Rational Catalan Combinatorics (Type A)

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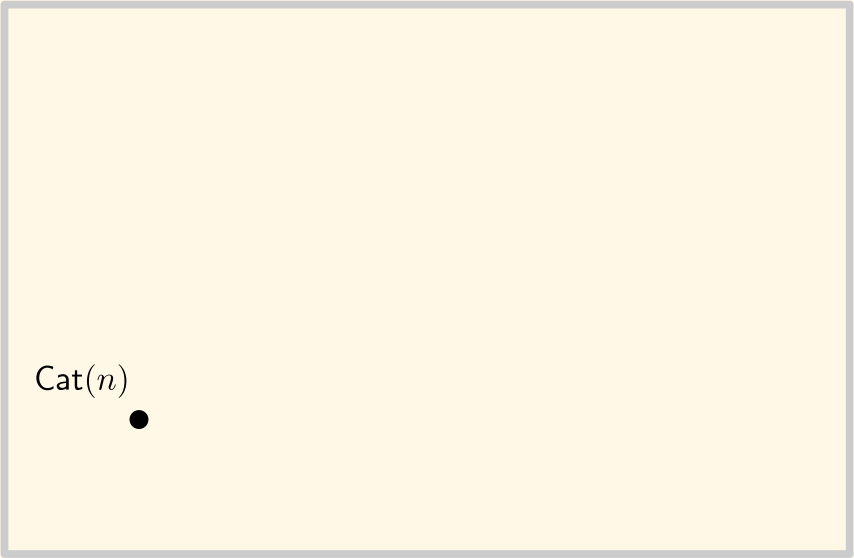
AIM Workshop
December 17–21, 2012

Catalan Combinatorics?



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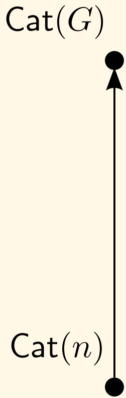

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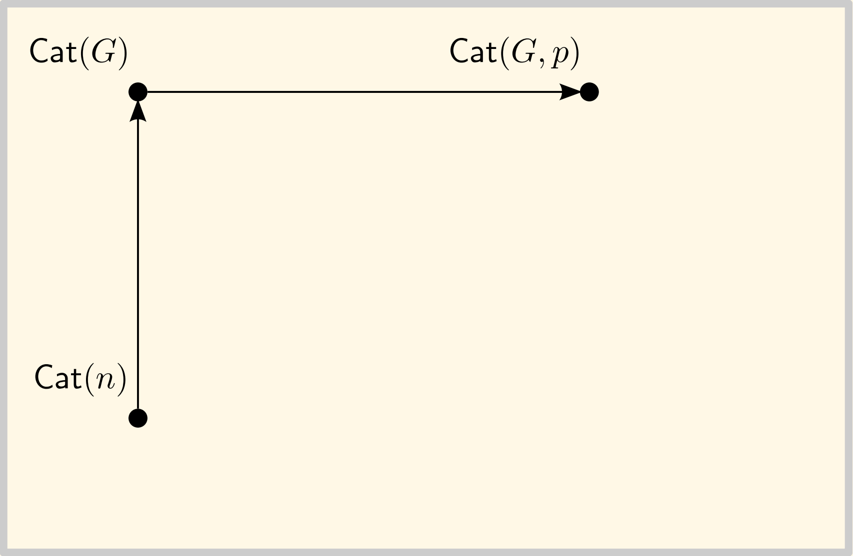

Catalan Combinatorics?

$$\begin{tikzcd}
\mathsf{Cat}(G) \arrow[r] & \mathsf{Cat}(G,p) \\
\mathsf{Cat}(n) \arrow[u] \arrow[r] & \mathsf{Cat}(x) \arrow[u]
\end{tikzcd}$$

Catalan Combinatorics?

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\begin{tikzcd}
\mathsf{Cat}(G) \arrow[rr] \arrow[dr] & & \mathsf{Cat}(G,p) \arrow[dr] & \\
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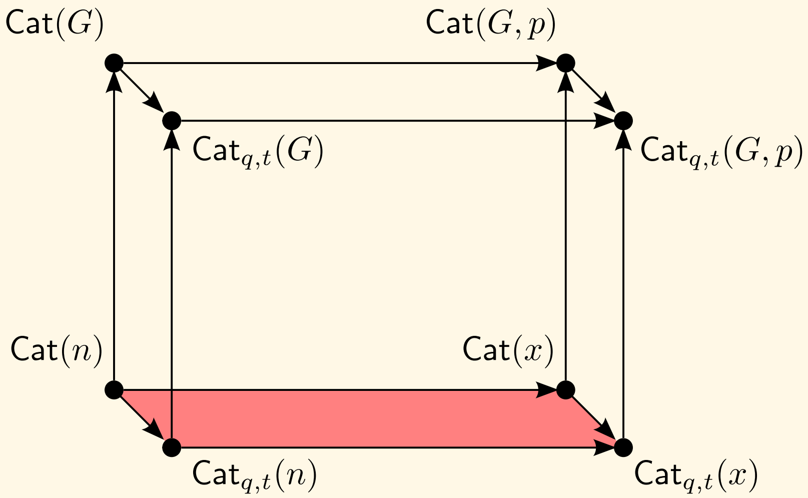
$$

Catalan Combinatorics? This talk is the red stuff.



Introduction to Everything Rational

Plan for the Talk

- Catalan Numbers
- Dyck Paths
- Noncrossing Partitions
- Associahedra
- Core Partitions
- Parking Functions
- Parking Spaces (q and t)
- Connect to Lie Theory

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Rational Catalan Numbers

CONVENTION

Given $x \in \mathbb{Q} \setminus [-1, 0]$, there exist **unique coprime** $(a, b) \in \mathbb{N}^2$ such that

$$x = \frac{a}{b-a}.$$

We will always identify $x \leftrightarrow (a, b)$.

Definition

For each $x \in \mathbb{Q} \setminus [-1, 0]$ we define the **Catalan number**:

$$\mathrm{Cat}(x) = \mathrm{Cat}(a, b) := \frac{1}{a+b}\binom{a+b}{a, b} = \frac{(a+b-1)!}{a!\,b!}.$$

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Special cases

When $b = 1 \bmod a \ldots$

- *Eugène Charles Catalan (1814-1894)*

  $(a, b) = (n, n+1)$ gives the **good old Catalan number**:

$$\mathrm{Cat}(n) = \mathrm{Cat}\left(\frac{n}{(n+1)-n}\right) = \frac{1}{2n+1}\binom{2n+1}{n}.$$

- *Nicolaus Fuss (1755-1826)*

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By definition we have $\mathrm{Cat}(a, b) = \mathrm{Cat}(b, a)$, which translates to

$$\mathrm{Cat}(x) = \mathrm{Cat}(-x - 1)$$

(i.e. symmetry about $x = -1/2$), which implies that

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$$\mathrm{Cat}'(1/x) = \mathrm{Cat}\left(\frac{1}{(1/x)-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right) = \mathrm{Cat}'(x).$$

We call this **rational duality**:

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Euclidean Algorithm

Observation

The process $\mathrm{Cat}(x) \mapsto \mathrm{Cat}'(x) \mapsto \mathrm{Cat}''(x) \mapsto \cdots$ is a categorification of the Euclidean algorithm.

Example: $x = 5/3$ and $(a, b) = (5, 8)$

Subtract the smaller from the larger:

$$\begin{aligned}
\mathrm{Cat}(5,8) &= 99,\\
\mathrm{Cat}'(5,8) &= \mathrm{Cat}(3,5) = 7,\\
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How to put it in Sloane's OEIS

Suggestion

The **Calkin-Wilf sequence** is defined by $q_1 = 1$ and

$$q_n := \frac{1}{2\lfloor q_{n-1} \rfloor - q_{n-1} + 1}.$$

Theorem: $(q_1, q_2, \ldots) = \mathbb{Q}_{>0}$.
Proof: See "Proofs from THE BOOK", Chapter 17.

