Rational Catalan Combinatorics

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This talk will advertise a definition.

Here is it.

Definition

Let x be a positive rational number written as $x = a/(b-a)$ for $0 < a < b$ coprime. Then we define the **Catalan number**

$$\mathrm{Cat}(x) := \frac{1}{a+b}\binom{a+b}{a,b} = \frac{(a+b-1)!}{a!b!}.$$

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Special cases.

When $b = 1 \bmod a \dots$

- *Eugène Charles Catalan (1814-1894)*

  $(a < b) = (n < n+1)$ gives the **good old Catalan number**

  $$\mathrm{Cat}(n) = \mathrm{Cat}\left(\frac{n}{1}\right) = \frac{1}{2n+1}\binom{2n+1}{n}.$$

- *Nicolaus Fuss (1755-1826)*

  $(a < b) = (n < kn+1)$ gives the **Fuss-Catalan number**

  $$\mathrm{Cat}\left(\frac{n}{(kn+1)-n}\right) = \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}.$$

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Euclidean Algorithm & Symmetry.

Definition

Again let $x = a/(b-a)$ for $0 < a < b$ coprime. Then we define the **derived Catalan number**

$$\mathrm{Cat}'(x) := \frac{1}{b}\binom{b}{a} = \begin{cases} \mathrm{Cat}(1/(x-1)) & \text{if } x > 1 \\ \mathrm{Cat}(x/(1-x)) & \text{if } x < 1 \end{cases}$$

This is a "categorification" of the Euclidean algorithm.

Remark

If we define $\mathrm{Cat} : \mathbb{Q} \setminus [-1, 0] \to \mathbb{N}$ by $\mathrm{Cat}(-x-1) := \mathrm{Cat}(x)$ then the formula is simpler:

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Catalan "Number Theory"?

Problem

Describe a recurrence for the Cat function, perhaps in terms of the *Calkin-Wilf sequence*

$$\frac{1}{1} \mapsto \frac{1}{2} \mapsto \frac{2}{1} \mapsto \frac{1}{3} \mapsto \frac{3}{2} \mapsto \frac{2}{3} \mapsto \frac{3}{1} \mapsto \frac{1}{4} \mapsto \frac{4}{3} \mapsto \cdots$$

which is defined by

$$x \mapsto \frac{1}{\lfloor x \rfloor + 1 - \{x\}}.$$

See Aigner and Ziegler: *"Proofs from THE BOOK"*, Chapter 17.

What?

Well, that was fun. But *perhaps untethered to reality...*

Motivation 1: Cores

Definition

- An integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \vdash n$ is called p-core if it has **no cell with hook length** p.
- Say $\lambda \vdash n$ is (a, b)-core if it has **no cell with hook length** a **or** b.

Example

The partition $(5, 4, 2, 1, 1) \vdash 13$ is $(5, 8)$-core.

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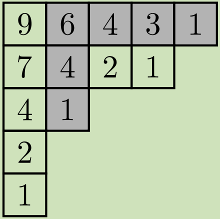

A few facts.

Theorem (Anderson, 2002)

The number of (a,b)-cores is finite if and only if a,b are coprime, in which case the number is

$$\mathrm{Cat}\left(\frac{a}{b-a}\right) = \frac{1}{a+b}\binom{a+b}{a,b}.$$

Theorem (Olsson-Stanton, 2005, Vandehey, 2008)

For a,b coprime $\exists$ unique largest (a,b)-core of size $\frac{(a^2-1)(b^2-1)}{24}$, which contains all others as subdiagrams.

Problem

Study Young's lattice restricted to (a,b)-cores.

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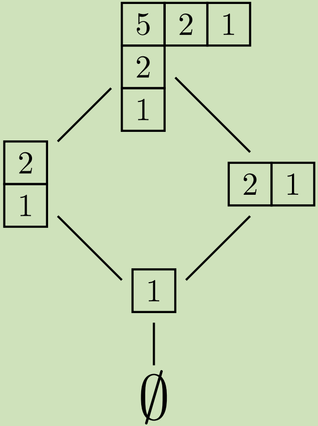

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Theorem (Ford-Mai-Sze, 2009)

For a, b coprime, the number of self-conjugate (a, b)-cores is $\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor}$.

