Introduction to the McKay Correspondence.
Lectures at Universidad de Talca, Chile
Nov 23 – 25, 2015.

These lectures will consider the following problem:

☆ Find all positive integers p, q, r such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

With a little thought you will find that the solutions are

$$\begin{array}{l} 1, *, * \\ 2, 2, * \\ 2, 3, 3 \\ 2, 3, 4 \\ 2, 3, 5 \end{array}$$

and that's all.

This problem shows up prominently in the following two topics:

1. Classification of finite groups of rotations in $\mathbb{R}^3$.

2. Classification of symmetric $(0,1)$ matrices with eigenvalues < 2.

But it is not immediately clear why these two problems should be related.

In the first two lectures I will explain the two problems and in the third lecture I will describe an observation of John McKay from 1979 that gives a beautiful connection between them.

This is all part of the mysterious "ADE phenomenon" that occurs throughout mathematics.

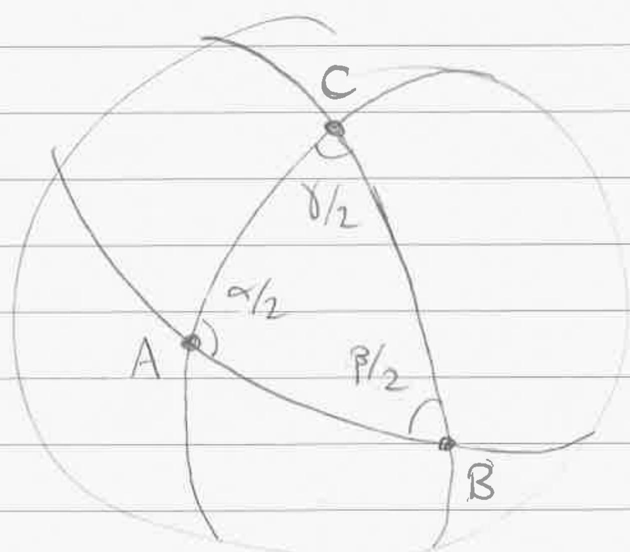
Lecture 1: Platonic solids and finite subgroups of $SU(2)$.

☆ Euler's Rotation Theorem (1776):

Let $R_A(\alpha), R_B(\beta) : \mathbb{R}^3 \to \mathbb{R}^3$ be rotations around vectors A, B by angles α, β counterclockwise.

Then $R_A(\alpha) \circ R_B(\beta) : \mathbb{R}^3 \to \mathbb{R}^3$ is also a rotation around some axis by some angle.

<u>Proof</u>: Intersect the vectors with a sphere and draw a triangle on the sphere as follows.

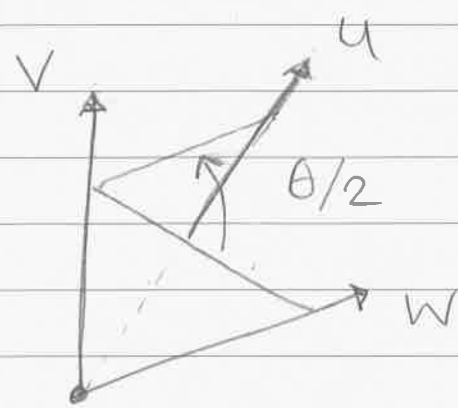


I claim that $R_A(\alpha) \circ R_B(\beta) = R_C(-\gamma)$.

To prove this let $F_{uv} : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection across the plane spanned by vectors u & v. It is not difficult to check that

$$F_{uv} \circ F_{vw} = R_v(\theta)$$

where θ is <u>twice</u> the angle c.c.w. from plane vw to plane uv, measured at v:



Thus we have

$$R_A(\alpha) = F_{CA} \circ F_{AB}$$

$$R_B(\beta) = F_{AB} \circ F_{BC}$$

and it follows that

$$
\begin{aligned}
R_A(\alpha) \circ R_B(\beta) &= (F_{CA} \circ F_{AB}) \circ (F_{AB} \circ F_{BC}) \\
&= F_{CA} \circ (\cancel{F_{AB} \circ F_{AB}}) \circ F_{BC} \\
&= F_{CA} \circ F_{BC} \\
&= (F_{BC} \circ F_{CA})^{-1} \\
&= R_C(\gamma)^{-1} \\
&= R_C(-\gamma).
\end{aligned}
$$

Let $SO(3)$ be the group of all rotations of $\mathbb{R}^3$, called the <u>special orthogonal</u> group. This is a Lie group which means it has the structure of a differentiable manifold. In fact it is isomorphic as a manifold to real projective 3-dimensional space.

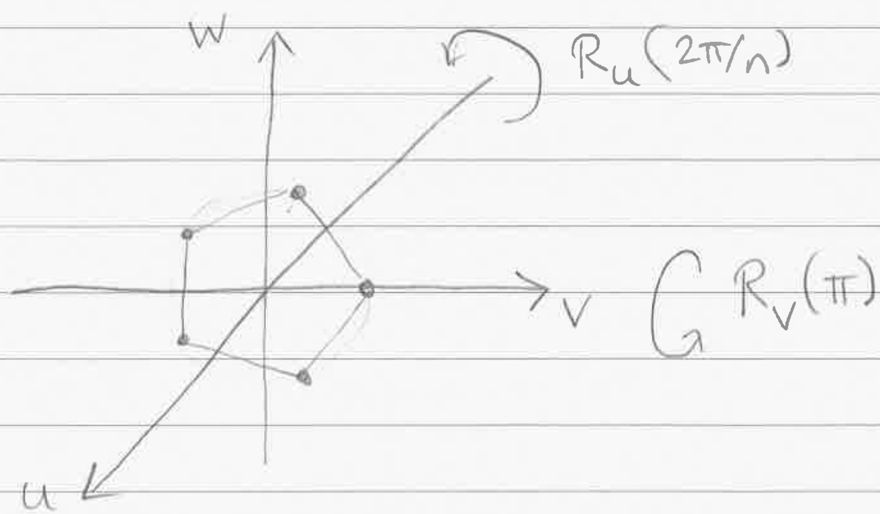
$$SO(3) \approx \mathbb{RP}^3.$$

Today I am interested in the discrete subroups of $SO(3)$, which since $SO(3)$ is compact are the same as the finite subgroups.

