Rational Catalan Combinatorics

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What is a Catalan Number?

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Convention

Given $x \in \mathbb{Q} \setminus [-1, 0]$ there exist unique *positive coprime* $a, b \in \mathbb{Z}$ with

$$x = \frac{a}{b - a}.$$

We will always identify $x \leftrightarrow (a, b)$.

Examples: Given $n \geq 1$ we have

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$$x = n = \frac{n}{1} = \frac{n}{(n+1)-n} \leftrightarrow (n, n+1)$$

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Examples: Given $n \geq 1$ we have

$$x = -n = \frac{n}{-1} = \frac{n}{(n-1)-n} \leftrightarrow (n, n-1) \quad \text{need } n \geq 2$$

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Examples: Given $n \geq 1$ we have

$$x = -\frac{1}{n} = \frac{1}{-n} = \frac{1}{(-n+1)-1} \leftrightarrow \ ? \quad \text{impossible!}$$

What is a Catalan Number?

Definition

For each $x \in \mathbb{Q} \setminus [-1, 0]$ we define the Catalan number:

$$\mathrm{Cat}(x) = \mathrm{Cat}(a, b) := \frac{1}{a+b}\binom{a+b}{a,b} = \frac{(a+b-1)!}{a!b!}.$$

Claim: This is an integer. (Proof postponed.)

Example:

$$\mathrm{Cat}\left(\frac{5}{3}\right) = \mathrm{Cat}\left(\frac{5}{8-5}\right) = \mathrm{Cat}(5, 8) = \frac{12!}{5!8!} = 99.$$

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Classical Cases

When $b = 1 \bmod a$ we have ...

- *Eugène Charles Catalan (1814-1894)*

  $(a, b) = (n, n+1)$ gives the **good old Catalan number**:

$$\mathrm{Cat}(n) = \mathrm{Cat}\left(\frac{n}{(n+1)-n}\right) = \frac{1}{2n+1}\binom{2n+1}{n}.$$

- *Nicolaus Fuss (1755-1826)*

  $(a, b) = (n, kn+1)$ gives the **Fuss-Catalan number**:

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Symmetry about $x = -1/2$

Definition

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$$\mathrm{Cat}(x) = \mathrm{Cat}(a, b) = \mathrm{Cat}(b, a) = \mathrm{Cat}(-x - 1).$$

This implies that for $0 < x \in \mathbb{Q}$ (i.e. $a < b$) we have

$$\mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

We will call this the derived Catalan number:

$$\mathrm{Cat}'(x) := \mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

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Note that $x > 0 \iff \frac{1}{x} > 0$ and we have

$$\mathrm{Cat}'(1/x) = \mathrm{Cat}\left(\frac{1}{(1/x) - 1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right) = \mathrm{Cat}'(x).$$

We call this rational duality:

$$\mathrm{Cat}'(x) = \mathrm{Cat}'(1/x).$$

In terms of coprime $0 < a < b$ this translates to

$$\mathrm{Cat}'(a, b) = \mathrm{Cat}'(b - a, b).$$

(This will appear later as **Alexander duality** of rational associahedra.)

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Euclidean Algorithm

Observation

Given $0 < a < b$ coprime, we observe that

$$\mathrm{Cat}'(a,b) = \frac{1}{b}\binom{b}{a} = \begin{cases} \mathrm{Cat}(a, b-a) & \text{for } a < (b-a) \\ \mathrm{Cat}(b-a, a) & \text{for } (b-a) < a \end{cases}$$

This allows us to define a sequence

$$\mathrm{Cat}(x) \mapsto \mathrm{Cat}'(x) \mapsto \mathrm{Cat}''(x) \mapsto \cdots$$

which is a Categorification of the Euclidean algorithm.

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Euclidean Algorithm

Example: $x = 5/3$ and $(a, b) = (5, 8)$

Subtract the smaller from the larger:

$$\begin{aligned} \mathrm{Cat}(5,8) &= 99, \\ \mathrm{Cat}'(5,8) &= \mathrm{Cat}(3,5) = 7, \\ \mathrm{Cat}''(5,8) &= \mathrm{Cat}'(3,5) = \mathrm{Cat}(2,3) = 2, \\ \mathrm{Cat}'''(5,8) &= \mathrm{Cat}''(3,5) = \mathrm{Cat}'(2,3) = \mathrm{Cat}(1,2) = 1 \quad \textbf{(STOP)} \end{aligned}$$

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Catalan Number Theorem?

