The McKay Correspondence

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Einfürung

Here is a basic problem:

Urfrage

Find all positive integers $p, q, r \in \mathbb{N}$ such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

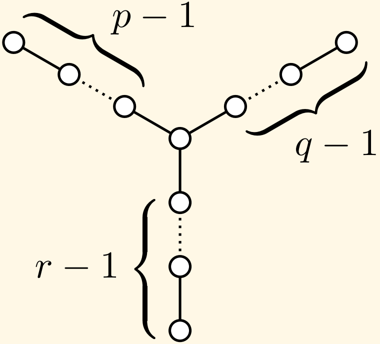
With a little thought you will find the following answer:

Urantwort

$$\{p, q, r\} \in \Big\{\{1, *, *\}, \{2, 2, *\}, \{2, 3, 3\}, \{2, 3, 4\}, \{2, 3, 5\}\Big\}$$

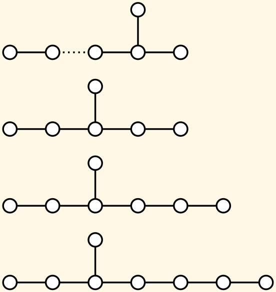
Einfürung

However, the answer is usually presented in graphical form. Consider the following graph with $(p-1)+(q-1)+(r-1)+1 = p+q+r-2$ vertices. For obvious reasons, we will call it Y_{pqr}:



Einfürung

The graphs Y_{pqr} satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ have special names:



A_n: $\{p, q, r\} = \{1, k, n - k + 1\}$

D_n: $\{p, q, r\} = \{2, 2, n - 2\}$

E_6: $\{p, q, r\} = \{2, 3, 3\}$

E_7: $\{p, q, r\} = \{2, 3, 4\}$

E_8: $\{p, q, r\} = \{2, 3, 5\}$

Einfürung

You may have noticed that these ‘ADE diagrams’ show up everywhere in mathematics.

- **Where** do these diagrams come from?
- **What** do they mean?
- **Why** do they show up everywhere?

Terry Gannon (in *Moonshine Beyond the Monster*) calls ADE a ‘meta-pattern’ in mathematics, i.e., a collection of seemingly different problems that have similar answers. Vladimir Arnold (in *Symplectization, Complexification and Mathematical Trinities*) describes ADE as ‘a kind of religion rather than mathematics’. The general topic of ADE is too vast for one colloquium talk.*

* I am (slowly) writing a book about it.

Einfürung

So today I will talk about two specific examples of ADE classification and a surprising connection between them.

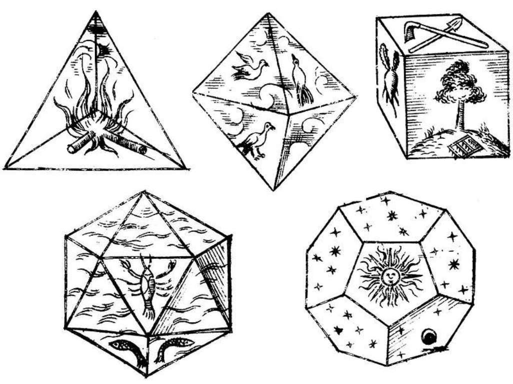
Problem 1. Classify finite subgroups of SO(3), SU(2), and $SL(2, \mathbb{C})$.

Problem 2. Classify symmetric $0, 1$ matrices with spectral radius < 2.

The equation $1/p + 1/q + 1/r > 1$ shows up naturally in Problem 1 via the Platonic solids. However, its occurrence in Problem 2 at first seemed mysterious. Then in 1980 the British/Canadian mathematician John McKay found a surprising bijection between the two problems. In this talk I will give an elementary introduction to both problems and then I will describe this 'McKay Correspondence'.

Problem 1: Platonic Solids

The earliest example of ADE is the classification of Platonic solids, as described in Plato's *Timaeus*. (This figure is taken from Kepler's *Mysterium Cosmographicum*, 1596.)