Note: Beautiful bijective proof! (omitted)

Observation/Problem

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor} = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q \Bigg|_{q=-1}$$

Conjecture (Armstrong, 2011)

The average size of an (a, b)-core and the average size of a self-conjugate (a, b)-core are **both equal** *to* $\frac{(a+b+1)(a-1)(b-1)}{24}$.

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Step 1

- Bijection: (a, b)-cores $\leftrightarrow$ Dyck paths in $a \times b$ rectangle

Example (The $(5, 8)$-core from earlier.)

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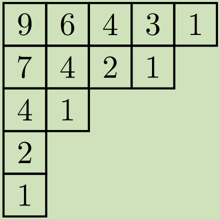

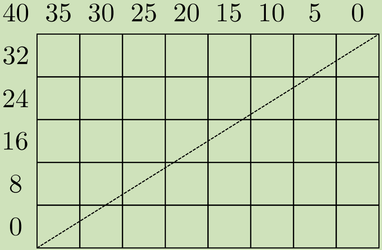

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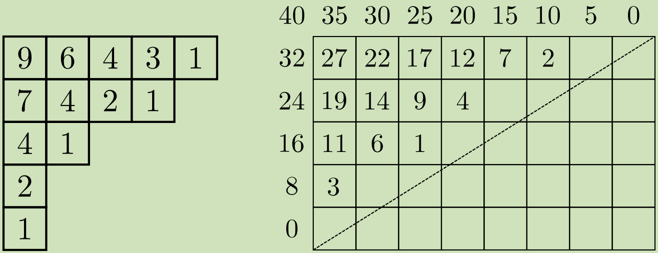

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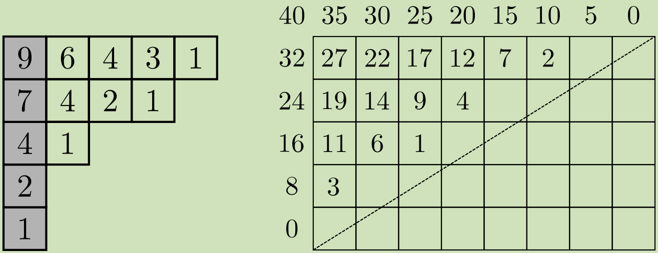

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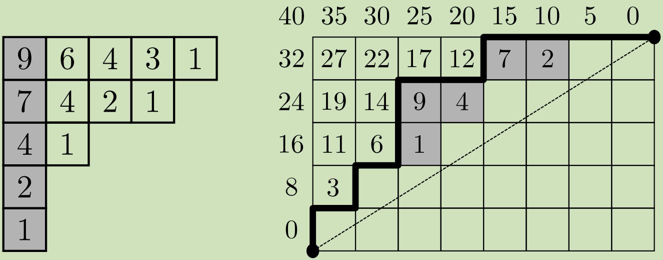

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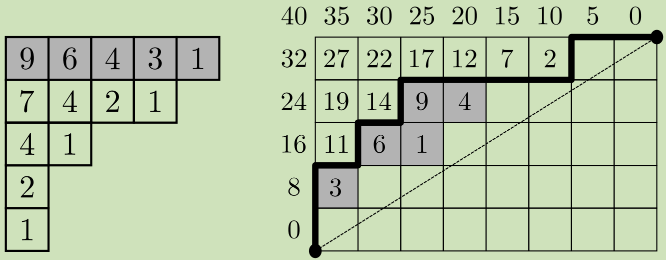
Example (NB: Conjugation is weird, but...)



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- **Theorem (Bizley, 1954):** # Dyck paths is $\frac{1}{a+b}\binom{a+b}{a,b}$.
- See Loehr's book: *"Bijective Combinatorics"*, page 497.

Proof idea.

- The $\binom{a+b}{a,b}$ lattice paths break into cyclic orbits of size $a+b$.
- Each orbit contains a unique Dyck path.
- Coprimality of a, b is necessary.

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Motivation 2: Parking Functions

(with Haglund, Haiman, Loehr, Warrington *et al.*)

Definition

Again let $x = a/(b-a)$ with $0 < a < b$ coprime.