There are some obvious examples coming from regular polygons and polyhedra in $\mathbb{R}^3$.

Two infinite families:

Consider the following picture



- Cyclic Group $C_n = \langle R_u(2\pi/n) \rangle$
- Dihedral Group $D_{2n} = \langle R_u(2\pi/n), R_v(\pi) \rangle$.

Three "exceptional" cases:

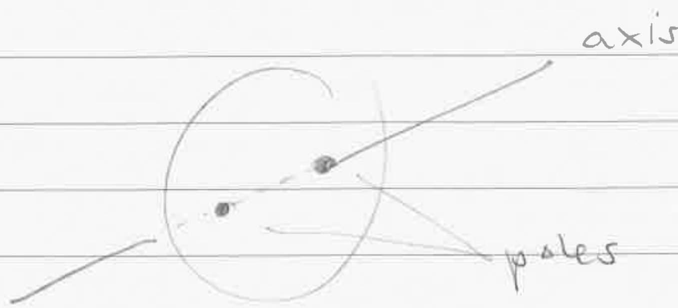
- T = 12 rotations of a tetrahedron
- O = 24 rotations of a cube/octahedron
- I = 60 rotations of dodec./icosahedron.

☆ Theorem: These are <u>all</u> of the finite subgroups of $SO(3)$.

Proof Sketch:

Let $G \subseteq SO(3)$ be finite and consider the set of "axes of rotation" for the non-identity elements of G.

Each axis intersects the sphere in two points, called "poles":



Then G acts on the set of poles P.
Consider the decomposition into orbits

$$P = \mathrm{Orb}_1 \cup \mathrm{Orb}_2 \cup \cdots \cup \mathrm{Orb}_m$$

and let $r_i := |\mathrm{Stab}_G(p)|$ for $p \in \mathrm{Orb}_i$.

By counting the set

$$\{(g,p) : 1 \neq g \in G,\ p \in P,\ g(p) = p\}$$

in two ways we obtain

$$(*) \qquad \underset{\substack{\uparrow \\ \text{choose } g \text{ first}}}{2(|G|-1)} = \underset{\substack{\uparrow \\ \text{choose } p \text{ first.}}}{\sum_{p \in P} (r_p - 1)}$$

Now let $o_i := |\mathrm{Orb}_i|$, so that

$$\sum_{p \in P} (r_p - 1) = \sum_{i=1}^{m} o_i (r_i - 1)$$

$$= \sum_{i=1}^{m} (o_i r_i - o_i)$$

$$= \sum_{i=1}^{m} \left(|G| - \frac{|G|}{r_i} \right)$$

$$= |G| \sum_{i=1}^{m} \left(1 - \frac{1}{r_i} \right).$$

Then from (*) we have

$$2(|G|-1) = |G| \sum_{i=1}^{m} \left(1 - \frac{1}{r_i} \right)$$

$$2 - \frac{2}{|G|} = \sum_{i=1}^{m} \left(1 - \frac{1}{r_i} \right)$$

$$\boxed{\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_m} = m - 2 + \frac{2}{|G|}}$$

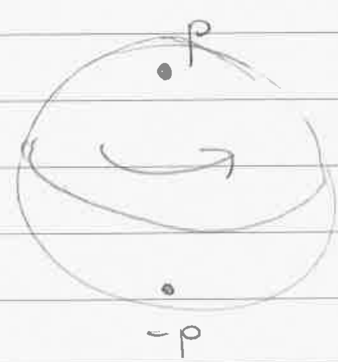
where m is the number of orbits. Since $r_i \geq 2$ one can check that $m=1$ and $m \geq 4$ are impossible, so there are two cases.

Case $m=2$: $\quad \dfrac{1}{r_1} + \dfrac{1}{r_2} = \dfrac{2}{|G|}.$

Since $r_i \leq |G|$ this implies $r_1 = r_2 = |G|$ hence $o_1 = o_2 = 1$.

$\mathrm{Orb}_1 = \{p\}$

$\mathrm{Orb}_2 = \{-p\}$



We conclude that G is cyclic.

Case $m=3$: $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = 1 + \frac{2}{|G|} > 1$.

We have already seen the solutions to this equation.

$2, 2, n \Longrightarrow G = D_{2n}$

$2, 3, 3 \Longrightarrow G = T$

$2, 3, 4 \Longrightarrow G = O$

$2, 3, 5 \Longrightarrow G = I$.

In these cases we can view r_1, r_2, r_3 as the amount of rotational symmetry around edges, vertices, faces of a regular polyhedron.

In fact we can define these groups abstractly via generators and relations.

☆ Definition: Given integers $p, q, r \geq 1$ we define the von Dyck group

$$D(p,q,r) = \langle X^p = Y^r = Z^q = XYZ = 1 \rangle.$$

The fact that $D(2,3,5) \approx I$ was discovered by Hamilton (1856). He called it the "icosian calculus" and tried to market it as a board game.