Problem 1: Platonic Solids

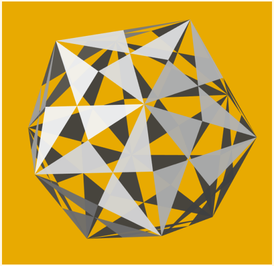
The correspondence goes as follows:

Type	Platonic Solid	$\{p, q, r\}$
A_n	1-sided n-gon	$\{1, 1, n\}$
D_n	2-sided $(n-2)$-gon	$\{2, 2, n-2\}$
E_6	tetrahedron	$\{2, 3, 3\}$
E_7	cube/octahedron	$\{2, 3, 4\}$
E_8	dodecahedron/icosahedron	$\{2, 3, 5\}$

The numbers p, q, r in this table describe the amount of rotational symmetry around vertices, edges, and faces of the polyhedron. (Note that types A and D are degenerate cases.)

Problem 1: Platonic Solids

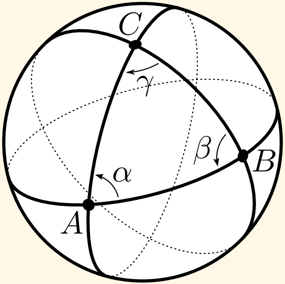
The occurrence of of the equation $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ in this classification is easy to explain. First we consider the barycentric subdivision of the Platonic solid and then project it onto the surface of a sphere. (Picture from Jeff Weeks' *KaleidoTile* software.)



This divides the sphere into two sets of isometric spherical triangles with internal angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$.

Problem 1: Platonic Solids

Now consider a general triangle on a sphere with vertices A, B, C and internal angles α, β, γ.



When R is the radius of the sphere, Thomas Harriot's (1603) formula says that the area of the triangle is

$$\text{area of triangle} = R^2(\text{angle excess}) = R^2(\alpha + \beta + \gamma - \pi)$$

Problem 1: Platonic Solids

In the case of a Platonic solid the triangle has positive area, so that

$$\begin{aligned}
\text{area of triangle} &> 0 \\
R^2\left(\frac{\pi}{p}+\frac{\pi}{q}+\frac{\pi}{r}-\pi\right) &> 0 \\
R^2\pi\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\right) &> 0 \\
\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1 &> 0 \\
\frac{1}{p}+\frac{1}{q}+\frac{1}{r} &> 1,
\end{aligned}$$

as desired. □

Problem 1: Finite Subgroups of SO(3)

In modern terms we encode the Platonic solids via their groups of symmetries. To describe this, first let me recall that the collection of all rotations $\mathbb{R}^3 \to \mathbb{R}^3$ is a group.

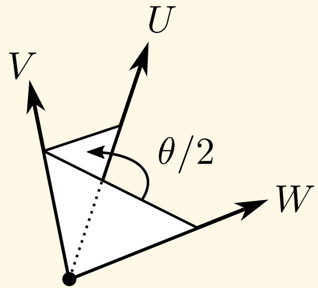
Euler's Rotation Theorem (1776)

The composition of two rotations $\mathbb{R}^3 \to \mathbb{R}^3$ is again a rotation.

To prove this, let $F_{UV} : \mathbb{R}^3 \to \mathbb{R}^3$ denote the reFlection across the plane spanned by vectors $U, V \in \mathbb{R}^3$ and let $R_V(\theta) : \mathbb{R}^3 \to \mathbb{R}^3$ denote the Rotation counterclockwise by angle θ around the vector $V \in \mathbb{R}^3$.

Problem 1: Finite Subgroups of SO(3)

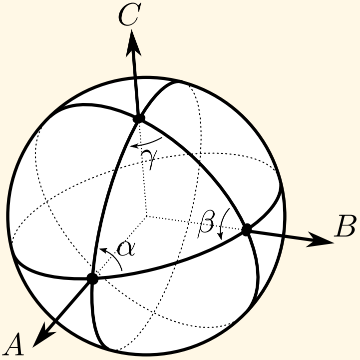
Now recall that the composition of two reflections is a rotation. In fact, if $\theta/2$ is the angle from the plane VW to the plane UV measured counterclockwise at V then we have



$$F_{UV} \circ F_{VW} = R_V(\theta)$$

Problem 1: Finite Subgroups of SO(3)

Finally, let $R_A(2\alpha)$ and $R_B(2\beta)$ be two arbitrary rotations, so that $\alpha, \beta \in [0, \pi/2]$. We want to show that the composition $F_A(2\alpha) \circ F_B(2\beta)$ is also a rotation. Indeed, there exists a unique direction C and a unique angle $\gamma \in [0, \pi/2]$ as in the following picture:



Problem 1: Finite Subgroups of SO(3)

And then since reflections are involutions we must have

$$\begin{aligned} R_A(2\alpha) \circ R_B(2\beta) &= (F_{CA} \circ F_{AB}) \circ (F_{AB} \circ F_{BC}) \\ &= F_{CA} \circ \cancel{(F_{AB} \circ F_{AB})} \circ F_{BC} \\ &= F_{CA} \circ F_{BC} \\ &= (F_{BC} \circ F_{CA})^{-1} \\ &= R_C(2\gamma)^{-1} \\ &= R_C(-2\gamma), \end{aligned}$$

which is a rotation as desired. $\square$

Problem 1: Finite Subgroups of SO(3)

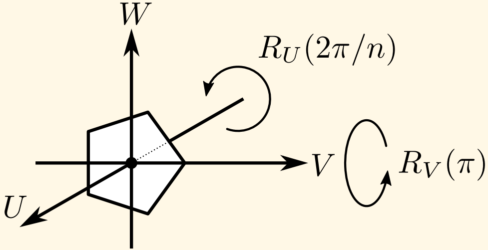
This completes the proof that rotations $\mathbb{R}^3 \to \mathbb{R}^3$ form a group. The standard name for this group is SO(3) (called a special orthogonal group). It is also a Lie group, which means that SO(3) carries a real manifold structure which is compatible with its group structure. As a real manifold, SO(3) is isomorphic to real projective 3-dimensional space:

$$\mathrm{SO}(3) \cong \mathbb{R}P^3.$$

Today I am interested in the discrete subgroups of SO(3), which since SO(3) is compact are the same as the finite subgroups. Note that there are some obvious finite subgroups coming from regular polygons and polyhedra in $\mathbb{R}^3$.

Problem 1: Finite Subgroups of SO(3)

We get two infinite families of groups from the following diagram:



- **Cyclic Group.** $\mathrm{Cyc}_n = \langle R_U(2\pi/n) \rangle$
- **Dihedral Group.** $\mathrm{Dih}_{2n} = \langle R_U(2\pi/n), R_V(\pi) \rangle$

Problem 1: Finite Subgroups of SO(3)

And we get three exceptional groups from the Platonic solids:

- $T = 12$ rotations of a tetrahedron
- $O = 24$ rotations of a cube/octahedron
- $I = 60$ rotations of a dodecahedron/icosahedron

The claim is that there are no other examples.

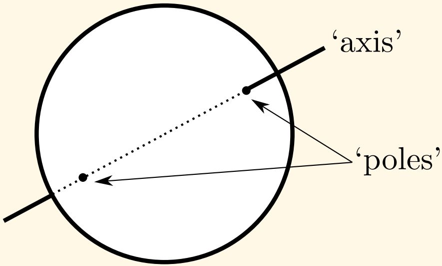
Theorem: Finite Subgroups of SO(3)

Every finite subgroup of SO(3) is isomorphic to one of the following:

$$\mathrm{Cyc}_n, \mathrm{Dih}_{2n}, T, O, I.$$

Problem 1: Finite Subgroups of SO(3)

Proof Sketch: Let $G \subseteq \mathrm{SO}(3)$ be a finite subgroup and consider the set of 'axes of rotation' for the non-identity elements of G. Each axis intersects the sphere in two points, called 'poles'.



Let P denote the set of poles and consider the action of G on this set. (Note: If G comes from a Platonic solid then the poles are just the barycenters of the vertices, edges, and faces.)

Problem 1: Finite Subgroups of SO(3)

Suppose that the action of G divides the set of poles into m orbits

$$P = \mathrm{Orb}_1 \sqcup \mathrm{Orb}_2 \sqcup \cdots \sqcup \mathrm{Orb}_m$$

and define $o_i := |\mathrm{Orb}_i|$ and $r_p = r_i := |\mathrm{Stab}_G(p)|$ for $p \in \mathrm{Orb}_i$. By counting the set

$$\{(g, p) : 1 \neq g \in G, p \in P, g(p) = p\}$$

in two different ways, we obtain the equation

$$2\,(|G| - 1) = \sum_{p \in P} (r_p - 1) = \sum_{i=1}^{m} o_i(r_i - 1).$$

Then applying the Orbit-Stabilizer Theorem (i.e., $o_i r_i = |G|$) gives

$$\boxed{\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_m} = m - 2 + \frac{2}{|G|}}$$