Study the function $n \mapsto \mathrm{Cat}(q_n)$.

q	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{3}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{3}{1}$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{5}$	$\frac{5}{2}$	$\frac{2}{5}$	$\frac{5}{3}$	$\cdots$
$\mathrm{Cat}(q)$	1	1	2	1	7	3	5	1	30	15	66	4	99	$\cdots$

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Pause

Well, that was fun.

The Prototype: Rational Dyck Paths

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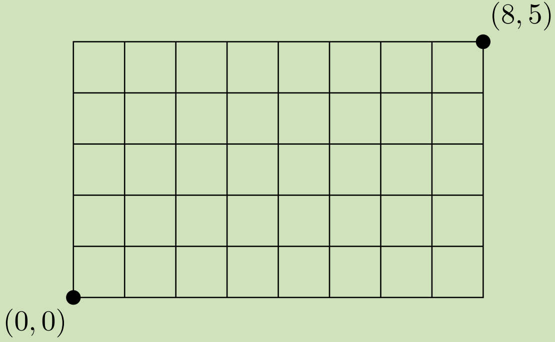
- Consider the "Dyck paths" in an $a \times b$ rectangle.

Example $(a, b) = (5, 8)$



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- Again let $x = a/(b-a)$ with a, b positive and coprime.

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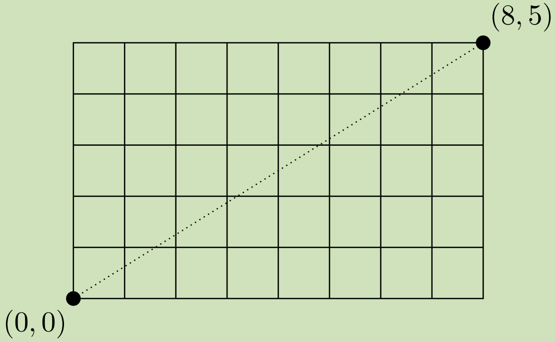

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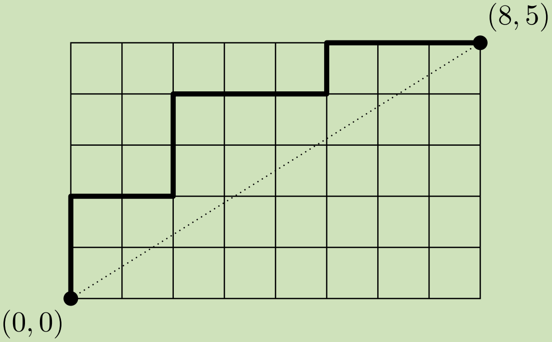
- Let $\mathcal{D}(x) = \mathcal{D}(a, b)$ denote the set of Dyck paths.

Example $(a, b) = (5, 8)$



The Prototype: Rational Dyck Paths

Theorem (Grossman 1950, Bizley 1954)

The number of Dyck paths is the Catalan number:

$$|\mathcal{D}(x)| = \mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a,b}.$$

- *Claimed by Grossman (1950), "Fun with lattice points, part 22".*
- *Proved by Bizley (1954), in Journal of the Institute of Actuaries.*
- *Proof: Break $\binom{a+b}{a,b}$ lattice paths into cyclic orbits of size $a+b$. Each orbit contains a unique Dyck path.*

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The Prototype: Rational Dyck Paths

Theorem (Armstrong 2010, Loehr 2010)

- *The number of Dyck paths with k vertical runs equals*

$$\mathrm{Nar}(x;k) := \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}.$$

Call these the **Narayana numbers**.

- *And the number with r_j vertical runs of length j equals*

$$\mathrm{Krew}(x;\mathbf{r}) := \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} = \frac{(b-1)!}{r_0!r_1!\cdots r_a!}.$$

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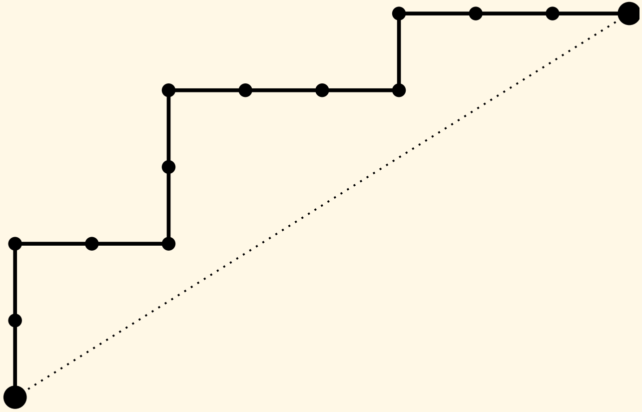
Next: Rational NC Partitions

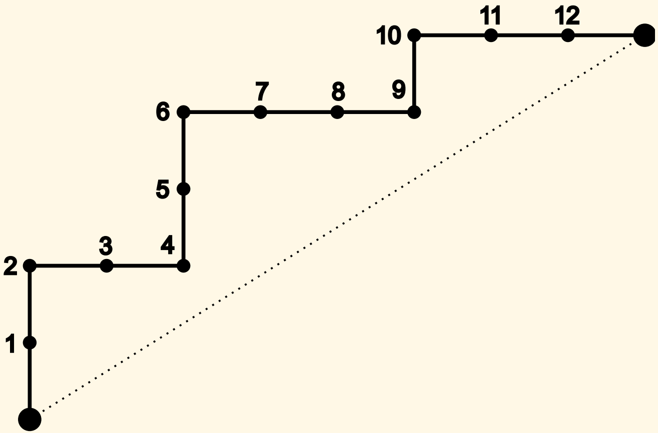
To **create** a noncrossing partition...

- Start with a Dyck path. Here $(a, b) = (5, 8)$.



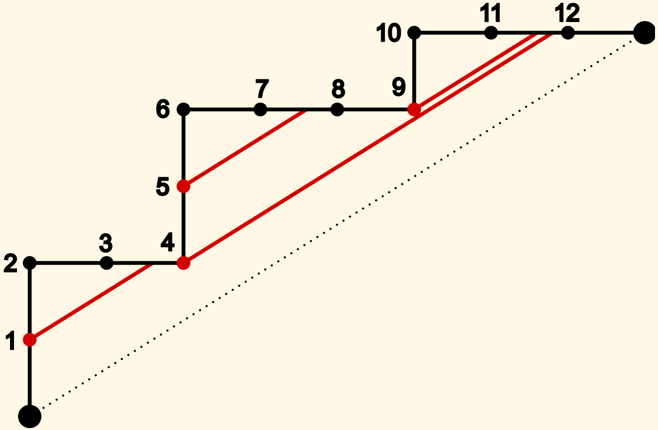
To **create** a noncrossing partition...

- Label the internal vertices by $\{1, 2, \ldots, a + b - 1\}$.



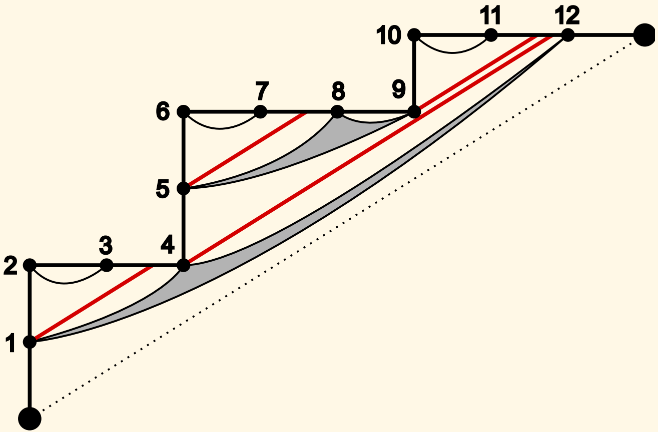
To **create** a noncrossing partition...