It turns out that $D(p,q,r)$ is finite if and only if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 :$$

Moreover, we have

$$\begin{aligned}
D(1,n,n) &\approx C_n \\
D(2,2,n) &\approx D_{2n} \\
D(2,3,3) &\approx T \\
D(2,3,4) &\approx O \\
D(2,3,5) &\approx I.
\end{aligned}$$

In 1940, Coxeter noticed something strange. Define the <u>binary von Dyck group</u>

$$\tilde{D}(p,q,r) := \langle X^p = Y^q = Z^r = XYZ \rangle.$$

He showed that $\tilde{D}(p,q,r)$ is finite if and only if $D(p,q,r)$ is finite, in which case

$$|\tilde{D}(p,q,r)| = 2\,|D(p,q,r)|.$$

In fact, he proved that there exists an element $\delta \in \tilde{D}(p,q,r)$ such that

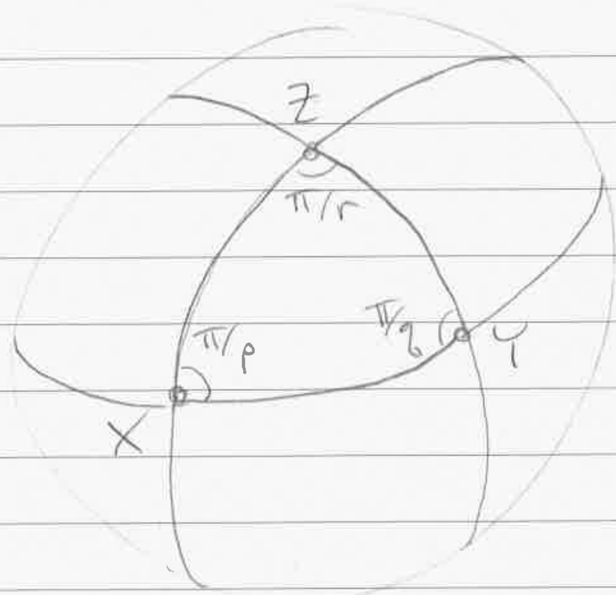
- $\delta^2 = 1$
- $\tilde{D}(p,q,r) = \langle X^p = Y^q = Z^r = XYZ = \delta \rangle$

Observe that $\tilde{D}(2,2,2)$ is the "quaternion group" Q_8 as defined by Hamilton:

$$\boxed{i^2 = j^2 = k^2 = ijk = -1}$$

You might wonder if this has a geometric interpretation ...

The Big Picture: Consider a triangle on the surface of a sphere



where p, q, r are positive integers. By the area formula for a spherical triangle we have

$$0 < \text{area} = \text{angle excess}$$

$$= \left(\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} \right) - \pi$$

$$\Longrightarrow \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

We can define three groups from this triangle

$$D(p,q,r) = \langle R_x(2\pi/p), R_y(2\pi/q), R_z(2\pi/r) \rangle$$

$$\Delta(p,q,r) = \langle F_{xy}, F_{yz}, F_{zx} \rangle$$

$$\widetilde{D}(p,q,r) = \langle \widetilde{R}_x(2\pi/p), \widetilde{R}_y(2\pi/q), \widetilde{R}_z(2\pi/r) \rangle$$

with $|\Delta(p,q,r)| = |\widetilde{D}(p,q,r)| = 2\,|D(p,q,r)|$.

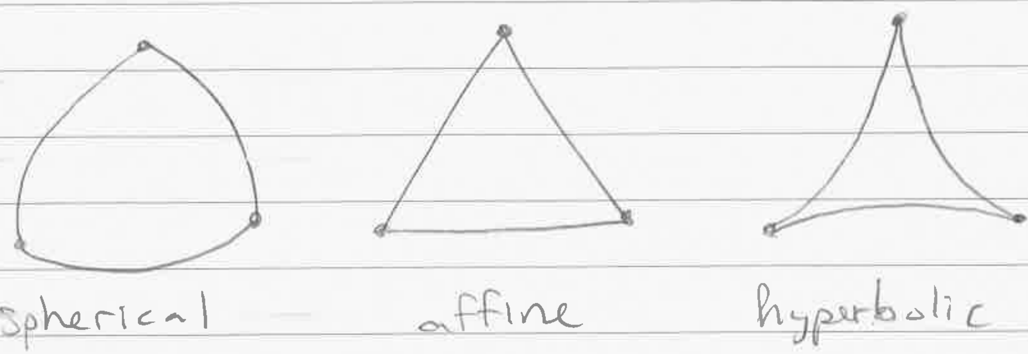
Moreover, $\Delta(p,q,r)$ and $\widetilde{D}(p,q,r)$ can be thought of as topological lifts of $D(p,q,r)$ to two different covering spaces of $\mathbb{RP}^3$.

$$\begin{tikzcd}
\widetilde{D}(p,q,r) & & \Delta(p,q,r) \\
SU(2) = S^3 \arrow[dr, "2:1"'] & & O(3) = \mathbb{RP}^3 \cup \mathbb{RP}^3 \arrow[dl, "2:1"] \\
& SO(3) = \mathbb{RP}^3 & \\
& D(p,q,r) &
\end{tikzcd}$$

Remark:

When $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$ or < 1,

then $D(p,q,r)$ & $\Delta(p,q,r)$ are discrete groups of isometries of the affine and hyperbolic planes, respectively.



I have no idea about the group $\widetilde{D}(p,q,r)$ in these other cases. If you know something about it please tell me.

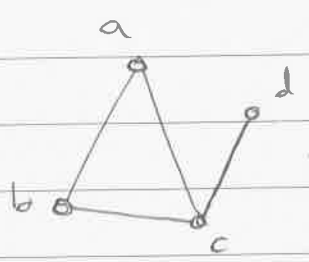
Lecture 2: Dynkin Diagrams of Type ADE.

In this Lecture "G" stands for "graph" instead of "group". To each finite simple graph G we associate a

$$\text{symmetric } (0,1)\text{-matrix } A_G$$

called its <u>adjacency matrix</u>. Conversely, any symmetric $(0,1)$-matrix is the adjacency matrix of some graph.

Example:

$G =$ 

, $A_G = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$

We define the <u>spectral radius</u> of G by

$$\|G\| := \{ |\lambda| : \lambda \text{ is an eigenvalue of } A_G \}.$$

This is some measure of the "complexity" of the graph.

For example, the graph above has

$$\|G\| \approx 2.17 .$$

The problem of today's lecture is to find the graphs with the smallest spectral radius.

First note that the spectral radius of a disconnected graph is given by

$$\|G \cup H\| = \max\{\|G\|, \|H\|\}.$$

Proof: We have

$$A_{G \cup H} = \left(\begin{array}{c|c} A_G & O \\ \hline O & A_H \end{array}\right)$$

so that the eigenvalues of $G \cup H$ are the union of the eigenvalues of G & H. ///

Thus we will restrict our attention to connected graphs.

