 1881 and 1900. Here is the notation that Coxeter used:

A_n

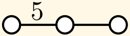
hypersimplex

B_n


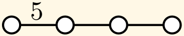
hypercube/hyperoctahedron

F_4


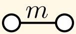
24-cell

H_3


dodecahedron/icosahedron

H_4


120-cell/600-cell

$I_2(m)$


m-gon

History

Question: Why did Coxeter choose such a bizarre notation (A,B,F,H,I) for the regular polytopes?

Answer: I lied. Actually he used the letters (A,C,D,F,G). Anyway, he was building on top of a notation (A,B,C,D,E,F) already established by someone else.

History

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Question: Who established this notation and what was it for?

Answer: A fellow named Wilhelm Killing, in 1887. He used the notation to desribe his classification of “space forms”.

History

Wilhelm Killing wrote his dissertation under Karl Weierstrass at Berlin in 1872. Thomas Hawkins (in his *Background to Killing's Work on Lie Algebras*) states that Killing was a geometer at heart but he was nontheless attracted to Weierstrass' analytic rigor.

Motivated by the recent revolution in non-Euclidean geometry, Killing's major idea was to apply Weierstrass' theory of **elementary divisors** (i.e., the Jordan canonical form) to the problem of infinitesimal rigid motions in geometry.

Killing set himself the task of classifying all possible **space forms** (today called **real Lie algebras**) and by 1884 he had made some significant progress. Killing sent his work to Felix Klein, who immediately recognized the similarity with the work of Sophus Lie on differential equations.

History

Lie's Galois Theory (1874)

- Attempt to develop a "Galois theory" of differential equations.
- Realized importance of invariant (normal) subgroups.
- Reduced the problem to simple groups.
- Called solvable groups "integrable".
- Integrability is equivalent to closure under the Poisson-Jacobi bracket (now also called the "Lie bracket").

Sophus Lie
1842-1899

Lie attempted to classify the simple groups but his linear algebra was not very sophisticated (far from Weierstrassian).

History

Lie was aware of three infinite families of simple groups, today called the "classical groups". In modern notation, they are

Type	Group
A_n	$\mathrm{SL}(n+1)$
B_n	$\mathrm{SO}(2n+1)$
C_n	$\mathrm{Sp}(n)$
D_n	$\mathrm{SO}(2n)$

But he got stuck trying to prove that these are the **only** simple groups. He was using a brute force method and ran into trouble above rank 3.

History

When Killing learned of Lie's work through Friedrich Engel, he was inspired to rededicate himself to the problem of classification. He also realized that he had algebraic tools unknown to Lie (namely, Weierstrass' elementary divisors).

At first Killing conjectured that every simple group is of classical type, but he soon began to see mysterious ghosts the calculations. On **May 23, 1887**, he sent a letter to Engel including the multiplication table for a certain "exceptional" 14-dimensional simple Lie algebra. Killing called it IIC; Élie Cartan later called it G_2.

> *Wenn ich mich nicht sehr irre, gibt es noch mehr einfache Gruppen.* [If I'm not mistaken, there are more simple groups.]

History

Engel strongly encouraged Killing to publish, and he relayed Klein's invitation to use the pages of *Mathematische Annalen* for this purpose. Five months later, on **October 18, 1887**, Killing sent a letter to Engel stating that he had succeeded in determining the structure (*Bildung*) of all simple groups.

His full classification appeared in 1888 as *Die Zusammensetzung der stetigen, endlichen Transformationsgruppen* [The composition of continuous, finite transformation groups], commonly known as Z.v.G.II. It was in this paper that Killing invented the A,B,C,D,E,F notation still used today. In addition to the classical groups (A,B,C,D) he announced the existence of five "exceptional groups":

$$\mathrm{IVF}^*, \mathrm{VIE}, \mathrm{VIIE}, \mathrm{VIIIE}, \mathrm{IIC}$$

∗ Actually, he announced two groups isomorphic to IVF

History

Killing's work was mostly correct but it was very much incomplete. In fact he had not proved the existence of the exceptional groups, only suggested that they "should" exist. He was despondent about the amount of work still to be done and he underestimated the significance of his results.