- Shoot lasers from the bottom left with slope a/b.



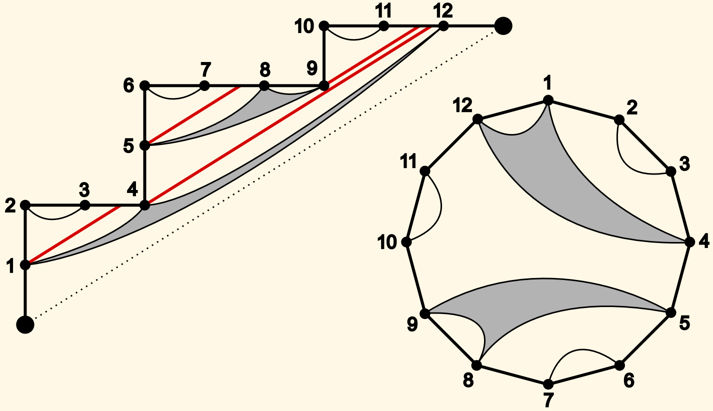
To **create** a noncrossing partition. . .

- Who can see each other?



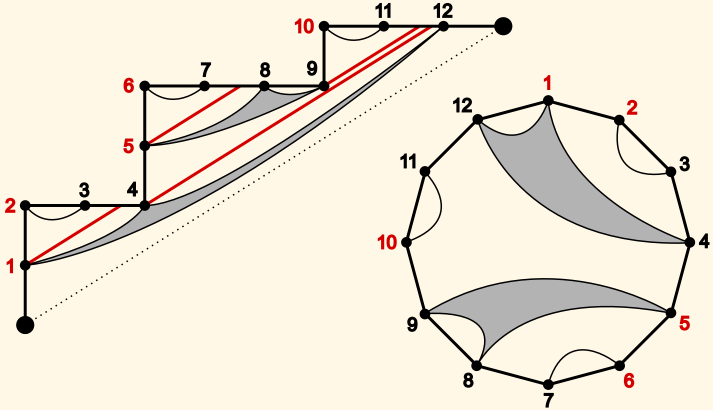
To **create** a noncrossing partition. . .

- There you go!



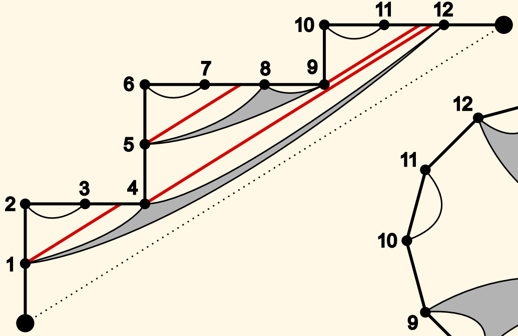
To **create** a noncrossing partition...

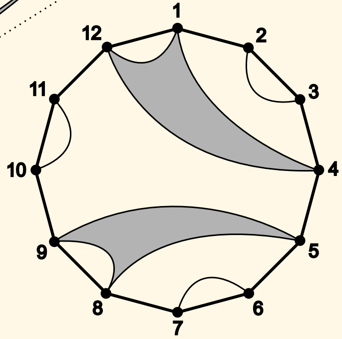
- We have created $\mathrm{Cat}(x) = \frac{1}{a}\binom{a+b}{a,b}$ different noncrossing partitions of the cycle $[a+b-1]$, and each of them has a blocks.



To **rotate** a noncrossing partition. . .

- Q: What does "rotation" of the partition correspond to?





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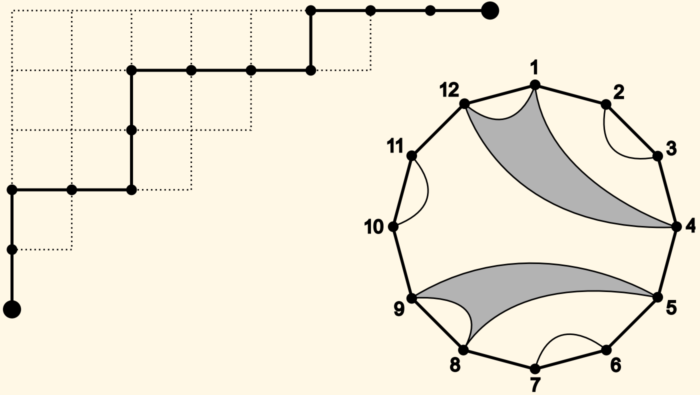
- A: Think of the path as a maximal chain in a poset.



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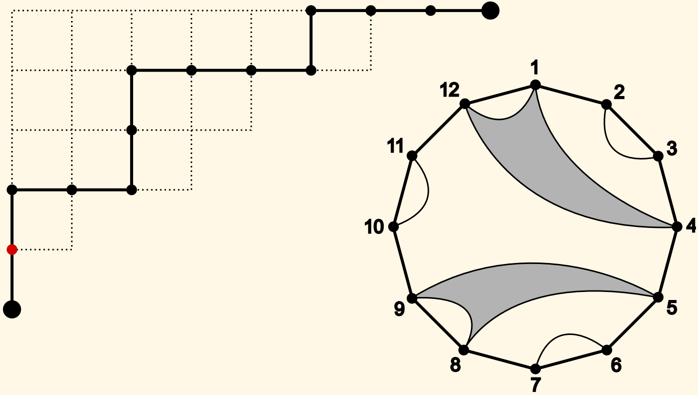

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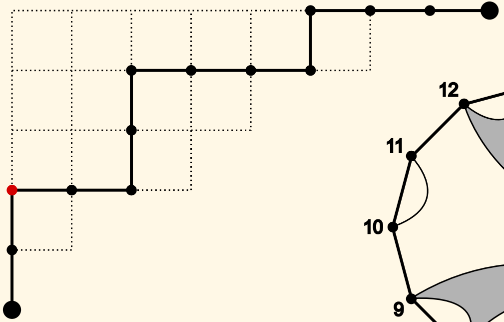
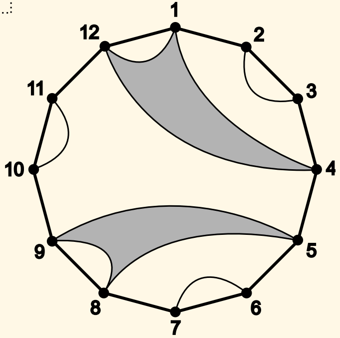

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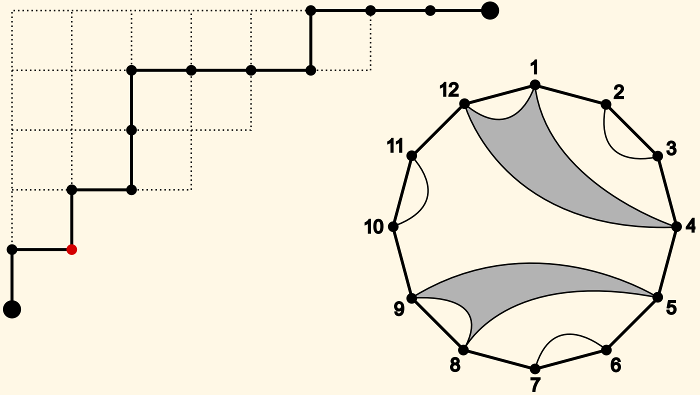