Problem 1: Finite Subgroups of SO(3)

$$\boxed{\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_m} = m - 2 + \frac{2}{|G|}}$$

Let's examine this equation. When $p \in \mathrm{Orb}_i$ then $r_i = r_p = |\mathrm{Stab}_G(p)|$ is just the amount of rotational symmetry around the pole p. Since $r_i \geq 2$ for all i, one can check that $m = 1$ and $m \geq 4$ are impossible. Thus there are two possible cases:

- **Two Orbits:** $\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{|G|}$

  This implies that $r_1 = r_2 = |G|$ and hence $G = \mathrm{Cyc}_{|G|}$.

- **Three Orbits:** $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1 + \frac{2}{|G|} > 1$

  This is the Urproblem again. It leads to $G \in \{\mathrm{Dih}_{|G|}, T, O, I\}$.

□

Problem 1: Finite Subgroups of SO(3)

Let me pause to record the following observation:

Observation

Let $G \subseteq \mathrm{SO}(3)$ be the group of symmetries of a Platonic solid and let p, q, r denote the amount of rotational symmetry around vertices, edges, and faces. Then we have

$$|G| = \frac{2}{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1}.$$

(Remark: Interesting things still happen when the denominator is negative. For example, when $\{p, q, r\} = \{2, 3, 7\}$ the strange formula

$$|G| = -84$$

is related to the Hurwitz Theorem on symmetries of Riemann surfaces.*)

∗ For more, see here: http://www.math.ucr.edu/home/baez/42.html

Problem 1: Finite Subgroups of $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbb{C})$

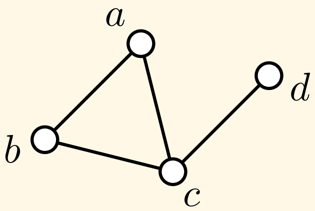
Let me also mention that the classification of finite subgroups of SO(3) lifts to the Lie groups SU(2) and $\mathsf{SL}(2,\mathbb{C})$:

$$\begin{array}{ccc} \mathsf{SU}(2) \cong S^3 & \hookrightarrow & \mathsf{SL}(2,\mathbb{C}) \\ \Big\downarrow{\scriptstyle 2:1} & & \\ \mathsf{SO}(3) \cong \mathbb{R}P^3 & & \end{array}$$

- The special unitary group SU(2) is topologically a 3-sphere. It is a double cover of SO(3) via the 'Hopf map' which identifies antipodal points. Each finite subgroup of SO(3) has a unique lift to SU(2).
- The special linear group $\mathsf{SL}(2,\mathbb{C})$ deformation retracts onto its subgroup SU(2). Furthermore, every finite subgroup of $\mathsf{SL}(2,\mathbb{C})$ can be conjugated into SU(2) by averaging over the group to obtain an invariant Hermitian inner product.

Problem 2: Symmetric $0, 1$ Matrices

Now we will consider a seemingly quite different problem. Every symmetric matrix with entries from $\{0, 1\}$ can be thought of as the adjacency matrix of a graph. For example:



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Given a graph G we let A_G denote its adjacency matrix. Today we will assume that the diagonal entries are zero (i.e., G has no loops) but this is not very important.

Problem 2: Symmetric $0, 1$ Matrices

Given a graph G we define its spectral radius as the size of the largest eigenvalue of its adjacency matrix:

$$\|G\| := \max\left\{|\lambda| : \lambda \text{ is an eigenvalue of } A_G\right\}.$$

In the previous example we had $\|G\| \approx 2.17$. The spectral radius is some kind of measure of the 'complexity' of the graph G.

Our goal today is to investigate the graphs that are 'least complicated', or which have the smallest spectral radius. Since the spectral radius of a disjoint union of graphs is given by

$$\|G \sqcup H\| = \max\left\{\|G\|, \|H\|\right\}$$

it will be enough to investigate connected graphs.

Problem 2: Symmetric $0, 1$ Matrices

Let me spoil the surprise by giving you the answer right away. The result was first stated in this way by J. H. Smith (1970), but I regard it as a 'folklore theorem' since it is really implicit in work of Coxeter and Dynkin.