Suggestion

Define the Catalan counting function for $n \in \mathbb{N}$ by

$$N_{\mathrm{Cat}}(n) := \#\{x \in \mathbb{Q}_{>0} : \mathrm{Cat}(x) \leq n\}.$$

Lemma: We have $N_{\mathrm{Cat}}(n) \geq n$ because $\mathrm{Cat}(2, 2n-1) = n$.

Investigate the asymptotics of $N_{\mathrm{Cat}}(n)$. Is it true that

$$\lim_{n\to\infty} \frac{\log N_{\mathrm{Cat}}(n)}{\log n} = \delta = \text{constant ?}$$

Call this δ the Hausdorff dimension of Cat.

See: *P. Sarnak, "Integral Apollonian Packings", AMM (2011)*

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Pause

Well, that was fun.

The Prototype: Rational Dyck Paths

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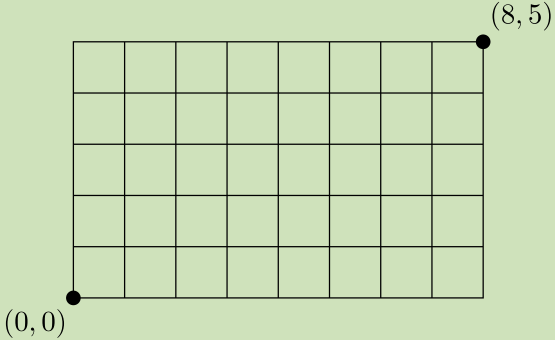
- Consider the "Dyck paths" in an $a \times b$ rectangle.

Example $(a, b) = (5, 8)$



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- Again let $0 < x = a/(b-a)$ with $0 < a < b$ coprime.

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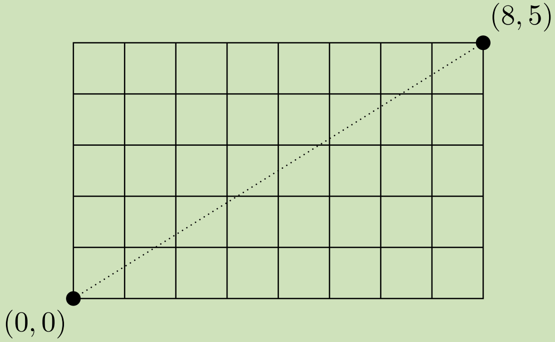

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- Let $\mathcal{D}(x) = \mathcal{D}(a, b)$ denote the set of Dyck paths.

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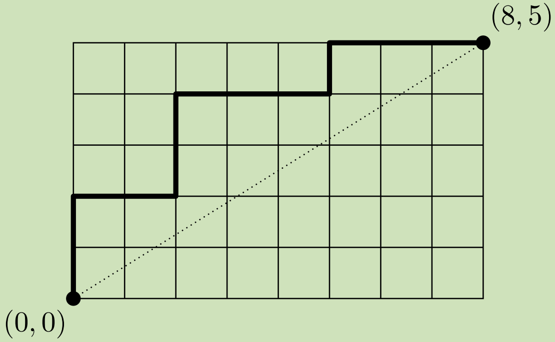

The Prototype: Rational Dyck Paths

Theorem (Grossman 1950, Bizley 1954)

*For a, b **coprime**, the number of Dyck paths is the Catalan number:*

$$|\mathcal{D}(x)| = \mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a,b}.$$

- *Claimed by Grossman (1950), "Fun with lattice points, part 22".*
- *Proved by Bizley (1954), in Journal of the Institute of Actuaries.*
- *Proof: Break $\binom{a+b}{a,b}$ lattice paths into cyclic orbits of size $a+b$. Each orbit contains a unique Dyck path.*

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Theorem (Armstrong 2010, Loehr 2010)

- *The number of Dyck paths with k vertical runs equals*

$$\mathrm{Nar}(x;k) := \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}.$$

Call these the **Narayana numbers**.

- *And the number with r_j vertical runs of length j equals*

$$\mathrm{Krew}(x;\mathbf{r}) := \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} = \frac{(b-1)!}{r_0!r_1!\cdots r_a!}.$$

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The Classical Associahedron

Definition

Let $n \geq 0$ and consider a convex $(n+2)$-gon C. Let $\mathrm{Ass}(n)$ be the abstract simplicial complex with

- vertices = chords of C
- faces = noncrossing sets of chords of C
- max. faces = triangulations of C

Theorem (Milnor, Haiman, C. Lee, etc.)