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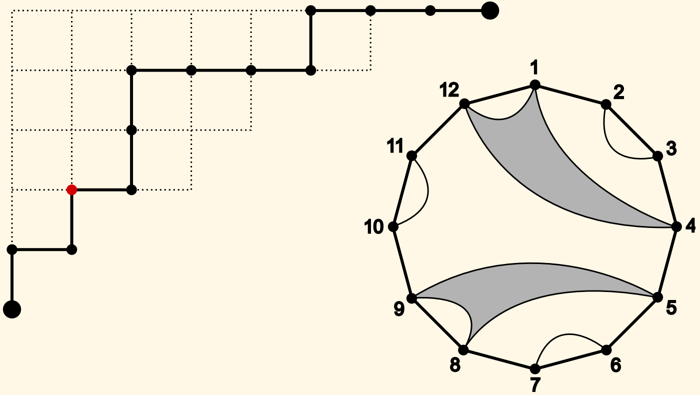

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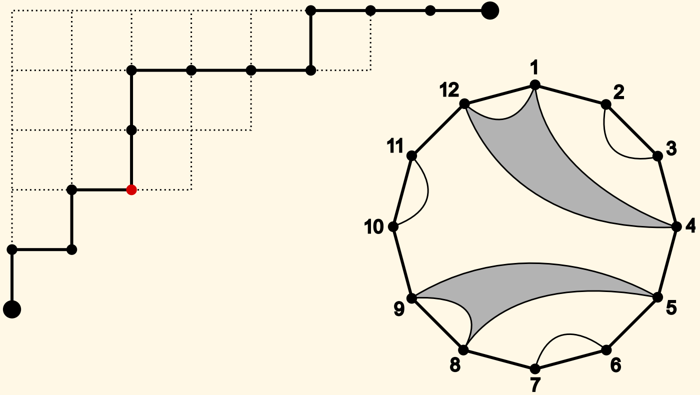

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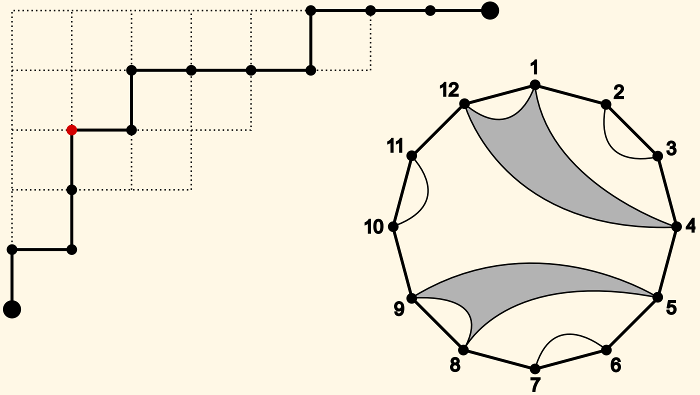

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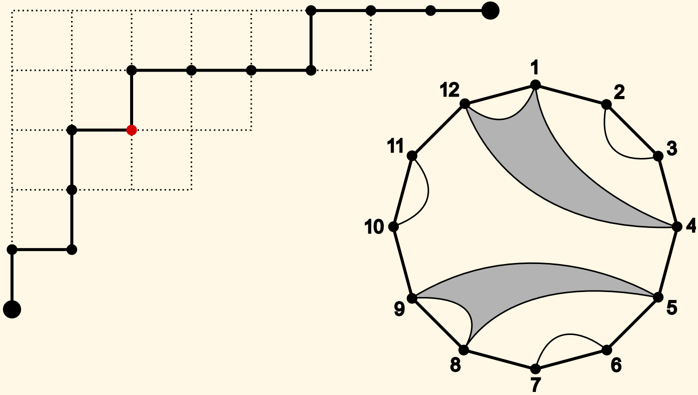

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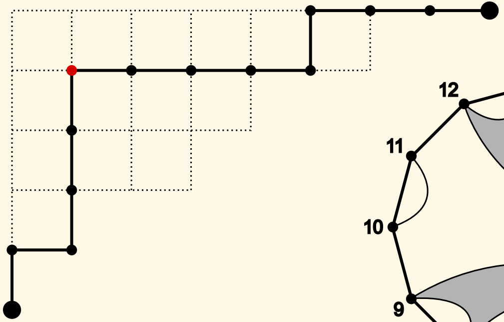

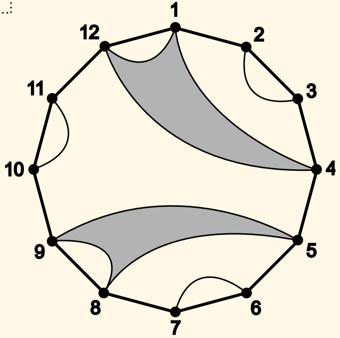

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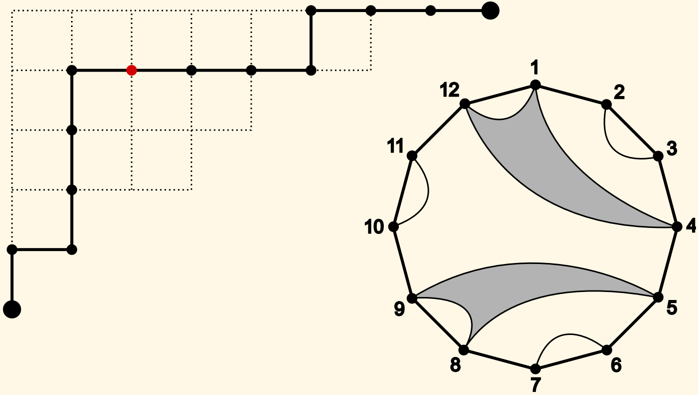

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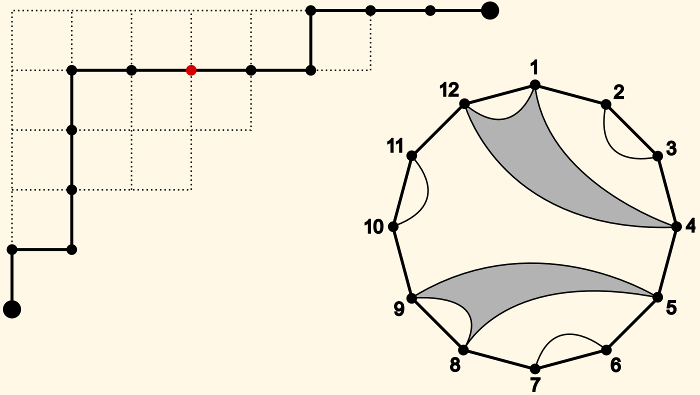

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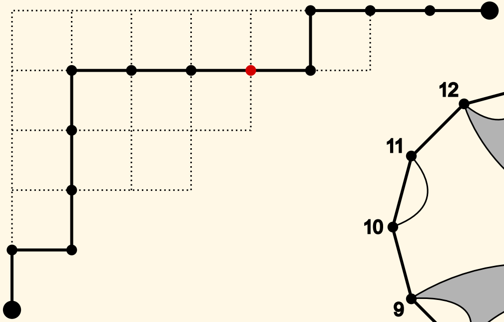

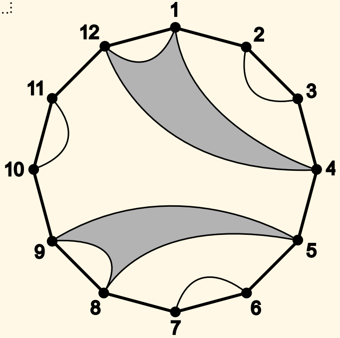

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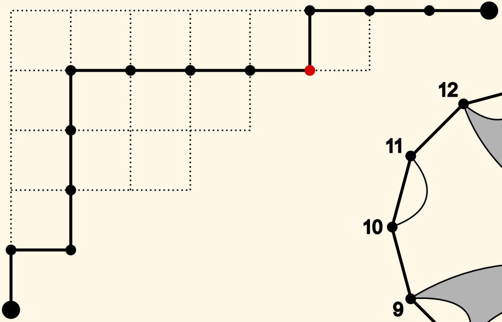