The most basic connected graph is a chain of length n

$$C_n = \circ\!-\!\circ\!-\!\circ \cdots \circ\!-\!\circ \,, \quad A_{C_n} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}$$

[Today "C_n" is for "chain" not "cyclic group". But wait until tomorrow...]

This matrix can be explicitly diagonalized using the trigonometric angle sum/difference formulas. For all $k, l \in \mathbb{Z}$ we have

$$\sin\left(\frac{lk\pi}{n+1} + \frac{k\pi}{n+1}\right) = \sin\left(\frac{lk\pi}{n+1}\right)\cos\left(\frac{k\pi}{n+1}\right) + \cos\left(\frac{lk\pi}{n+1}\right)\sin\left(\frac{k\pi}{n+1}\right)$$

$$+ \quad \sin\left(\frac{lk\pi}{n+1} - \frac{k\pi}{n+1}\right) = \sin\left(\frac{lk\pi}{n+1}\right)\cos\left(\frac{k\pi}{n+1}\right) - \cos\left(\frac{lk\pi}{n+1}\right)\sin\left(\frac{k\pi}{n+1}\right)$$

$$(*) \quad \sin\left(\frac{(l-1)k\pi}{n+1}\right) + \sin\left(\frac{(l+1)k\pi}{n+1}\right) = \left(2\cos\left(\frac{k\pi}{n+1}\right)\right)\sin\left(\frac{lk\pi}{n+1}\right).$$

Now for each $1 \le k \le n$ define the vector

$$v_k = \left(\sin\left(\frac{k\pi}{n+1}\right), \sin\left(\frac{2k\pi}{n+1}\right), \ldots, \sin\left(\frac{(n-1)k\pi}{n+1}\right)\right).$$

Then the equation (*) becomes

$$\boxed{A_{C_n}(V_k) = 2\cos\left(\frac{k\pi}{n+1}\right) V_k}$$

This result was first obtained by Lagrange in 1759 ("Researches on the Nature and Propagation of Sound") as part of his solution to the wave equation. It was probably the first explicit diagonalization of a matrix.

We conclude that $2\cos\left(\frac{k\pi}{n+1}\right)$ are the eigenvalues of A_{C_n} and hence

$$\|C_n\| = 2\cos\left(\frac{\pi}{n+1}\right) < 2$$

with $\|C_n\| \longrightarrow 2$ as $n \longrightarrow \infty$. ///

We have obtained an infinite family of graphs G with $\|G\| < 2$.

Are there any others ?

At this point we will use (without proof) a very important technical lemma. It is a special case of the the theorem of Perron-Frobenius.

☆ Lemma: Let G be a connected graph.

(1) If A_G has a positive eigenvector with eigenvalue λ then we have

$$\|G\| = \lambda.$$

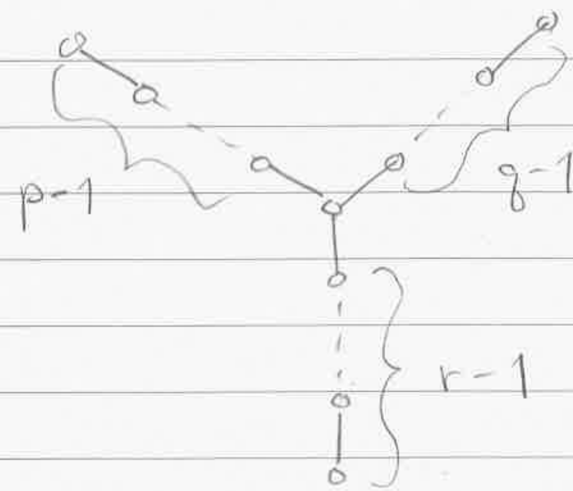
(2) If $H \subsetneq G$ is any proper subgraph of G then we have

$$\|H\| \lneq \|G\|.$$

[Remark: Part (1) can be proved with the Banach fixed-point theorem and Part (2) follows from (1).]

Now let's use this to investigate graphs with small spectral radius.

☆ Definition : Given integers $p, q, r \geq 1$ we define the graph $Y(p,q,r)$ as follows



☆ Theorem (Folklore, Smith 1970) :

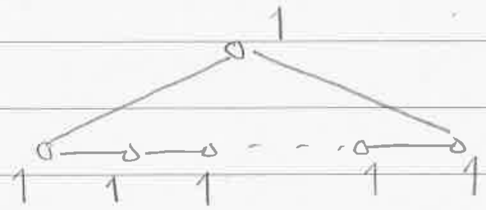
The graphs G with $\|G\| < 2$ are disjoint unions of $Y(p,q,r)$ with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 .$$

Proof : Let G be a connected graph with $\|G\| < 2$.

There are 4 steps.

Step 1 : G has no cycle. Otherwise G contains a subraph of the form

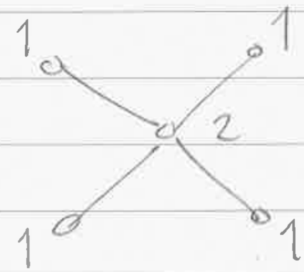
$H =$ 

Note that the vertex labels are a positive eigenvector for A_H with eigenvalue 2. By Lemma ① this implies

$$2 = \|H\| \leq \|G\| < 2.$$

Contradiction. ///

Step 2 : G contains no vertex of degree ≥ 4. Otherwise G contains the subgraph

$H =$ 

which has $\|H\| = 2$ via the displayed eigenvector.