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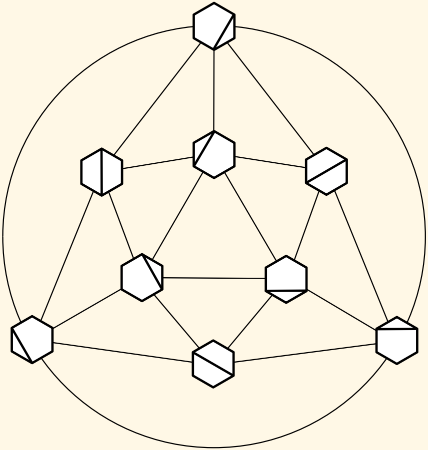
- Example: Here is Ass(4).



The Classical Associahedron

Theorem (Euler, 1751)

The f-vector and h-vector of Ass(n) *are given by the* **Kirkman numbers**

$$\mathrm{Kirk}(n;k) = \frac{1}{n}\binom{n}{k}\binom{n+k}{k-1}$$

and the **Narayana numbers**

$$\mathrm{Nar}(n;k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

The Classical Associahedron

- Example: Here are the *f*-vector and *h*-vector of Ass(4) .

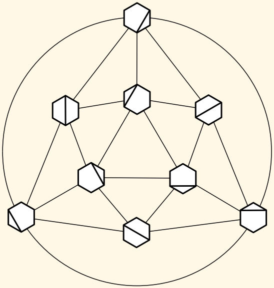
1

1 6

1 7 6

1 8 13 1

1 9 21 14



The Rational Associahedron?

Question

Given $0 < x = a/(b-a)$ with $0 < a < b$ coprime, can one define a "rational associahedron"

$$\mathrm{Ass}(x) = \mathrm{Ass}(a, b)$$

with the "correct" numerology and structure?

Answer

Yes.

The Rational Associahedron?

Question

Given $0 < x = a/(b-a)$ with $0 < a < b$ coprime, can one define a "rational associahedron"

$$\mathrm{Ass}(x) = \mathrm{Ass}(a, b)$$

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Answer

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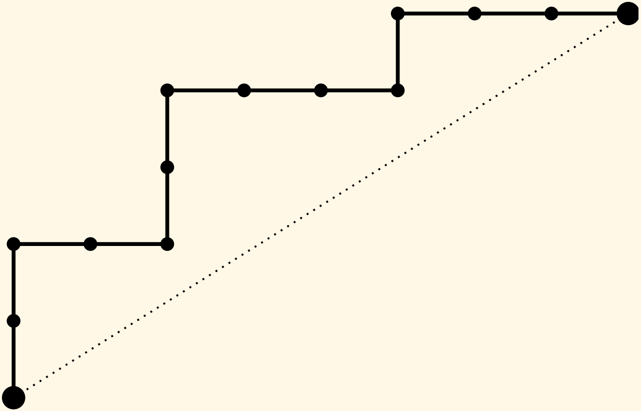
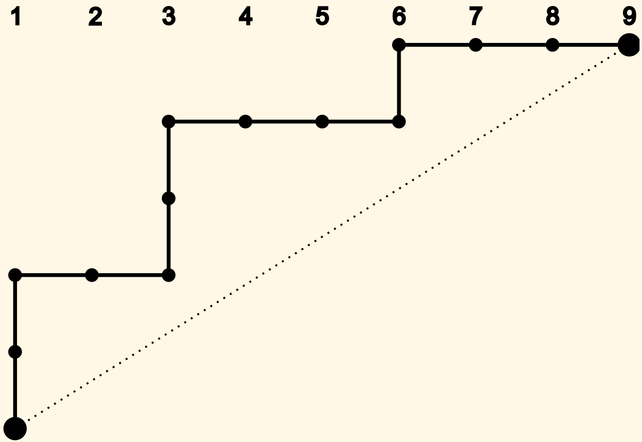
To define a "rational triangulation" . . .

- Start with a Dyck path. Here $(a, b) = (5, 8)$.



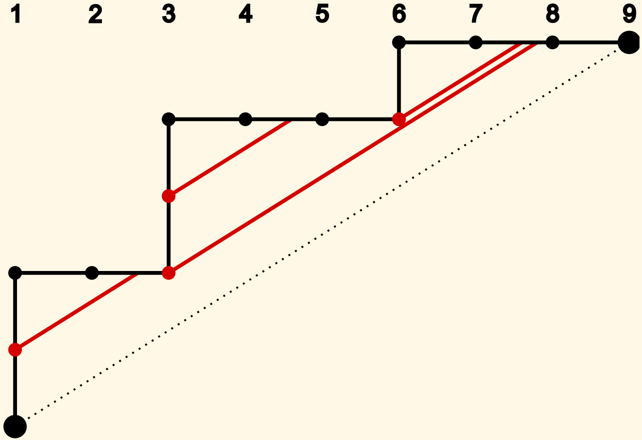
To define a “rational triangulation” . . .