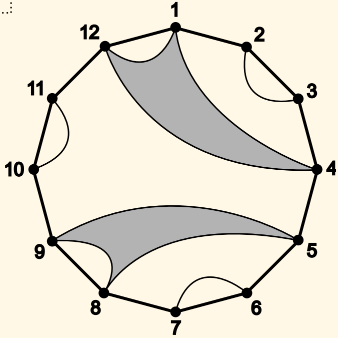

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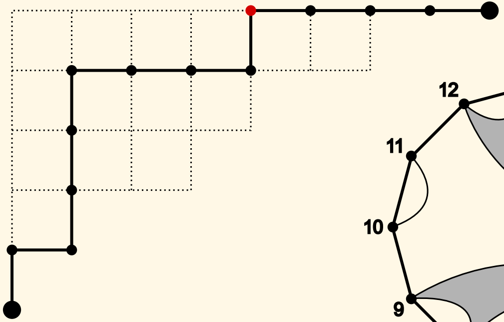

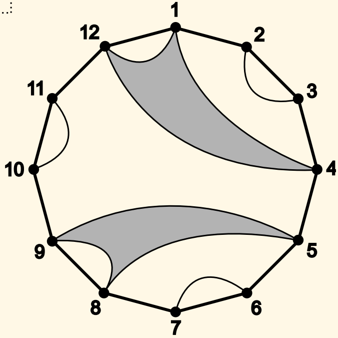

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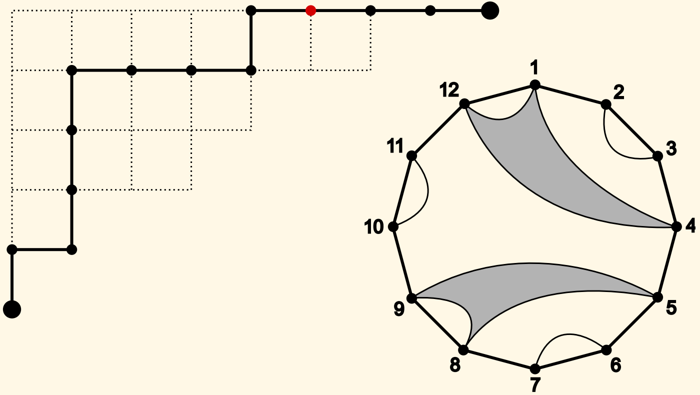

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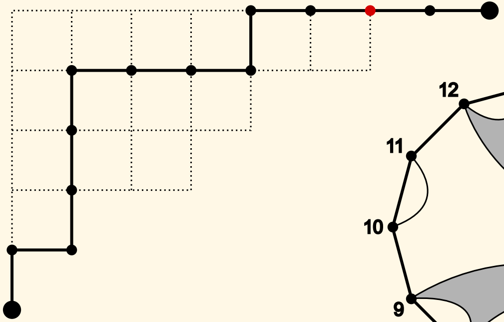

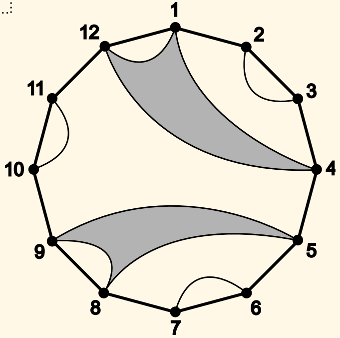

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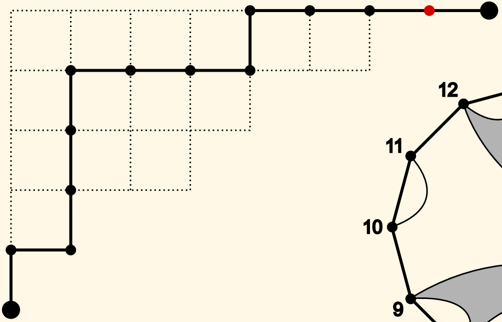

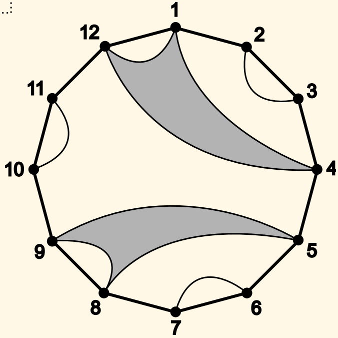

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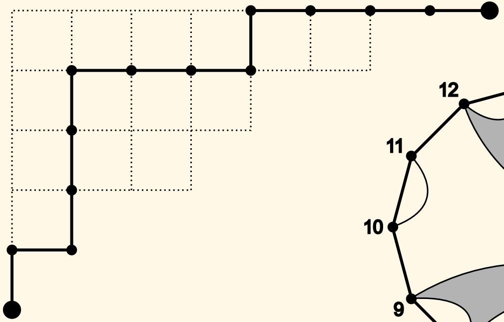
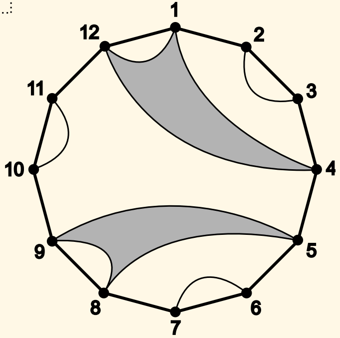
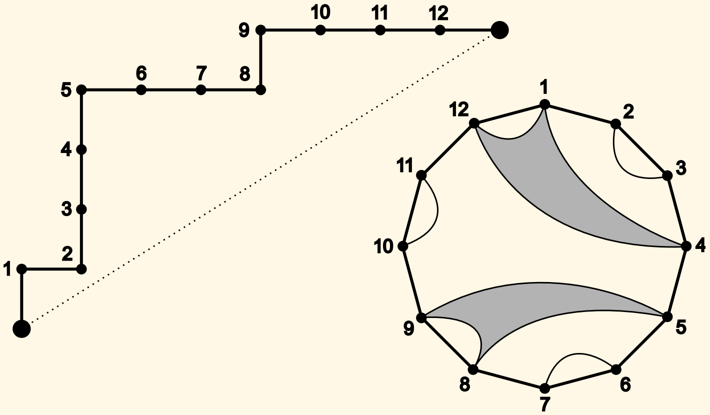




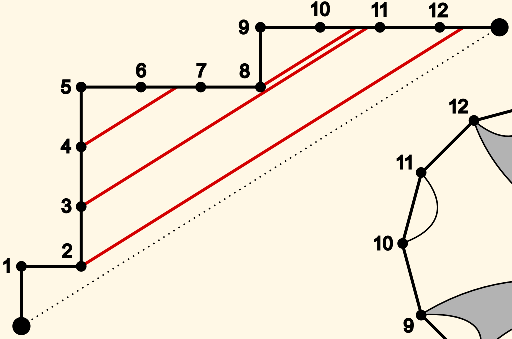
To **rotate** a noncrossing partition...

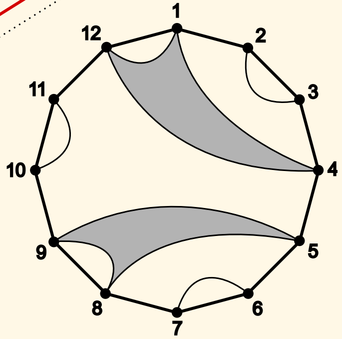
- Think of it as a path again.



To **rotate** a noncrossing partition...