Folklore Theorem

Let G be a connected graph. Then we have

$$\|G\| < 2 \quad \Longleftrightarrow \quad G = \mathsf{Y}_{pqr} \text{ for some } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

In other words, the graphs with spectral radius less than 2 are precisely the diagrams of type ADE.

(Remark: If we also allow loops then the only additional graphs with $\|G\| < 2$ are the 'lollipops':  .)

Problem 2: Symmetric $0, 1$ Matrices

The proof is quite easy to write down if you will allow me to assume the following facts, which are part of the Perron-Frobenius Theorem.

Lemma (Perron-Frobenius):

- **PF1.** Let G be a connected graph. If A_G has a λ-eigenvector with positive real entries, then the eigenvalue λ equals the spectral radius:

$$\|G\| = \lambda.$$

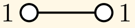
- **PF2.** Let G be a connected graph. If $H \subsetneq G$ is any proper sugraph then the spectral radius of H is strictly smaller than that of G:

$$\|H\| \lneq \|G\|.$$

Problem 2: Symmetric $0, 1$ Matrices

To illustrate how the Lemma is used, to observe the following:

If a graph G contains an edge then we must have $\|G\| \geq 1$.

Proof: If G has an edge then it contains $H =$  as a subgraph. Note that the displayed vertex labeling $(1, 1)$ is a positive real eigenvector for the adjacency matrix of H. Thus from the Lemma we conclude that

$$1 \overset{\text{PF1}}{=} \|H\| \overset{\text{PF2}}{\leq} \|G\|.$$

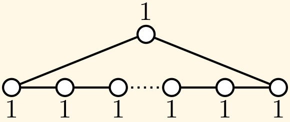
□

Now we prove theorem.

Proof: Let G be a connected graph with $\|G\| < 2$. We will show that $G = Y_{pqr}$ for some p, q, r such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

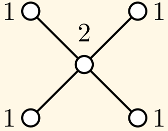
Problem 2: Symmetric $0, 1$ Matrices

Step 1 (G contains no cycle): Otherwise G has a subgraph of the form



which has spectral radius 2 via the displayed eigenvector. Contradiction.

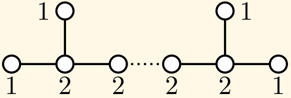
Step 2 (G contains no vertex with degree ≥ 4): Otherwise G contains a subgraph of the form



which has spectral radius 2 via the displayed eigenvector. Contradiction.

Problem 2: Symmetric $0, 1$ Matrices

Step 3 (G has at most one vertex of degree 3): Otherwise G contains a subgraph of the form

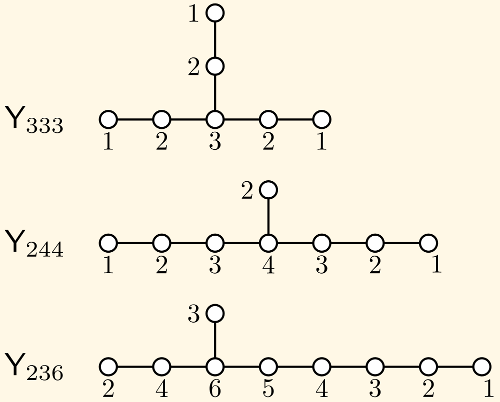


which has spectral radius 2 via the displayed eigenvector. Contradiction.

We now know that G is of the form Y_{pqr} for some p, q, r.

Step 4 ($\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$): Otherwise G contains a subgraph of the form Y_{333}, Y_{244} or Y_{236}, and each of these has spectral radius 2 via the following displayed eigenvectors:

Problem 2: Symmetric 0, 1 Matrices



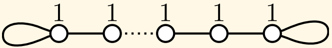
This completes the proof. $\square$

Mysteries

Note that this proof leaves two mysteries unexplained:

- Where did the special 2-eigenvectors come from?
- Where did the equation $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ come from?

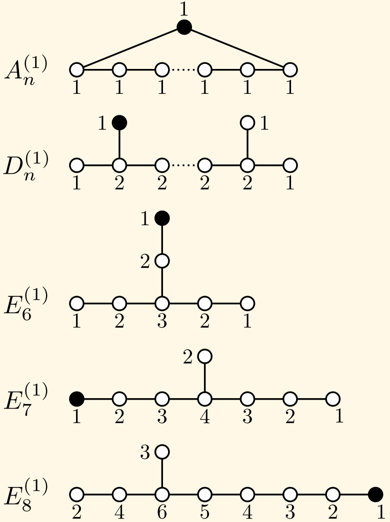
In fact, these two mysteries are deeply related and they are the clues that led John McKay to his Correspondence. With careful scrutiny we might notice that there is a bijection between the connected graphs of spectral radius < 2 and the connected graphs of spectral radius $= 2$.