- Label the columns by $\{1, 2, \ldots, b+1\}$.



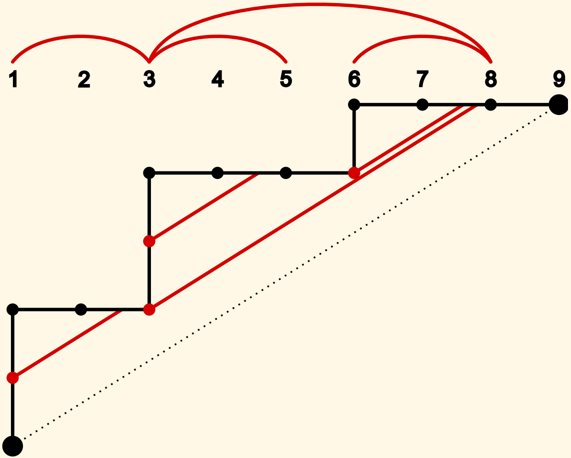
To define a "rational triangulation" . . .

- Shoot lasers from the bottom left with slope a/b.



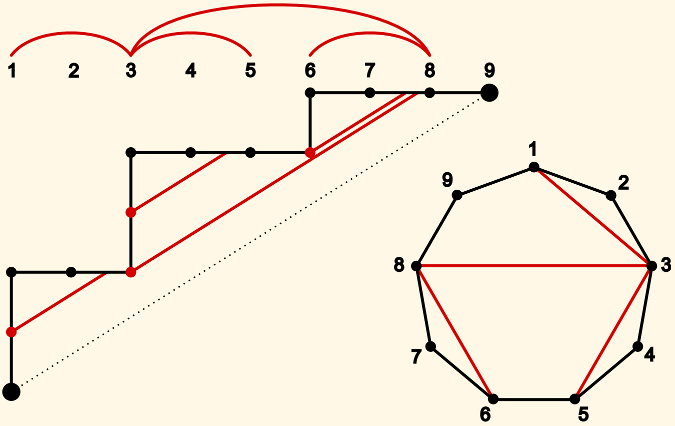
To define a “rational triangulation” ...

- Lift the lasers up.



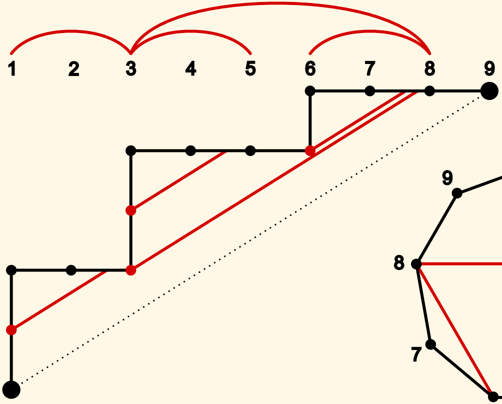
To define a "rational triangulation" ...

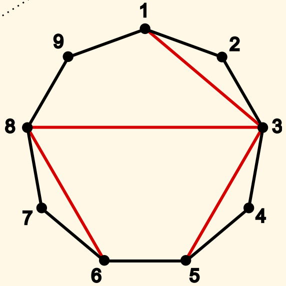
- There you go!



To define a “rational triangulation” ...

- We have constructed Cat(a, b) many “rational triangulations” of a convex $(b+1)$-gon, and each of them has $a-1$ chords.





The Rational Associahedron

Definition

Given $0 < x = a/(b-a)$, let $\mathrm{Ass}(x) = \mathrm{Ass}(a,b)$ be the abstract simplicial complex whose maximal faces are the "rational triangulations".

Geometric Realization

Note that $\mathrm{Ass}(a,b)$ is a pure $(a-1)$-dimensional subcomplex of the $(b-1)$-dimensional polytope $\mathrm{Ass}(b-1)$.

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Theorems (with B. Rhoades and N. Williams)

- $\mathrm{Ass}(n, n+1)$ is the **classical associahedron** $\mathrm{Ass}(n)$.
- $\mathrm{Ass}(n, (k-1)n+1)$ is the **generalized cluster complex** of Athanasiadis-Tzanaki and Fomin-Reading.
- $\mathrm{Ass}(x)$ has $\mathrm{Cat}(x)$ max. faces and Euler characteristic $\mathrm{Cat}'(x)$.
- $\mathrm{Ass}(x)$ is shellable and hence homotopy equivalent to a wedge of $\mathrm{Cat}'(x)$ many $(a-1)$-dimensional spheres.
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Rational Duality?