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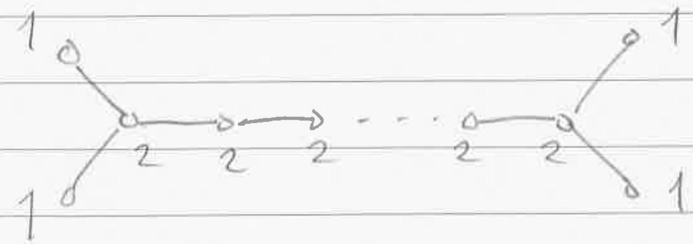
Step 3: G has at most one vertex of degree 3.
Otherwise G contains the subgraph

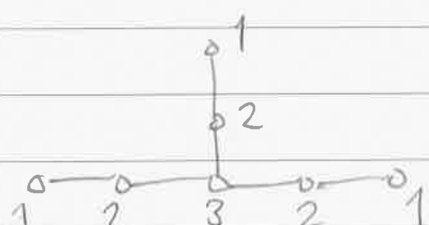
$H =$ 

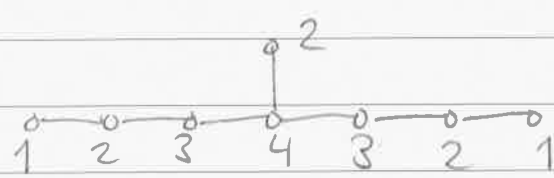
which has $\|H\| = 2$. ///

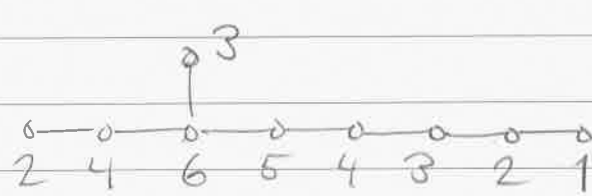
At this point we know that
$G = Y(p,q,r)$ for some p,q,r.

~~Step~~ 4: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

Otherwise G contains one of the following graphs of spectral radius 2.

$Y(3,3,3) =$ 

$Y(2,4,4) =$ 

$Y(2,3,6) =$ 

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This completes the proof. But this proof raises more questions than it answers.

For example:

- It does not explain *why* we should have

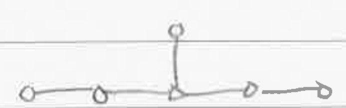
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 .$$

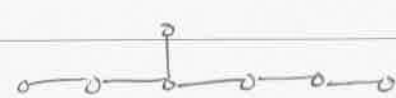
- The vertex labelings of the obstruction graphs are very mysterious. Where do they come from?

I will answer these questions tomorrow. Right now let me give you the standard names for these graphs

$A_\ell := Y(1, k, \ell-k+1) =$ 

$D_\ell := Y(2, 2, \ell-2) =$ 

$E_6 := Y(2, 3, 3) =$ 

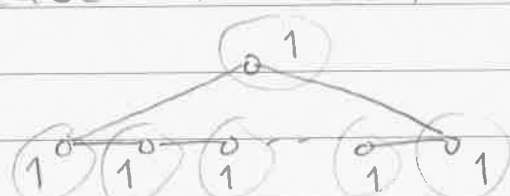
$E_7 := Y(2, 3, 4) =$ 

$E_8 := Y(2, 3, 5) =$ 

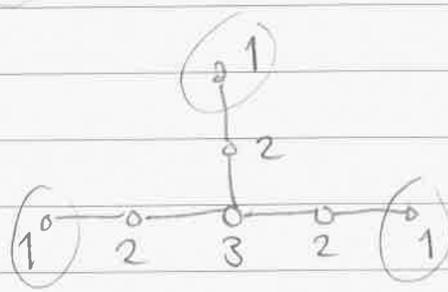
These graphs are commonly known as the Dynkin diagrams of type ADE. The "ADE" notation goes back to Wilhelm Killing's work of 1888 on classification of "space forms", now called "complex semisimple Lie algebras".

The graphs occurring as obstructions in the proof also have standard names.

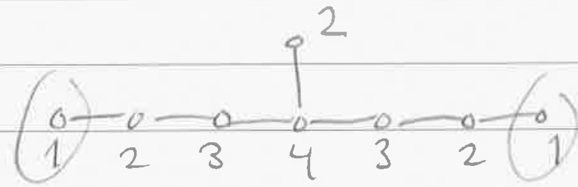
☆ **Theorem**: There is a bijection between connected graphs with $\|G\| = 2$ and connected graphs with $\|G\| < 2$. The bijection is given by deleting any vertex with label 1 below.

$\tilde{A}_\ell =$  any vertex

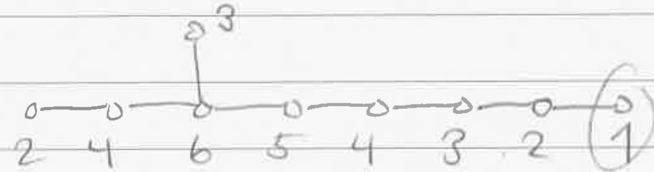
$\tilde{D}_\ell =$ 

$\tilde{E}_6 =$ 

$\tilde{E}_7 =$



$\tilde{E}_8 =$



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These are called the <u>affine</u> Dynkin diagrams of type ADE.

Now let me record some numerological observations. Let

$$G \in \{ A_\ell, D_\ell, E_6, E_7, E_8 \}$$

If G has ℓ vertices then $\tilde{G}$ has $\ell+1$ vertices and we index them by the set

$$I = \{0, 1, 2, \ldots, \ell\}$$

where $I \setminus \{0\}$ index the vertices of G.

Let $\{n_i : i \in I\}$ be the vertex labels on the affine diagram $\tilde{G}$, i.e., the entries of the eigenvector of eigenvalue 2, scaled so that $n_0 = 1$. We will call these the marks of type G.

☆ Observation: The spectral radius of G is given by

$$\|G\| = 2\cos\left(\frac{\pi}{h}\right)$$

where $h := n_0 + n_1 + n_2 + \cdots + n_\ell$ is called the Coxeter number.

Proof: We have seen that this is true for $G = A_\ell$ with $h = \ell + 1$ and it can be checked directly for the other types. ///

I don't know an easy conceptual proof of this fact. It follows from a theorem of Kostant on root systems which requires too many definitions to state today.

Instead I'll tell you a combinatorial game version, communicated to me by Allen Knutson.