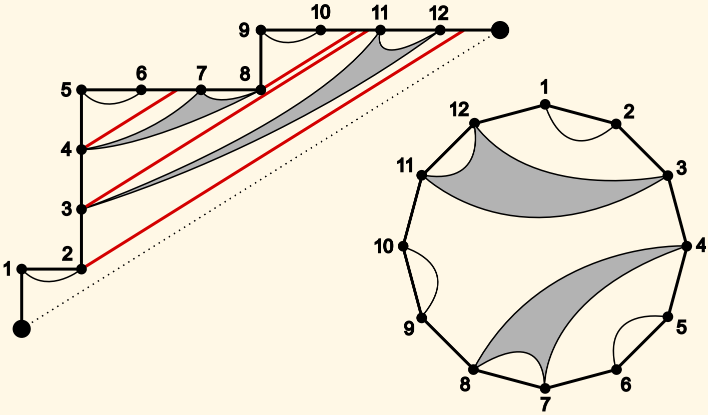
- Again the lasers.





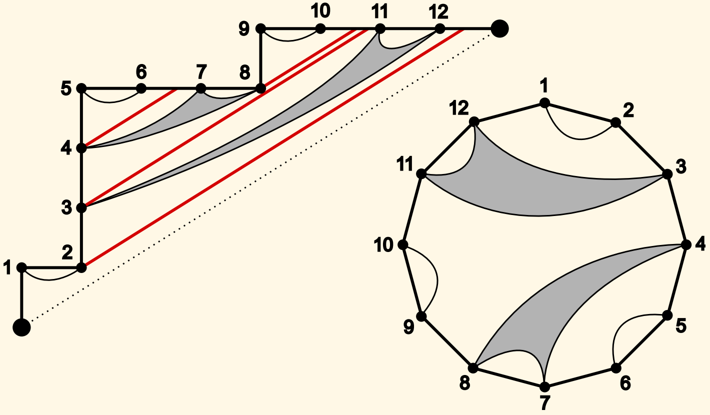
To **rotate** a noncrossing partition. . .

- And there you go!



To **rotate** a noncrossing partition. . .

- *Drew: mention the case* $(a, b) = (n, (k-1)n + 1)$.



A Few Facts

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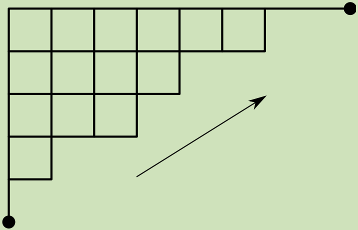
Definition

For (a, b) coprime, consider the **triangle poset**

$$\mathcal{T}(a,b) := \{(x,y) \in \mathbb{Z}^2 : y \leq a,\ x \leq b,\ yb - xa \geq 0\}.$$

As you see here.



A Few Facts

Results (with Nathan Williams)

- Promotion on $\mathcal{T}(a, b)$ has order $a + b - 1$.
- Conjecture: The number of chains invariant under promotion^d is the q-**Catalan number** evaluated at a root of unity:

$$\frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q \Bigg|_{q=e^{\frac{2\pi i d}{a+b-1}}}$$

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- $\mathcal{T}(n, n+1)$ is related to the type A root poset.
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Next: Rational NC Partition Posets

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Our rational NC partitions don't form a nice poset. Indeed, they each have the same number of blocks! (i.e., a)

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Can one define a **nice poset** of rational NC partitions?

Answer

Yes.

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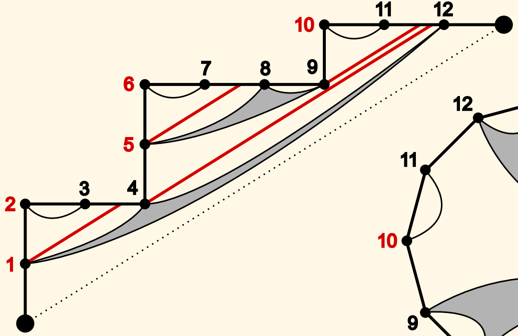
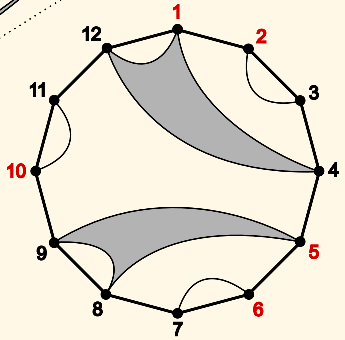
To **de-homogenize** a noncrossing partition. . .

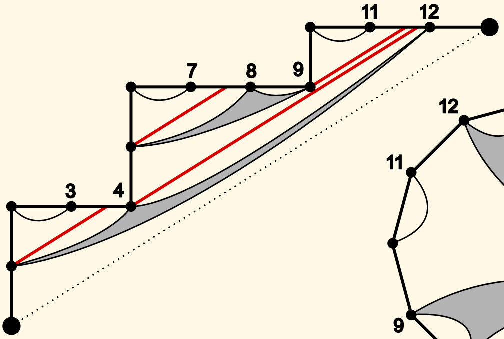
- Recall this.

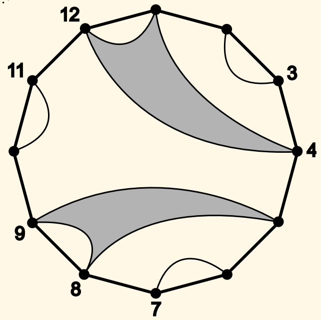




To **de-homogenize** a noncrossing partition. . .

- Now we label only the horizontal steps.





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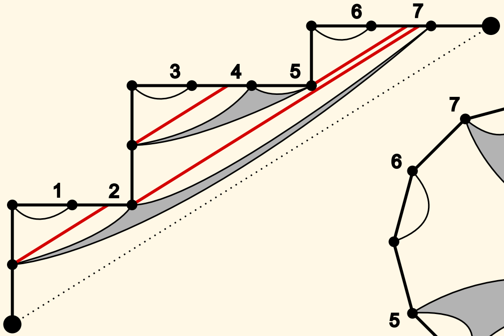
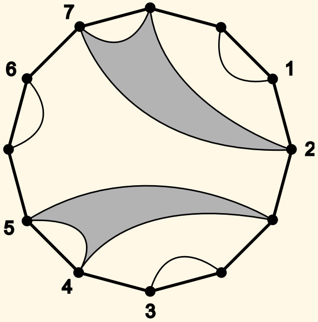
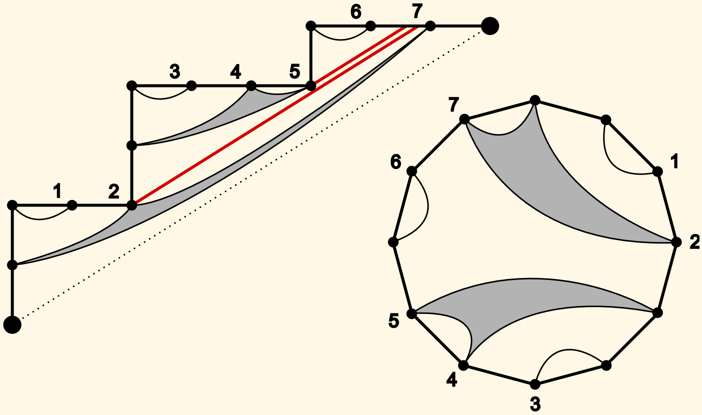




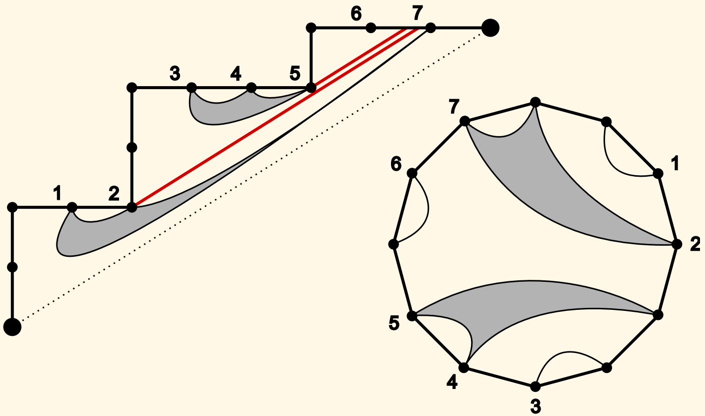
To **de-homogenize** a noncrossing partition. . .