(Yes, even for the 'lollipops': 

.)

Thus we will extend the ADE notation as on the following slide:

Mysteries



$A_n^{(1)}$
1
1 1 1 1 1 1
$D_n^{(1)}$
1 1
1 2 2 2 2 1
$E_6^{(1)}$
1
2
1 2 3 2 1
$E_7^{(1)}$
2
1 2 3 4 3 2 1
$E_8^{(1)}$
3
2 4 6 5 4 3 2 1

Mysteries

If G is a diagram of type ADE (with $\|G\| < 2$) then we let $G^{(1)}$ denote the corresponding augmented diagram (with $\|G^{(1)}\| = 2$). Note that the augmented diagram has one extra vertex. We will label the vertices with the 2-eigenvector, scaled so the new vertex get the label '1'. Through some miracle, it turns out that these labels are integers; we call these vertex labels the marks of the diagram.

Here are two mysterious properties of the marks:

- Let G be a diagram of type ADE and let n_i be the marks of the augmented diagram $G^{(1)}$. If we define $h := \sum_i n_i$ then

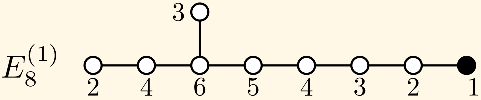
$$2\cos(\pi/h) \text{ is the spectral radius of } G.$$

- Furthermore, if $G = \mathrm{Y}_{pqr}$ for some $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ then we have

$$\sum_i n_i^2 = \frac{4}{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1}.$$

Mysteries

Example (Type E_8):



In this case we have:

- $1 + 2 + 3 + 4 + 5 + 6 + 2 + 4 + 3 = 30$
- $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 2^2 + 4^2 + 3^2 = 120$

Indeed, the spectral radius of the E_8 diagram is $2\cos(\pi/30)$ and the size of the icosahedral group $I \subseteq \mathrm{SO}(3)$ is $120/2 = 60$.

But is there really a direct relationship between the icosahedron and the E_8 diagram? McKay says ‘yes’.

The McKay Correspondence

The McKay Correspondence gives an explicit bijection

$$\boxed{\text{finite subgroups of SU(2)}} \longleftrightarrow \boxed{\text{diagrams of type ADE}}$$

So let $\Gamma \subseteq \mathrm{SU}(2)$ be a finite subgroup (the letter Γ is traditional here) and let $\Gamma' \subseteq \mathrm{SO}(3)$ be its projection under the Hopf map, so that $|\Gamma| = 2|\Gamma'|$. Since these $\Gamma' \subseteq \mathrm{SO}(3)$ come from Platonic solids, the finite groups $\Gamma \subseteq \mathrm{SU}(2)$ are called binary polyhedral groups.

McKay (1980) showed how to construct a graph from each binary polyhedral group and then Steinberg (1985) generalized this construction to all finite groups. To describe Steinberg's construction I must first remind you of the representation theory of finite groups.

The McKay Correspondence

Let Γ be any finite group and consider the complex group algebra $\mathbb{C}[\Gamma]$. Recall that a $\mathbb{C}[\Gamma]$-module is decomposable if it is a non-trivial direct sum and it is reducible if it has a non-trivial submodule. In general we have decomposable $\Rightarrow$ reducible.

Fundamental Theorem of $\mathbb{C}[\Gamma]$-Modules

- For the algebra $\mathbb{C}[\Gamma]$ we also have reducible $\Rightarrow$ decomposable, hence every f.d. $\mathbb{C}[\Gamma]$-module can be expressed uniquely as a direct sum of irreducibles.
- If $\mathbb{C}[\Gamma] \cong \bigoplus_i V_i^{\oplus n_i}$ with V_i irreducible, then every irreducible $\mathbb{C}[\Gamma]$-module is isomorphic to one of the V_i. The number of distinct irreducibles equals the number of conjugacy classes of Γ.
- The multplicities in the above formula are $n_i = \dim V_i$, and hence
$$\sum_i n_i^2 = |\Gamma|.$$

That last formula suggests how we should proceed. We want a graph whose vertices are indexed by the irreducible $\mathbb{C}[\Gamma]$-modules.