☆ Kostant's Game :

Let G be a finite graph with all vertices labeled by 0. Start the game by putting a 1 on any vertex. Now divide the vertices into three types :

- Happy if the label is $1/2$ the sum of its neighbours.
- Unhappy if it's less.
- Manic if it's more.

To play the game, find an unhappy vertex. Replace its label with the sum of its neighbors' labels, minus the old label. Repeat until there are no unhappy vertices.

Optional Ending : Add a new vertex with label 1 and connect it to any remaining manic vertices.

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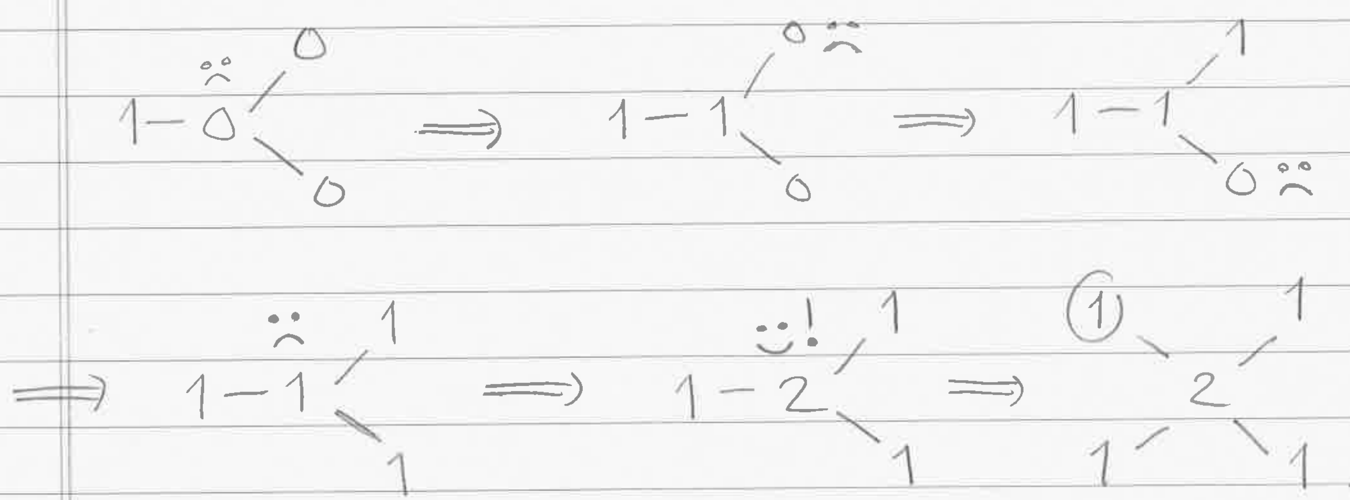
☆ Theorem (Kostant, Gabriel, etc.):

The game will end if and only if G is of finite type ADE, in which case the ending will be the affine diagram $\widetilde{G}$ labeled by its marks. //

Example: $G = D_4 =$  .



Now every vertex is happy.

[Remark: This game can be thought of as moving up in the partial order on positive roots of a root system. The moves are also called "reflection functors" and are related to "Gabriel's theorem" in representation theory.]

☆ Final Observation :

consider integers $p, q, r \geq 1$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

One can check case-by-case that the marks of the graph $\tilde{Y}(p,q,r)$ satisfy

$$\sum_{i=0}^{\ell} n_i^2 = \frac{4}{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1}$$

$$= |\tilde{D}(p,q,r)|$$

where $\tilde{D}(p,q,r)$ is the binary polyhedral group mentioned yesterday.

This is a very mysterious observation that I will attempt to explain tomorrow.

Lecture 3 : The McKay Correspondence.

We have seen two kinds of objects classified by integers $p,q,r \geq 1$ such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

(1) Finite subgroups of $SO(3)$ & $SU(2)$

$$D(p,q,r) \subseteq SO(3)$$

$$\tilde{D}(p,q,r) \subseteq SU(2)$$

(2) Finite & affine Dynkin diagrams of type ADE

$$Y(p,q,r) \ \& \ \tilde{Y}(p,q,r).$$

We have also seen the mysterious observation that

$$\sum_{i=0}^{l} n_i^2 = |\tilde{D}(p,q,r)|$$

where $n_0, n_1, \ldots, n_l$ are the entries of the special eigenvector for $\tilde{Y}(p,q,r)$ with eigenvalue 2.

This was just a mysterious coincidence until in 1979 observed a phenomenon giving a direct connection between the objects (1) and (2).

[Remark: Du Val had given another connection in 1934 via the quotient singularities of the actions $\tilde{D}(p,q,r) \circlearrowright \mathbb{C}^2$ but it was less direct and did not explain all of the numerology.]

Today I will describe McKay's observation.

To do this I need to outline the theory of representations of finite groups.

Let G be a group. A $\mathbb{C}G$-module consists of a $\mathbb{C}$-vector space V and a group homomorphism

$$f : G \longrightarrow GL(V).$$

A subspace $\mathcal{U} \subseteq V$ is called a $\mathbb{C}G$-submodule if for all $u \in \mathcal{U}$ and $g \in G$ we have

$$f_g(u) \in \mathcal{U},$$

in which case we obtain a homomorphism

$$f|_{\mathcal{U}} : G \longrightarrow GL(\mathcal{U}).$$

We say that V is a <u>simple</u> $\mathbb{C}G$-module if it has no $\mathbb{C}G$-submodules except for 0 & V.

The <u>character</u> of the module is defined as the composition of φ with the <u>trace</u>

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[rr, bend right, "\chi_f"'] & GL(V) \arrow[r, "\mathrm{Tr}"] & \mathbb{C} \end{tikzcd}.$$

It follows that $\chi_f : G \longrightarrow \mathbb{C}$ is a <u>class function</u>, i.e., for all $g, h \in G$ we have

$$\chi_f(ghg^{-1}) = \chi_f(h).$$

More generally, we can form the $\mathbb{C}$-vector space of class functions

$$\mathbb{C}[G]^G := \{ \chi : G \to \mathbb{C} \mid \chi(ghg^{-1}) = \chi(h) \ \forall g, h \in G \}$$

If G is finite then this space comes with a natural Hermitian form

$$\langle \chi, \varphi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\varphi(g)} .$$

I'll explain the normalization factor $1/|G|$ in a moment. But first an example.