- Now we shoot lasers only from the corners.



To **de-homogenize** a noncrossing partition...

- Now who can see each other?



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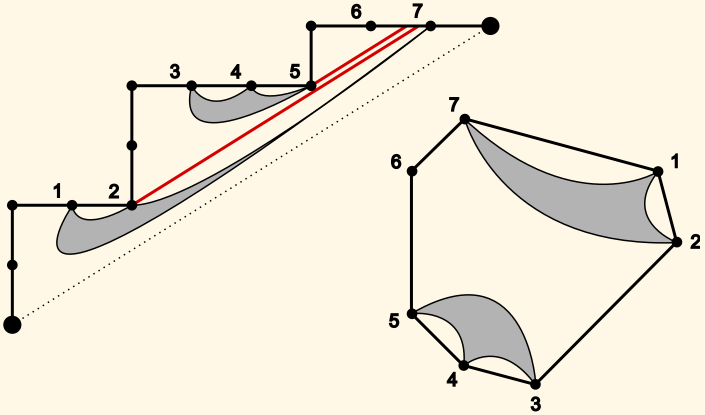

A Few Facts

Definition

Let $\mathrm{NC}(x) = \mathrm{NC}(a, b)$ be the poset of non-homogeneous NC partitions.

Facts (with Nathan Williams)

- $\mathrm{NC}(n, n+1) = \mathrm{NC}(n)$ is the **good old noncrossing partitions**.
- $\mathrm{NC}(n, (k-1)n+1)$ is the k-**divisible noncrossing partitions**.
- $\mathrm{NC}(a, b)$ is a (graded) order filter in $\mathrm{NC}(b-1)$.
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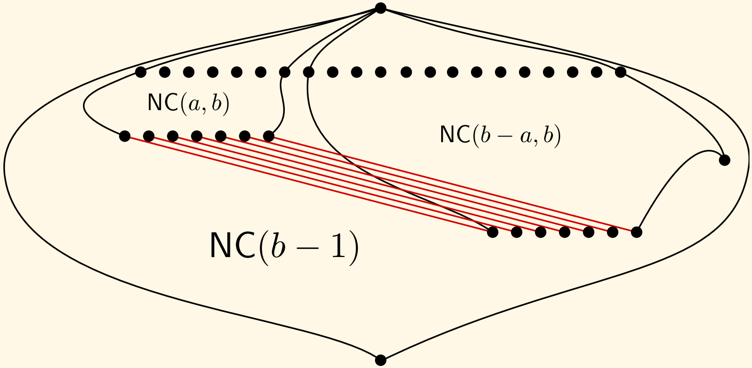
Let $\mathrm{NC}(x) = \mathrm{NC}(a, b)$ be the poset of non-homogeneous NC partitions.

Facts (with Nathan Williams)

- $\mathrm{NC}(n, n+1) = \mathrm{NC}(n)$ is the **good old noncrossing partitions**.
- $\mathrm{NC}(n, (k-1)n+1)$ is the k-**divisible noncrossing partitions**.
- $\mathrm{NC}(a, b)$ is a (graded) order filter in $\mathrm{NC}(b-1)$.
- $\mathrm{NC}(a, b)$ is ranked by the Narayana numbers $\frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}$.
- $\mathrm{NC}(x)$ has $\mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a,b}$ elements.
- $\mathrm{NC}(x)$ has $\mathrm{Cat}'(x) = \frac{1}{b}\binom{b}{a}$ elements of minimum rank.

Rational Duality

- Note that $x \leftrightarrow 1/x$ is the same as $(a < b) \leftrightarrow (b - a < b)$.



Next: Rational Associahedra

Observation

The good old associahedron is a nice polytope with h-vector given by the good old Narayana numbers.

Question

Can one define a rational associahedron with h-vector given by

$$\mathrm{Nar}(x;k) = \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}?$$

Answer

Yes. (But it's not a polytope.)

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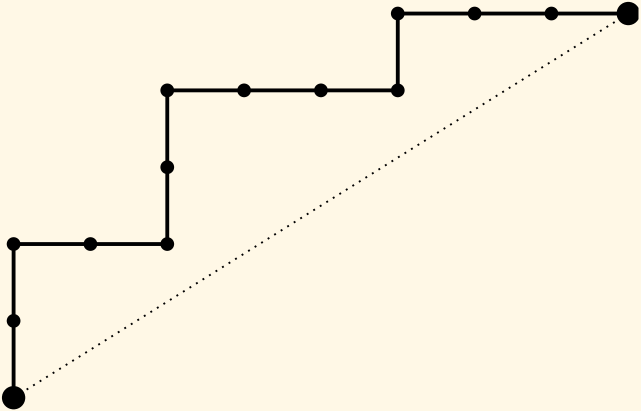
Answer

Yes. (But it's not a polytope.)

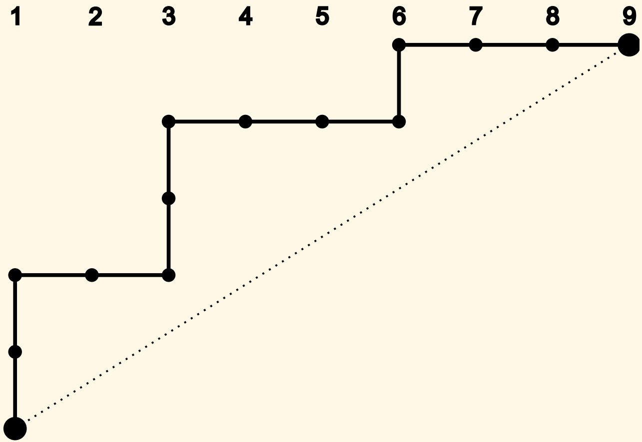
To **create** a polygon dissection...

- Start with a Dyck path. Here $(a, b) = (5, 8)$.



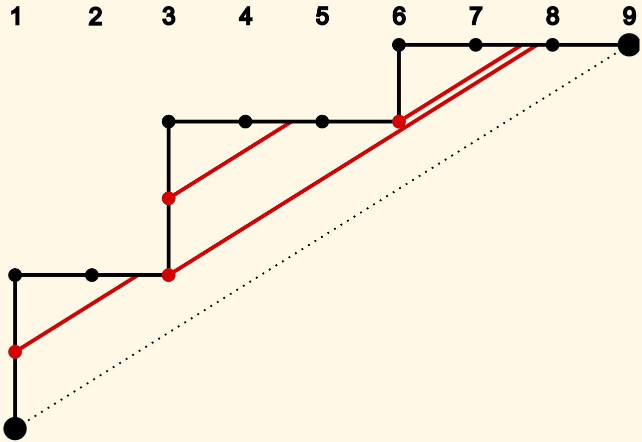
To **create** a polygon dissection...

- Label the columns by $\{1, 2, \ldots, b+1\}$.



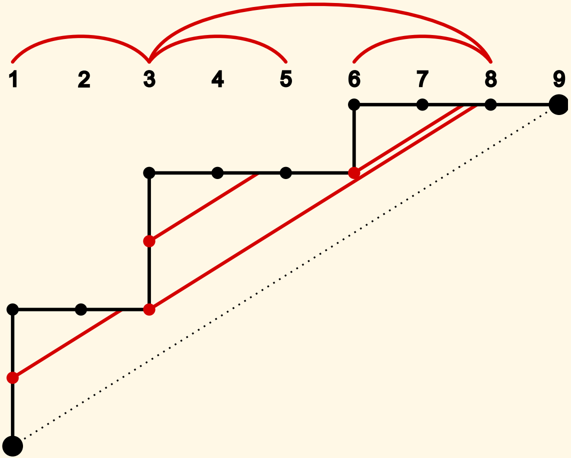
To **create** a polygon dissection. . .