Definition of the McKay Graph (Steinberg, 1985)

Let Γ be any finite group, let $\{V_i\}$ be the irreducible $\mathbb{C}[\Gamma]$-modules, and let U be any f.d. $\mathbb{C}[\Gamma]$-module. We define a graph $\mathrm{McK}_U(\Gamma)$ as follows:

- The vertices of $\mathrm{McK}_U(\Gamma)$ are indexed by the irreducibles V_i.
- For each pair of irreducibles V_i, V_j let a_{ij} denote the multiplicity of V_i in the irreducible decomposition of the tensor product $U \otimes V_j$. Then we define a weighted directed edge $V_i \xrightarrow{a_{ij}} V_j$.

Thus $\mathrm{McK}_U(\Gamma)$ is a weighted, directed graph with adjacency matrix $A = (a_{ij})$. (Remark: The dual module U^* corresponds to the transpose matrix $A^\top$. Hence if U is self-dual then we can think of $\mathrm{McK}_U(\Gamma)$ as an undirected graph.)

The McKay Correspondence

Finally, here is the big theorem:

Theorem (McKay, 1980)

Let $\Gamma \subseteq \mathrm{SU}(2)$ be a finite group and let U be its defining representation. Then the McKay graph $\mathrm{McK}_U(\Gamma)$ is a diagram of type ADE. Furthermore, this establishes a bijection

$$\boxed{\text{finite subgroups of SU(2)}} \longleftrightarrow \boxed{\text{diagrams of type ADE}}$$

(Remark: Since every matrix in SU(2) is unitary we can think of $\mathrm{McK}_U(\Gamma)$ as an undirected graph.)

John McKay proved this theorem with a case-by-case argument and then Robert Steinberg gave a uniform argument in 1985.

The McKay Correspondence

This finally gives us a good reason for the occurrence of the equation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

in the classification of graphs with small spectral radius. If $\Gamma' \subseteq SO(3)$ is a polyhedral group corresponding to triple $\{p, q, r\}$ then it also gives us a good reason for the observation

$$\sum_i n_i^2 = \frac{4}{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1} = 2|\Gamma'| = |\Gamma|.$$

Now we see that the mysterious marks of the diagram $Y_{pqr}^{(1)}$ are equal to the dimensions of the irreducible modules for the binary polyhedral group $\Gamma \subseteq SU(2)$. Hence this formula comes from decomposing the group algebra $\mathbb{C}[\Gamma]$ into irreducible $\mathbb{C}[\Gamma]$-modules.

The McKay Correspondence

There is much more to say about the McKay Correspondence. For example, the full set of eigenvectors for the adjacency matrix of the diagram $E_8^{(1)}$ is given by the columns of the character table for the binary icosahedral group:

Class	1_+	1_-	30	20_+	20_-	12_{a+}	12_{b+}	12_{a-}	12_{b-}
Order	1	2	4	6	3	10	5	5	10
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	-2	0	1	-1	μ	ν	$-\mu$	$-\nu$
χ_3	2	-2	0	1	-1	$-\nu$	$-\mu$	ν	μ
χ_4	3	3	-1	0	0	$-\nu$	μ	$-\nu$	μ
χ_5	3	3	-1	0	0	μ	$-\nu$	μ	$-\nu$
χ_6	4	4	0	1	1	-1	-1	-1	-1
χ_7	4	-4	0	-1	1	1	-1	-1	1
χ_8	5	5	1	-1	-1	0	0	0	0
χ_9	6	-6	0	0	0	-1	1	1	-1

Here, $\mu = \frac{\sqrt{5}+1}{2}$, and $\nu = \frac{\sqrt{5}-1}{2}$.

And this data may have been detected in an experiment.*

∗ Borthwick and Garibaldi, *Did a 1-Dimensional Magnet Detect a 248-Dimensional Lie Algebra?*, Notices of the AMS, 2011.

Ende dem Vortrag

But I'll save the rest for another time.*

Vielen Dank!

* Expanded notes on the subject can be found here: `http://www.math.miami.edu/~armstrong/Talks/McKay_Talca.pdf`.