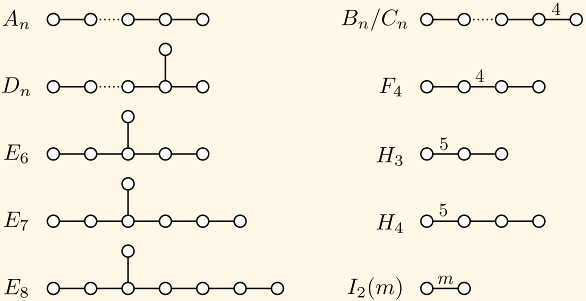
The work of filling in the details was continued by Engel and brought to a satisfying conclusion in the doctoral thesis of Élie Cartan (1894).

That being said, the subject was still very abstruse and far from the mainstream. The important work of popularization (*vulgarization*, as the French say) was performed independently by two men: Eugene Dynkin (1946) and HSM Coxeter (1934). They both came up with essentially the same graphical scheme to present the classification.* (I prefer Coxeter's notation because it is more general.)

$*$ To hear Dynkin's 1978 interview with Coxeter go to: `http://hdl.handle.net/1813/17339`

In 1934, Coxeter distilled the essence of Schläfli's and Killing's classifications into the following result.

Classification of Reflection Groups: Every finite group generated by reflections is a direct product of irreducible such groups. The irreducible groups are classified by the following diagrams.



Pause

Pause

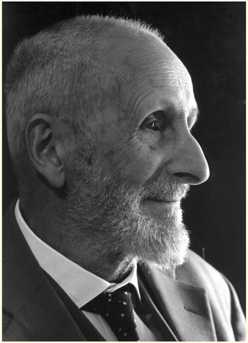
And that was just the beginning.

ADE type classifications touch every area of modern mathematics and seem to provide the material out of which mathematics is built (or at least *real mathematics*, as Ivan Cherednik would call it).

I can not explain this, of course. All I can do is share with you the **most basic** version of ADE classification that I know.

Ahistory

Quiz: Who are these people?



Ahistory

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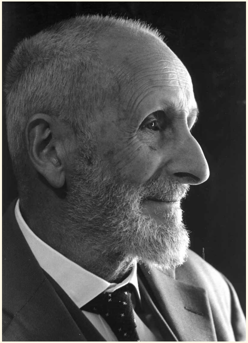
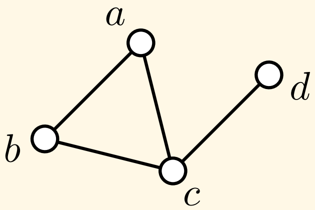


Oskar Perron
1880-1975

Georg Frobenius
1849-1917

Ahistory

A **simple graph** G is the same thing as a symmetric matrix A_G of zeroes and ones with zeroes on the diagonal. A_G is the **adjacency matrix** of G.



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Define the **spectral radius** of the graph G by

$$\|G\| := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A_G\}.$$

This is some measure of the "complexity" of G. In the example we have

$$\|G\| \approx 2.17$$

Ahistory

Which graphs have the **smallest spectral radius**? Since

$$\|G \sqcup H\| = \max\{\|G\|, \|H\|\}$$

we can restrict our attention to **connected** graphs. The following theorem was first written down in this form by J.H. Smith (1970), but it was implicit in every example of ADE classification.

Folklore Theorem: Let G be a connected simple graph. Then we have

$$\boxed{\|G\| < 2 \quad \Longleftrightarrow \quad G = Y_{pqr} \text{ for some } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.}$$

I will present the proof because it is instructive. It is non-trivial but fortunately the non-trivial parts can be placed in a black box labeled "Perron-Frobenius".

Ahistory

Here is just what we will need.