Observation

Note that Ass$(b-1)$ has this many vertices:

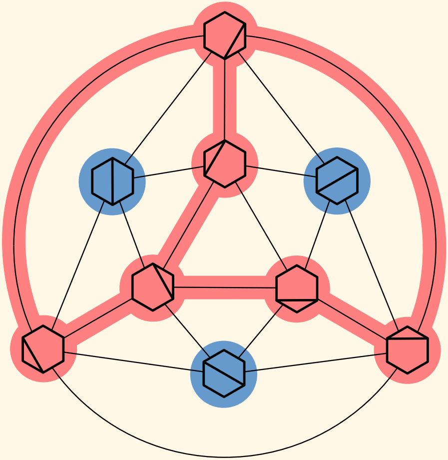
$$\binom{b+1}{2} - (b+1) = \frac{(b+1)b}{2} - \frac{2(b+1)}{2} = \frac{(b-2)(b+1)}{2}.$$

For all $0 < a < b$ coprime, the subcomplexes Ass(a, b) and Ass$(b-a, b)$ **bipartition** the vertices of Ass$(b-1)$ because

$$\frac{(a-1)(b+1)}{2} + \frac{(b-a-1)(b+1)}{2} = \frac{(b-2)(b+1)}{2}.$$

Rational Duality?

- Example: Here are subcomplexes Ass(2, 5) and Ass(3, 5) in Ass(4).



Rational Duality = Alexander Duality

Conjecture (with B. Rhoades and N. Williams)

We know that $\mathsf{Ass}(a, b)$ *and* $\mathsf{Ass}(b-a, b)$ *have the same number of homotopy spheres (of complementary dimensions) because*

$$\mathsf{Cat}'(a, b) = \mathsf{Cat}'(b-a, b).$$

We conjecture that the homotopy spheres are "intertwined" in a nice way. In particular, we conjecture that $\mathsf{Ass}(a, b)$ *and* $\mathsf{Ass}(b-a, b)$ *are* **Alexander dual** *inside the sphere* $\mathsf{Ass}(b-1)$*.*

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The conjecture is true.

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Euclidean Algorithm = ?

Definition

Given $0 < a < b$ coprime, if we define

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then

$$\# \textbf{ homotopy spheres } \mathrm{Ass}(a,b) = \# \textbf{ maximal faces } \mathrm{Ass}'(a,b).$$

Question

What does the following mean?

$$\mathrm{Ass}(a,b) \mapsto \mathrm{Ass}'(a,b) \mapsto \mathrm{Ass}''(a,b) \mapsto \cdots \mapsto \textbf{ a point}$$

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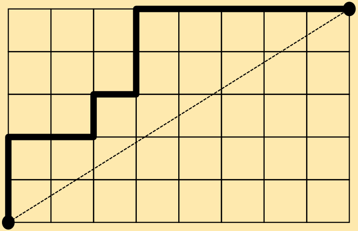
What is a Parking Function?

The Rational Parking Space

Definition

- Label the up-steps by $\{1, 2, \ldots, a\}$, increasing up columns.



- Call this a **parking function**.
- Let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote the set of parking functions.
- **Classical form** $(z_1, z_2, \ldots, z_a)$ has label z_i in column i.
- Example: $(3, 1, 4, 4, 1)$

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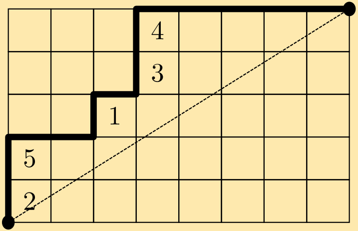

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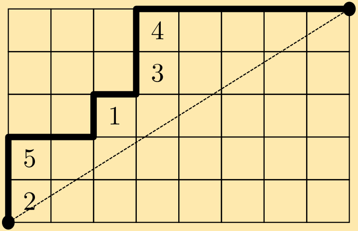

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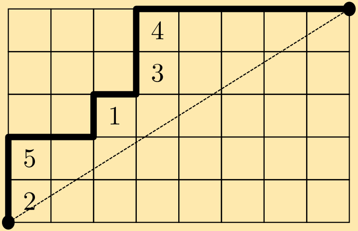

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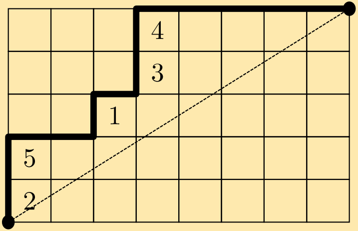