Let $G = \mathbb{Z}/2\mathbb{Z} = \langle a \rangle$. The space of class functions has an obvious basis

$$e_1 : \langle a \rangle \longrightarrow \mathbb{C} \qquad \& \qquad e_2 : \langle a \rangle \longrightarrow \mathbb{C}$$
$$1 \longmapsto 1 \qquad\qquad\qquad\qquad 1 \longmapsto 0$$
$$a \longmapsto 0 \qquad\qquad\qquad\qquad a \longmapsto 1$$

But it also has a more interesting basis. Since $\mathbb{Z}/2\mathbb{Z}$ is abelian, any irreducible module is 1-dimensional.

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[Essentially this is because commuting diagonalizable matrices are simultaneously diagonalizable.]

So let $\chi : G \longrightarrow GL_1(\mathbb{C}) \xrightarrow[\sim]{Tr} \mathbb{C}$ be the character of a simple module. Since $a^2 = 1$ and since χ is a homomorphism [$Tr : GL_n(\mathbb{C}) \to \mathbb{C}$ is a hom. $\Leftrightarrow n = 1$] we have

$$\chi(a)^2 = \chi(a^2) = \chi(1) = 1,$$

and hence $\chi(a) = +1$ or $\chi(a) = -1$. We obtain two simple characters

$$\chi_1 : \langle a \rangle \longrightarrow \mathbb{C}^\times \qquad \& \qquad \chi_2 : \langle a \rangle \longrightarrow \mathbb{C}^\times.$$
$$1 \longmapsto +1 \qquad\qquad\qquad\qquad 1 \longmapsto +1$$
$$a \longmapsto +1 \qquad\qquad\qquad\qquad 1 \longmapsto -1.$$

Observe that these are orthonormal with respect to the Hermitian form:

$$\langle \chi_1, \chi_1 \rangle = \frac{1}{2}(1 \cdot \bar{1} + 1 \cdot \bar{1}) = 1$$

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{2}(1 \cdot \bar{1} + 1 \cdot \overline{(-1)}) = 0$$

$$\langle \chi_2, \chi_2 \rangle = \frac{1}{2}(1 \cdot \bar{1} + (-1) \cdot \overline{(-1)}) = 1.$$

[This explains the normalization factor $1/|G|$.]

Furthermore, if U & V are $\mathbb{C}G$-modules we can define their direct sum and tensor product

$$U \oplus V \quad \& \quad U \otimes V.$$

The corresponding characters are just the pointwise sum & product

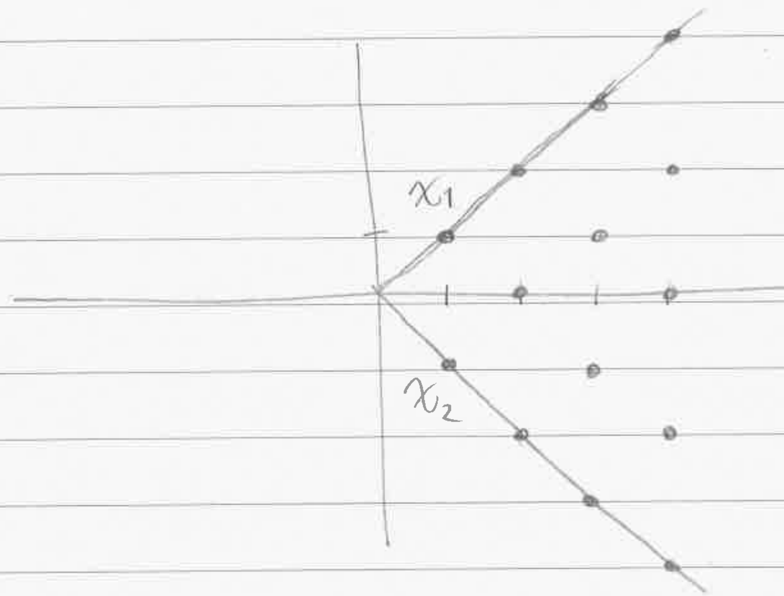
$$\chi_{U \oplus V} = \chi_U + \chi_V$$

$$\chi_{U \otimes V} = \chi_U \cdot \chi_V.$$

If G is finite then [by Maschke's Theorem] it turns out that every finite dimensional $\mathbb{C}G$-module is a direct sum of simples.

Thus the characters form an integer cone in the space of class functions, generated by the simple characters.

Here is the picture for $\mathbb{Z}/2\mathbb{Z}$. Recall that the space of class functions has basis e_1, e_2 and the simple characters are $\chi_1 = e_1 + e_2$ & $\chi_2 = e_1 - e_2$. These generate the cone of all characters:



The general situation is described by the following theorem.

☆ Fundamental Theorem of $\mathbb{C}G$-modules.

A $\mathbb{C}G$-module is determined up to isomorphism by its character. Moreover, if G is <u>finite</u> then the simple characters are an othonormal basis for the space of class functions. //

It follows that

$$\#\text{ simple } \mathbb{C}G\text{-modules} = \#\text{ conjugacy classes of } G$$

If $c(G)$ is the # of conjugacy classes then we can record all information about $\mathbb{C}G$-modules in a $c(G) \times c(G)$ array called the <u>character table</u>.

The rows are indexed by simple characters χ_i and the columns are indexed by conjugacy classes c_j. The (i,j) enty of the table is the complex number

$$\chi_i(c_j).$$

Example: The character table of $\mathbb{Z}/2\mathbb{Z}$.

	1	a
χ_1	1	1
χ_2	1	-1

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Why am I telling you this?