- An *x*-parking function is a "decorated" Dyck path in the $a \times b$ rectangle. (Decorate the vertical runs with the labels $\{1, 2, \ldots, a\}$.)
- Classical form: $(z_1, z_2, \ldots, z_a)$ where label i occurs in column z_i.
- Symmetric group $\mathfrak{S}_a$ acts on classical forms by permutation. Let $\mathrm{PF}(x)$ denote the corresponding $\mathfrak{S}_a$-module.

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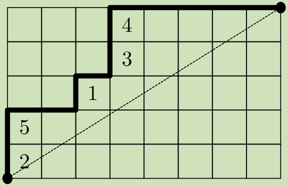
Motivation 2: Parking Functions

Examples for $x = 5/(8-5)$. ($\mathrm{Cat}(x) = 99$.)

- Here's the 5/3-parking function with classical form $(3, 1, \color{red}{4}, 4, \color{red}{1})$.



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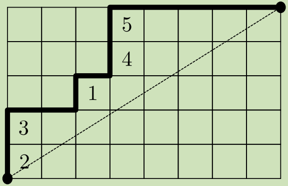

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Theorems

- # x-parking functions is b^{a-1}.
- # x-Dyck paths with r_i **vertical runs of length** i is $\frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a}$:

$$\mathrm{PF}(x) = \sum_{\mathbf{r} \vdash a} \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} h_{\mathbf{r}},$$

where the sum is over $\mathbf{r} = 0^{r_0}1^{r_1}\cdots a^{r_a} \vdash a$ with $\sum_i r_i = b$.

- # x-parking functions **fixed** by $\sigma \in \mathfrak{S}_a$ is $b^{\#\mathrm{cycles}(\sigma)-1}$:

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$\exists$ q and t?

Idea

Define a quasisymmetric function with coefficients in $\mathbb{N}[q,t]$ by

$$\mathrm{PF}_{q,t}(x) := \sum_{P} q^{\mathrm{qstat(P)}} t^{\mathrm{tstat(P)}} F_{\mathrm{iDes(P)}}.$$

- Sum over x-parking functions P.
- F is fundamental (Gessel) quasisymmetric function.
  — *natural refinement of Schur functions*
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  — *natural refinement of Schur functions*
- Must define qstat, tstat, iDes for x-parking function P.

$\exists\ q$ and t?

Idea

Define a quasisymmetric function with coefficients in $\mathbb{N}[q,t]$ by

$$\mathrm{PF}_{q,t}(x) := \sum_{P} q^{\mathrm{qstat(P)}} t^{\mathrm{tstat(P)}} F_{\mathrm{iDes(P)}}.$$

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qstat is easy.

Definition

- Let qstat := area := # boxes between the path and diagonal.
- Note: Maximum value of area is $(a-1)(b-1)/2$. (Frobenius) — *see Beck and Robins, Chapter 1*

Example

- This 5/3-parking function has area = 6.

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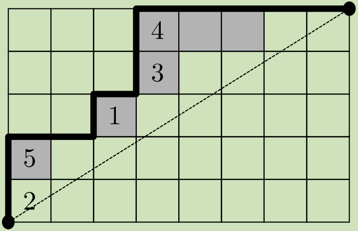

iDes is reasonable.

Definition

- Read labels by increasing "height" to get permutation $\sigma \in \mathfrak{S}_a$.
- iDes := the descent set of σ^{-1}.

Example

- This is a secret message.
- iDes $= \{1, 4\}$.

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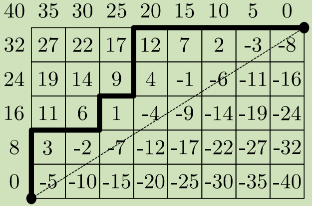
- Remember the "height"?



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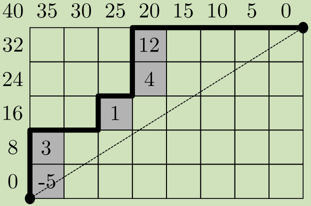
Example

- Look at the heights of the vertical step boxes.



- $\mathrm{iDes} = \{1, 4\}$.

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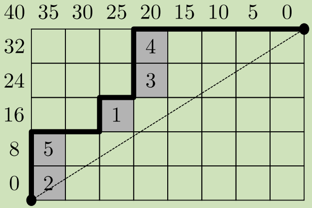
- iDes $:=$ the descent set of σ^{-1}.