- Example: $(3, 1, 4, 4, 1)$ versus $(3, 1, 1, 4, 4)$
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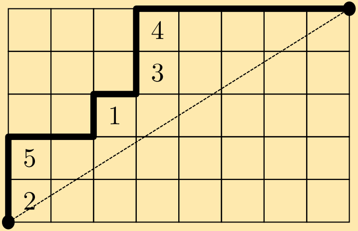

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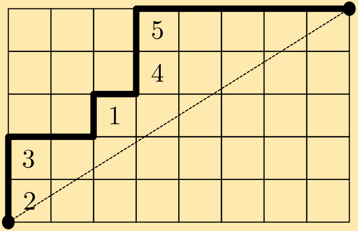

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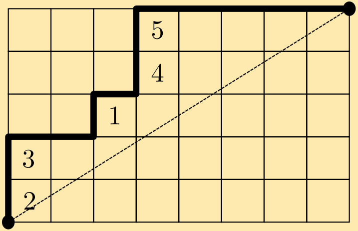

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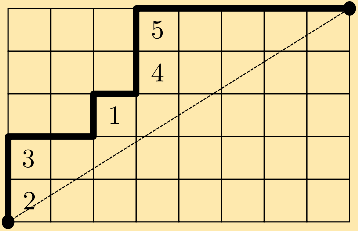

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Theorems (with N. Loehr and G. Warrington)

- The dimension of $\mathrm{PF}(a,b)$ is b^{a-1}.
- The **complete homogeneous expansion** is

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Observation/Definition

The multiplicities of the **hook Schur functions** $s[k+1, 1^{a-k-1}]$ in $\mathrm{PF}(a, b)$ are given by the **rational Schröder numbers**:

$$\mathrm{Schrö}(a, b; k) := \frac{1}{b} s_{[k+1, 1^{a-k-1}]}(1^b) = \frac{1}{b}\binom{a-1}{k}\binom{b+k}{a}.$$

Special Cases:

- Trivial character: $\mathrm{Schrö}(a, b; a-1) = \mathrm{Cat}(a, b)$.
- Smallest k that occurs is $k = \max\{0, a-b\}$, in which case

$$\mathrm{Schrö}(a, b; k) = \mathrm{Cat}'(a, b).$$

- Hence $\mathrm{Schrö}(x; k)$ interpolates between $\mathrm{Cat}(x)$ and $\mathrm{Cat}'(x)$.

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What does switching $a \leftrightarrow b$ mean?

Problem

Given a, b coprime we have an $\mathfrak{S}_a$-module $\mathrm{PF}(a, b)$ of dimension b^{a-1} and an $\mathfrak{S}_b$-module $\mathrm{PF}(b, a)$ of dimension a^{b-1}.

- What is the relationship between $\mathrm{PF}(a, b)$ and $\mathrm{PF}(b, a)$?
- Note that hook multiplicities are the same:

$$\mathrm{Schrö}(a, b; k) = \mathrm{Schrö}(b, a; k + b - a).$$

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Summary of Catalan Numerology

- The Kirkman/Narayana/Schröder numbers are equivalent. They contain information about rank. $(1 < k < a-1)$

$$\left.\begin{array}{c} \mathrm{Kirk}(x;k) = \frac{1}{a}\binom{a}{k}\binom{b+k-1}{k-1} \\ \\ \mathrm{Nar}(x;k) = \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1} \\ \\ \mathrm{Schr\ddot{o}}(x;k) = \frac{1}{b}\binom{a-1}{k}\binom{b+k}{a} \end{array}\right\} \begin{array}{c} f\text{-vector} \\ \\ h\text{-vector} \\ \\ \text{“dual” } f\text{-vector} \end{array}$$

- The Kreweras numbers are more refined. They contain parabolic information. $(\mathbf{r} \vdash a)$

$$\mathrm{Krew}(x;\mathbf{r}) = \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a}$$

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But what about q and t?

Tease

There **exists** a bigraded version $\mathrm{PF}_{q,t}(a,b)$. Here is the coefficient of the (non-hook) Schur function $s[2,2,1]$ in $\mathrm{PF}_{q,t}(5,8)$:

$$\begin{pmatrix}
 & & & & & & 1 & 1 & 1 & 1 & 1 \\
 & & & & 1 & 3 & 4 & 3 & 2 & 1 & \\
 & & & 2 & 6 & 6 & 4 & 2 & 1 & & \\
 & & 2 & 7 & 7 & 4 & 2 & 1 & & & \\
 & 1 & 6 & 7 & 4 & 2 & 1 & & & & \\
 & 3 & 6 & 4 & 2 & 1 & & & & & \\
1 & 4 & 4 & 2 & 1 & & & & & & \\
1 & 3 & 2 & 1 & & & & & & & \\
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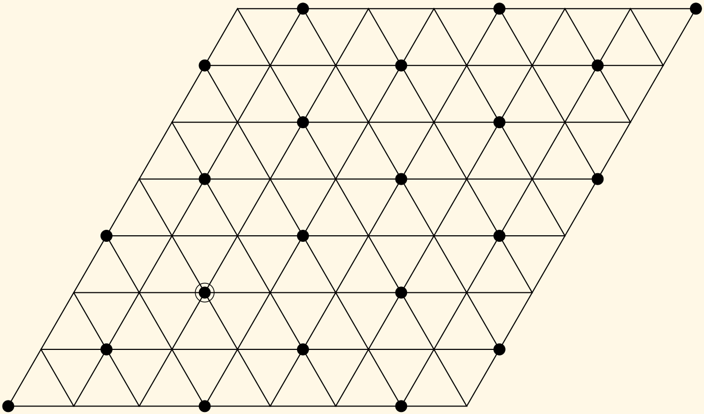
1 & & & & & & & & & &
\end{pmatrix}
$$

Epilogue: Lie Theory

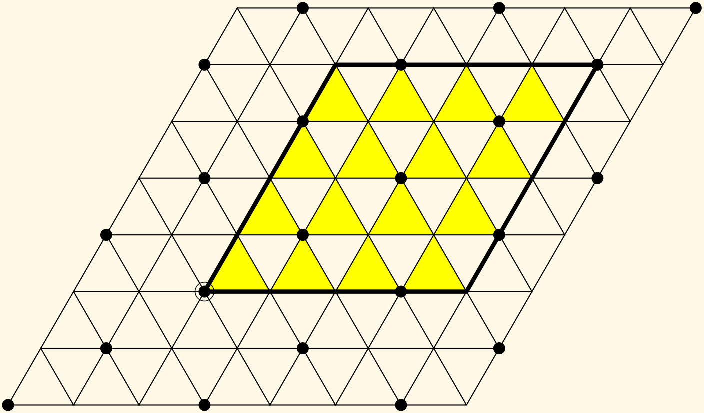
Consider Weyl Group S_a with a, b coprime.

- These are the root and weight lattices $Q \subseteq \Lambda$ of $\mathfrak{S}_a$.



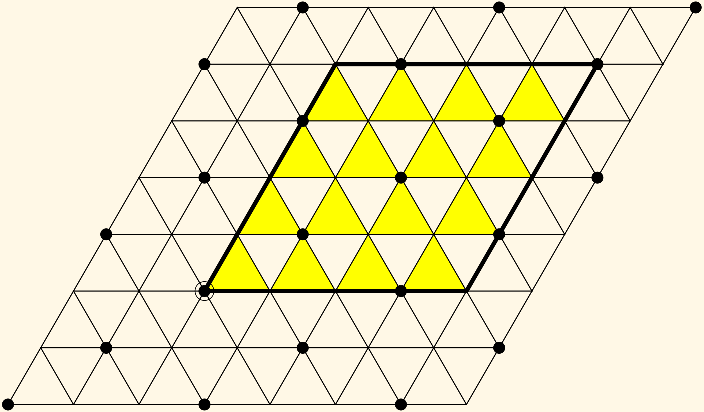
Consider Weyl Group S_a with a, b coprime.

- Here is a fundamental parallelepiped for $\Lambda/b\Lambda$.



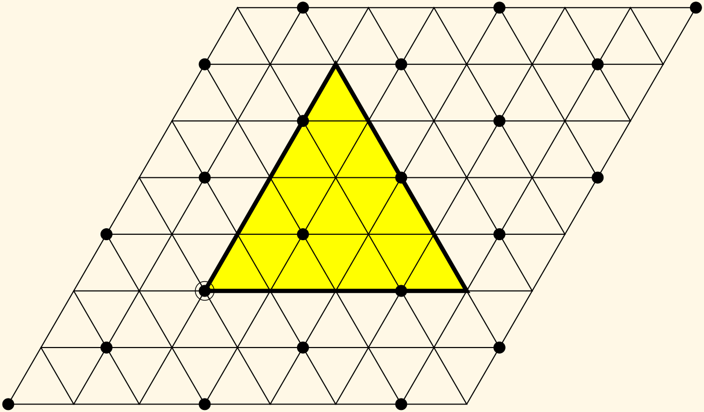
Consider Weyl Group S_a with a, b coprime.

- It contains b^{a-1} elements (these are the "parking functions").