☆ The McKay Correspondence:

Consider integers $p, q, r \geq 1$ such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

Let $\tilde{Y}(p,q,r)$ be the affine Dynkin diagram with $p+q+r-1$ vertices and let $\tilde{D}(p,q,r) \leq SU(2)$ be the binary polyhedral group of size

$$\frac{4}{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1}.$$

Then there exists a bijection between vertices of $\tilde{Y}(p,q,r)$ and simple characters of $\tilde{D}(p,q,r)$ such that

eigenvectors of the adjacency matrix of $\tilde{Y}(p,q,r)$ = columns of the character table of $\tilde{D}(p,q,r)$.

In particular, the column of the identity class consists of the numbers $\chi_i(1) = n_i$ where χ_i are the simple characters of the group and n_i are the marks of the diagram.

Since $\chi_V(1) = \dim V$, we conclude that

☆ the marks of $\tilde{Y}(p,q,r)$ are the dimensions of the simple modules of $\tilde{D}(p,q,r)$.

Now it follows from a standard theorem [decomposition of the regular module] that

$$\sum_{i=0}^{l} n_i^2 = \sum_{i=0}^{l} \chi_i(1)^2 = |\tilde{D}(p,q,r)|.$$

as we observed yesterday. ///

This is a pretty amazing observation. How could it possibly be proved?

Proof Idea:

Let G be a finite group on fix a finite dimensional $\mathbb{C}G$-module U.
Let $V_0, V_1, \dots, V_\ell$ be the simple modules.

We define a directed multigraph on the vertex set

$$I = \{0, 1, 2, \dots, \ell\}$$

by drawing k edges from $i \longrightarrow j$ if k is the multiplicity of V_j in the simple decomposition of the tensor product $V_j \otimes U$.

We will call this the McKay graph

$$McK_U(G).$$

Let A_U be the corresponding adjacency matrix, which is not necessarily symmetric [it will be if U is self-dual].

Let $\chi_i = \chi_{V_i}$ be the simple characters and let $c_0, c_1, \dots, c_l$ be the conjugacy classes of G.

McKay observed for $G \subseteq SU(2)$ and Steinberg observed in general that the vectors

$$(\chi_i(c_j))_i$$

(i.e. the columns of the character table of G) are the eigenvectors of the matrix A_U, with eigenvalues $\chi_U(c_j)$.

To complete the proof let $G = \tilde{D}(p,q,r)$ and let U be the embedding of $\tilde{D}(p,q,r)$ into $SU(2)$.

Then the McKay correspondence says that

$$McK_U(\tilde{D}(p,q,r)) = \tilde{Y}(p,q,r).$$

McKay (1977) proved this case-by-case. Steinberg (1985) gave a uniform proof.

QED.

How about an example?

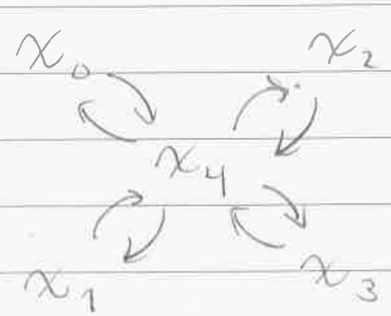
Consider the group $\widetilde{D}(2,2,2) = Q_8$ of size 8 consisting of

$$\pm 1, \pm i, \pm j, \pm k.$$

It has 5 simple modules and 5 conjugacy classes with character table:

	1	-1	$\pm i$	$\pm j$	$\pm k$
χ_0	1	1	1	1	1
χ_1	1	1	-1	1	-1
χ_2	1	1	-1	-1	1
χ_3	1	1	1	-1	-1
χ_4	2	-2	0	0	0

The McKay graph is $\widetilde{Y}(2,2,2) = \widetilde{D}_4$:



$$\begin{array}{c c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

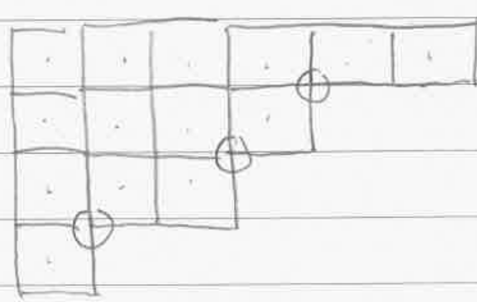
If T is the character table and A is the adjacency matrix then we have

$$T^{-1} A T = \begin{pmatrix} 2 & & & & \\ & -2 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

But Steinberg's construction is much more general than this.

Epilogue: The fun way to compute the character table of the symmetric group S_n.

Recall that the conjugacy classes of S_n are indexed by Young diagrams with n boxes, for example

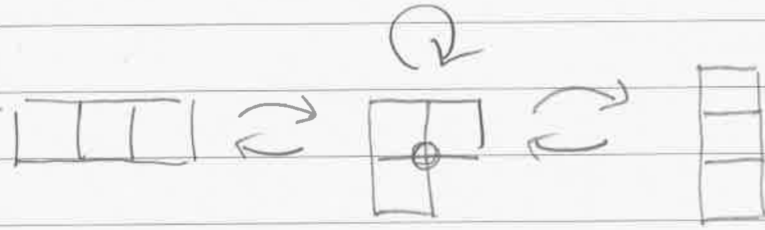


has 14 boxes, and 3 "internal corners".

Form a graph on the set $YD(n)$ of Young Diagrams with n boxes by drawing

- an edge $\lambda \longrightarrow \mu$ if we can get from λ to μ by moving one box.
- a loop $\lambda \circlearrowleft$ for each "internal corner" of λ.

Example: $YD(3)$



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I claim that $YD(n)$ is the McKay graph for S_n with respect to the standard "reflection representation".

It follows that the eigenvectors of the adjacency matrix are the columns of the character table.

$\downarrow$

The adjacency matrix of $YD(3)$ and character table of S_3 are given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \& \quad T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Note that we have

$$T^{-1} A T = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

where the diagonal entries are the character of the reflection representation. ///

Problem: In general the graph $YD(n)$ has eigenvalues of multiplicity ≥ 2 so it is not sufficient to compute the character table. Is there a nice way to compute the characters using the graph $McK_U(S_n)$ for various U?