Example

- Remember the labels we had before.



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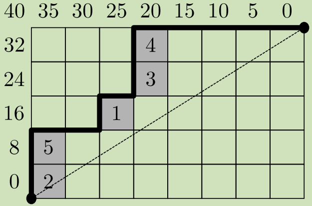
Example

- Read them by increasing height to get $\sigma = 2\bar{1}53\bar{4} \in \mathfrak{S}_5$.



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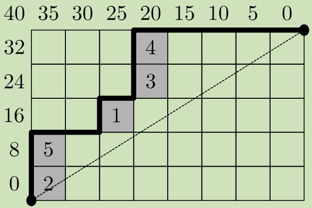

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tstat is bizarre (as usual).

Definition

- "Blow up" the x-parking function.
- Compute "dinv" of the blowup.

Example

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Definition

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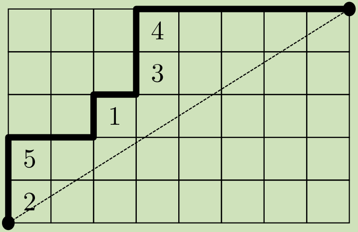
Example

- Remember our friend the 5/3-parking function.



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Definition

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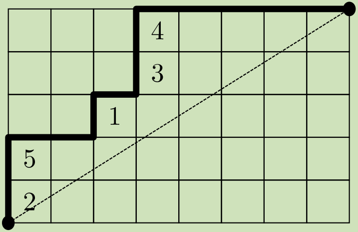
Example

- Since $2 \cdot 8 - 3 \cdot 5 = 1$ we "blow up" by 2 horiz. and 3 vert....



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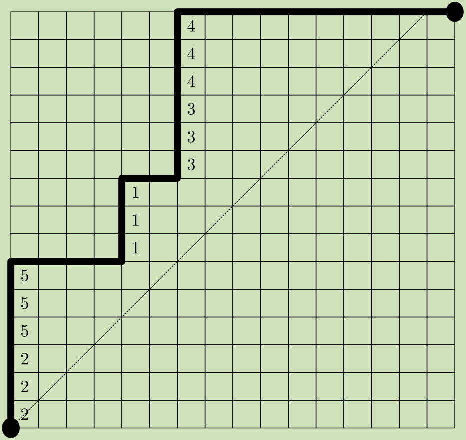
- To get this!



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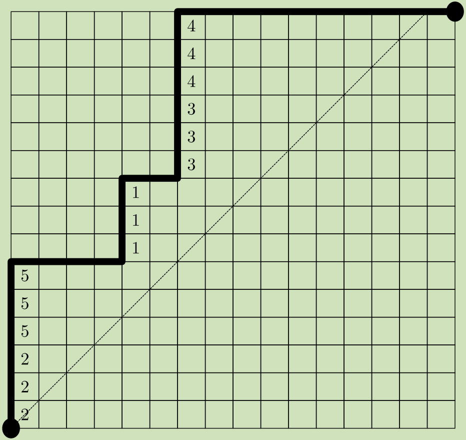
Example

- To get this! Now compute “dinv”. (Computation omitted.)



Some things.

Things

- $\mathrm{PF}_{1,1}(x) = \mathrm{PF}(x)$.
- $\mathrm{PF}_{q,t}(x)$ is symmetric and Schur-positive with coeffs $\in \mathbb{N}[q,t]$.
  — *via LLT polynomials*
- Probably $\mathrm{PF}_{q,t}(x) = \mathrm{PF}_{t,q}(x)$.
  — *this will be impossible to prove (see Loehr's Maxim)*
- The coefficient of **sgn** is some $\mathrm{Cat}_{q,t}(x)$.
- Probably $q^{(a-1)(b-1)/2}\mathrm{Cat}_{q,\frac{1}{q}}(x) = \frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q$.

Problems

- Does $\mathrm{PF}_{q,t}(x)$ occur "in nature"?
- How are $\mathrm{PF}_{q,t}(x)$ and $\mathrm{PF}_{q,t}(-x-1)$ related?