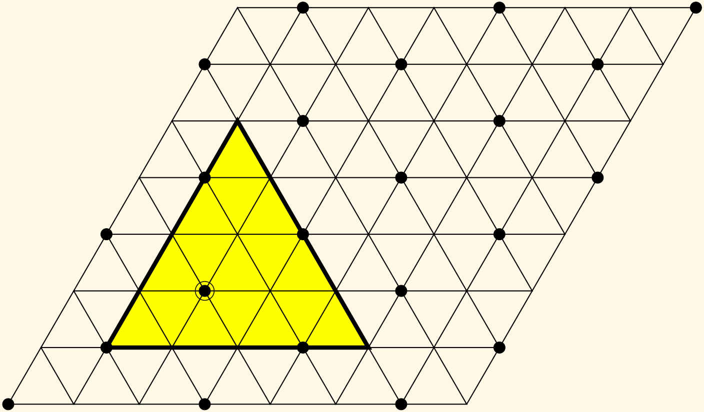
Consider Weyl Group S_a with a, b coprime.

- But they look better as a simplex...



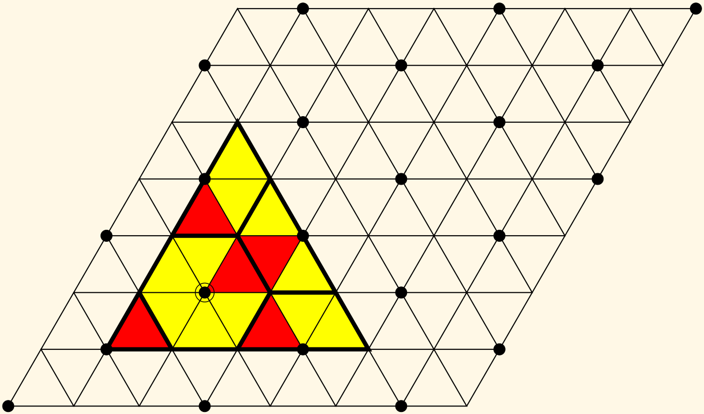
Consider Weyl Group S_a with a, b coprime.

- ...which is congruent to a nicer simplex.



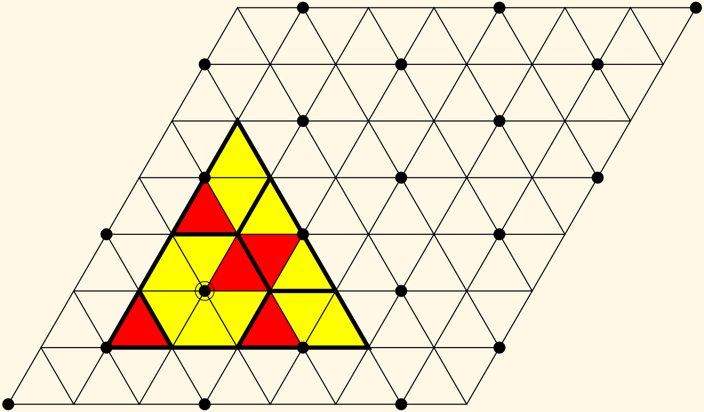
Consider Weyl Group S_a with a, b coprime.

- There are $\mathrm{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a,b}$ elements of the root lattice inside.



Consider Weyl Group S_a with a, b coprime.

- These are the (a, b)-Dyck paths (via Anderson, James-Kerber).



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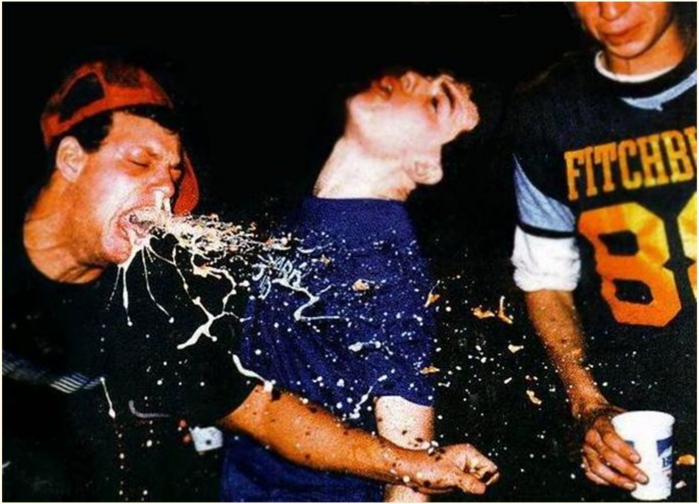

picture courtesy Allen Knutson