Black Box ("Perron-Frobenius"):

- If G is a connected simple graph and if A_G has a **positive real eigenvector** with eigenvalue λ then
$$\|G\| = \lambda.$$
- If G is a connected simple graph and $H \subsetneq G$ is **any proper subgraph** then
$$\|H\| \lneq \|G\|.$$

Ahistory

Before proving the theorem observe the following:

$$\text{If } G \text{ has an edge then } \|G\| \geq 1.$$

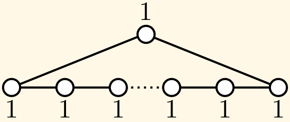
Proof: If G has an edge then it contains $H = 1 \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\, 1$ as a subgraph. Note that $\|H\| = 1$ because it has a positive real eigenvector (the displayed vertex labeling) with eigenvalue 1. We conclude that $1 = \|H\| \leq \|G\|$. $\square$

Now we prove theorem.

Proof: Let G be a **connected simple graph** with $\|G\| < 2$. We will show that $G = Y_{pqr}$ for some p, q, r such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

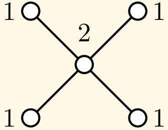
Ahistory

Step 1 (G contains no cycle): Otherwise G has a subgraph of the form



which has spectral radius 2 via the displayed eigenvector. Contradiction.

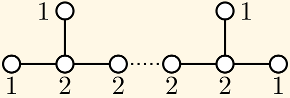
Step 2 (G contains no vertex of degree ≥ 4): Otherwise G contains a subgraph of the form



which has spectral radius 2 via the displayed eigenvector. Contradiction.

Ahistory

Step 3 (G has at most one vertex of degree 3): Otherwise G contains a subgraph of the form



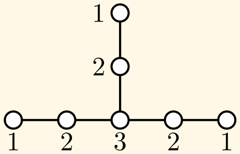
which has spectral radius 2 via the displayed eigenvector. Contradiction.

We now know that G is of the form Y_{pqr} for some p, q, r.

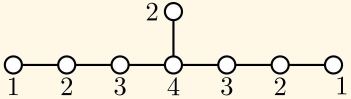
Step 4 ($\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$): Otherwise G contains a subgraph of the form Y_{333}, Y_{244} or Y_{236}, and each of these has spectral radius 2 via the following displayed eigenvectors.

Ahistory

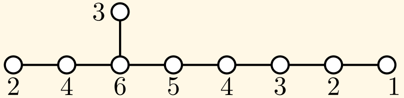
Y_{333}



Y_{244}



Y_{236}



This completes the proof. $\square$

Ahistory

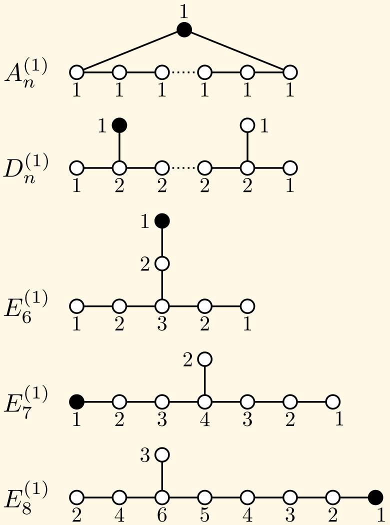
That was a very strange proof. In particular, the vertex labelings of Y_{333}, Y_{244} and Y_{236} seem to come from nowhere.

With careful scrutiny we notice that there is a **bijection** between the connected simple graphs with spectral radius < 2 and the connected simple graphs with spectral radius $= 2$.

I'll leave you with a puzzle:

Can you explain the diagrams on the next slide?

Puzzle



$A_n^{(1)}$
1
1 1 1 1 1 1
$D_n^{(1)}$
1 1
1 2 2 2 2 1
$E_6^{(1)}$
1
2
1 2 3 2 1
$E_7^{(1)}$
2
1 2 3 4 3 2 1
$E_8^{(1)}$
3
2 4 6 5 4 3 2 1

The End

Thanks!

If you would like to see a scary picture you can go to: `http://www.math.miami.edu/~armstrong/scary.html`. Warning: It's scary!