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Motivation 3: Lie Theory

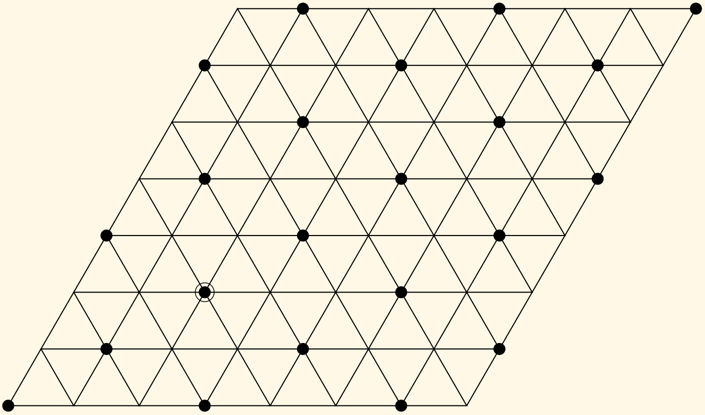
(quoting from: Cellini-Papi, Haiman, Shi, Sommers *et al.*)

Consider Weyl group $\mathfrak{S}_a$ with a, b coprime.

- Please disregard this.

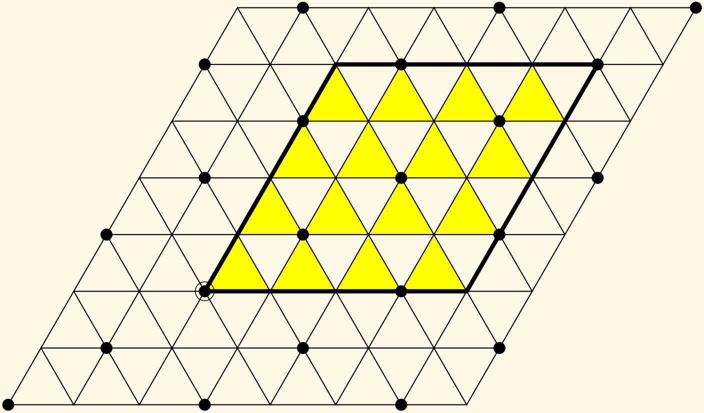
Consider Weyl group $\mathfrak{S}_a$ with a, b coprime.

- These are the weight and root lattices $\Lambda < Q$ of $\mathfrak{S}_a$.



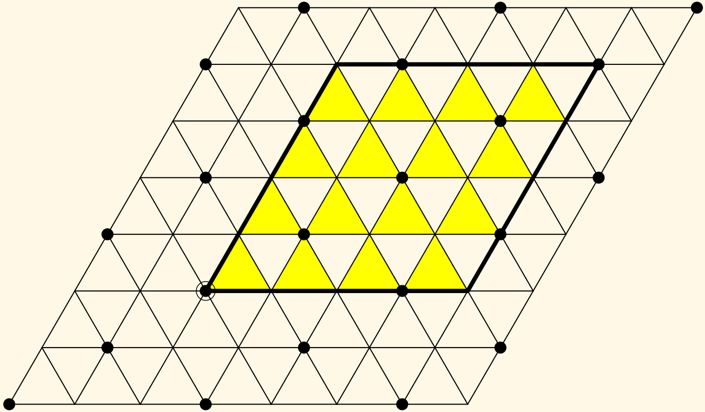
Consider Weyl group $\mathfrak{S}_a$ with a, b coprime.

- Here is a fundamental parallelepiped for $\Lambda/b\Lambda$.



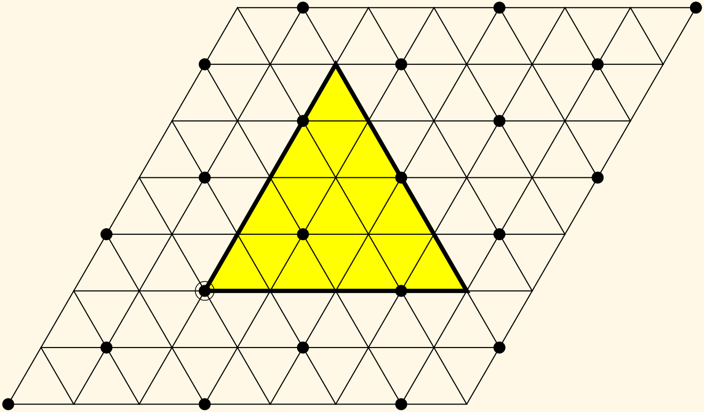
Consider Weyl group $\mathfrak{S}_a$ with a, b coprime.

- It contains b^{a-1} elements (the "parking functions").



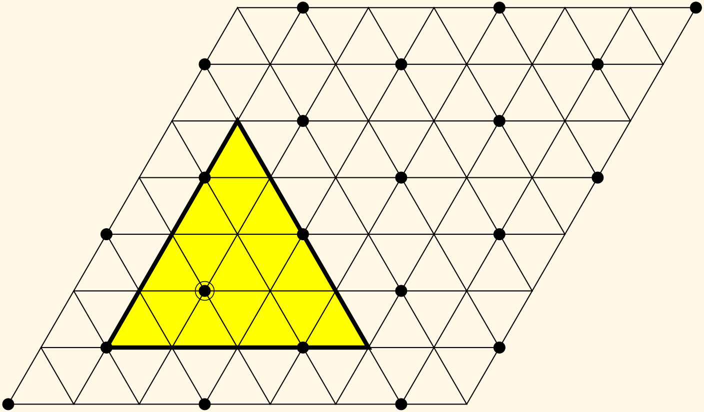
Consider Weyl group $\mathfrak{S}_a$ with a, b coprime.

- But they look better as a simplex...



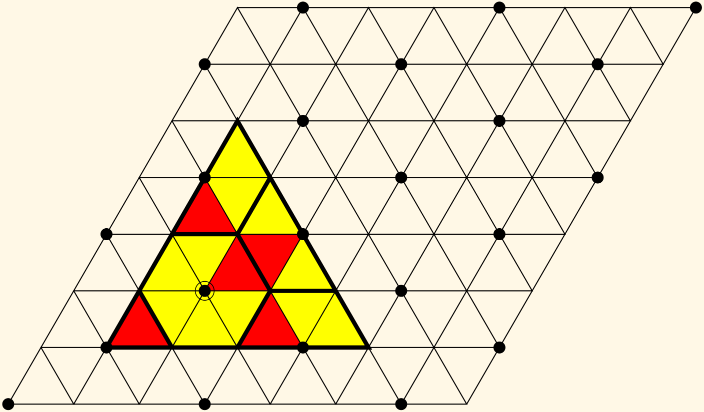
Consider Weyl group $\mathfrak{S}_a$ with a, b coprime.

- ...which is congruent to a nicer simplex.



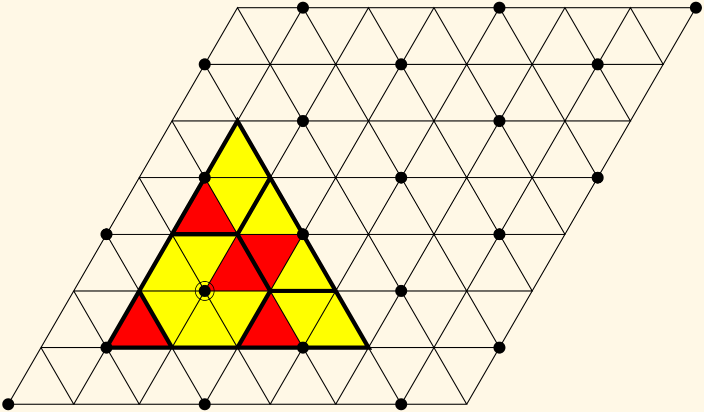
Consider Weyl group $\mathfrak{S}_a$ with a, b coprime.

- There are $\frac{1}{a+b}\binom{a+b}{a,b}$ elements of the root lattice inside.



Consider Weyl group $\mathfrak{S}_a$ with a, b coprime.

- These are called (a, b)-cores (or x-Dyck paths).



"The same" works for all Weyl groups...

Definition

Consider a Weyl group W with Coxeter number h and let $p \in \mathbb{N}$ coprime to h. We define the **Catalan number**

$$\mathrm{Cat}_q(W,p) := \prod_j \frac{[p+m_j]_q}{[1+m_j]_q}$$

where $e^{2\pi i m_j/h}$ are the eigenvalues of a Coxeter element.

...but I'm out of time.