- Shoot some lasers from the bottom left with slope a/b.



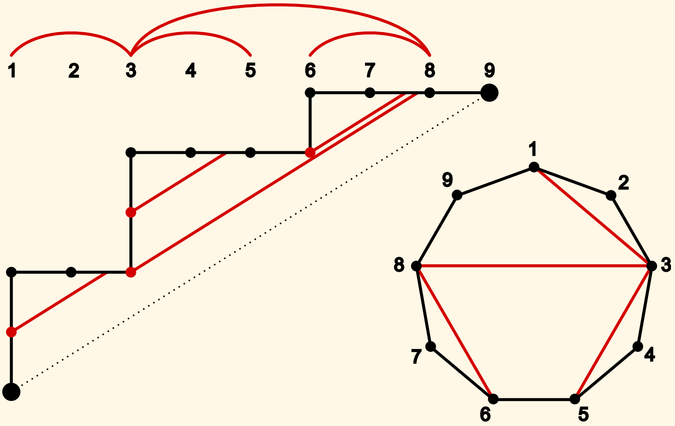
To **create** a polygon dissection. . .

- Lift the lasers up.



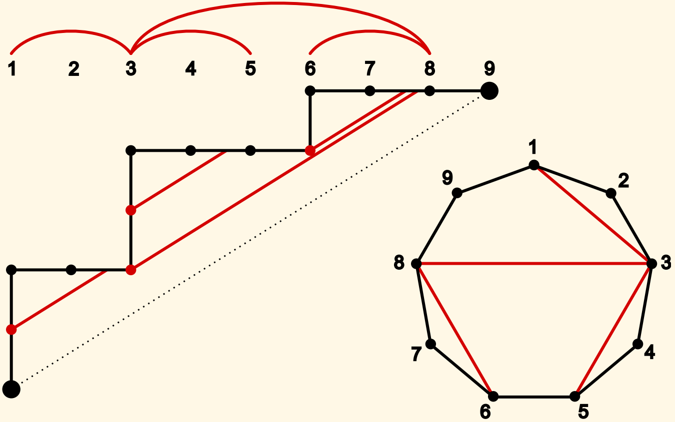
To **create** a polygon dissection. . .

- There you go!



To **create** a polygon dissection...

- We have created $\mathrm{Cat}(x) = \frac{1}{a}\binom{a+b}{a,b}$ different "rational dissections" of the cycle $[b+1]$, and each of them has a diagonals.



A Few Facts

Definition

Let $\mathrm{Ass}(x) = \mathrm{Ass}(a, b)$ be the simplicial complex with the desired facets.

Facts (with B. Rhoades and N. Williams)

- $\mathrm{Ass}(n, n+1) = \mathrm{Ass}(n)$ is the **good old associahedron**.
- $\mathrm{Ass}(n, (k-1)n+1)$ is the **generalized cluster complex** of Athanasiadis-Tzanaki and Fomin-Reading.
- $\mathrm{Ass}(x)$ has $\mathrm{Cat}(x)$ facets, and Euler characteristic $\mathrm{Cat}'(x)$.
- $\mathrm{Ass}(x)$ is shellable with h-vector $\mathrm{Nar}(x; k) = \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}$.
- Hence its f-vector is given by the **Kirkman numbers**:

$$\mathrm{Kirk}(x; k) := \frac{1}{a}\binom{a}{k}\binom{b+k-1}{k-1}.$$

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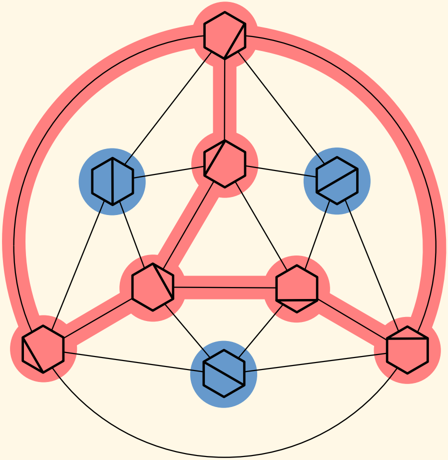
Rational Duality = Alexander Duality

- E.g. Ass(2/3) and Ass(3/2) are Alexander dual inside Ass(4).



Motivation: Core Partitions

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Definition

Let $\lambda \vdash n$ be an integer partition of "size" n.

- Say λ is a p-core if it has **no cell with hook length** p.
- Say λ is an (a, b)-core if it has **no cell with hook length** a **or** b.

Example

The partition $(5, 4, 2, 1, 1) \vdash 13$ is a $(5, 8)$-core.

9	6	4	3	1
7	4	2	1	
4	1			
2				
1				

Motivation: Core Partitions

Theorem (Anderson 2002)

The number of (a, b)-cores (of any size) is finite if and only if (a, b) are coprime, in which case they are counted by the Catalan number

$$\mathrm{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a, b}.$$

Theorem (Olsson-Stanton 2005, Vandehey 2008)

For (a, b) coprime $\exists$ unique largest (a, b)-core of size $\frac{(a^2-1)(b^2-1)}{24}$, which contains all others as subdiagrams.

Suggestion

Study Young's lattice restricted to (a, b)-cores.

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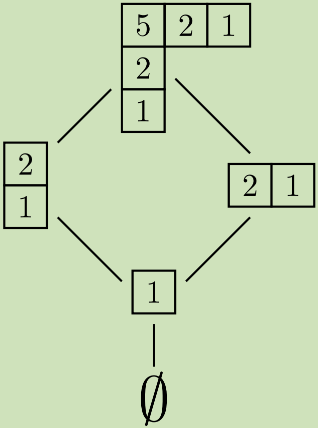
Example: The poset of $(3, 4)$-cores.



Motivation: Core Partitions

Theorem (Ford-Mai-Sze 2009)

For a, b coprime, the number of self-conjugate (a,b)-cores is $\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor}$.

Note: Beautiful bijective proof! (omitted)

Observation/Problem

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor} = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q \Bigg|_{q=-1}$$

Conjecture (Armstrong 2011)

The average size of an (a,b)-core and the average size of a self-conjugate (a,b)-core are **both equal** *to* $\frac{(a+b+1)(a-1)(b-1)}{24}$.

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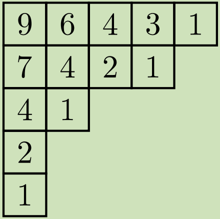
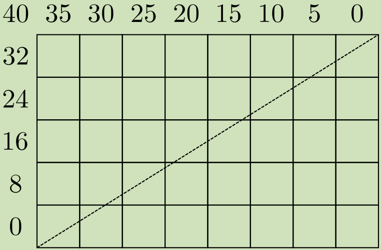
Anderson's Beautiful Proof

Proof.

Bijection: (a, b)-cores $\leftrightarrow$ Dyck paths in $a \times b$ rectangle $\square$

Example (The $(5, 8)$-core from earlier.)





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Example (Label the rectangle cells by "height".)

9	6	4	3	1
7	4	2	1	
4	1			
2				
1				

40	35	30	25	20	15	10	5	0
32	27	22	17	12	7	2		
24	19	14	9	4				
16	11	6	1					
8	3							
0								

Anderson's Beautiful Proof

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Bijection: (a, b)-cores $\leftrightarrow$ Dyck paths in $a \times b$ rectangle $\square$

Example (Label the first column hook lengths.)

9	6	4	3	1
7	4	2	1	
4	1			
2				
1				

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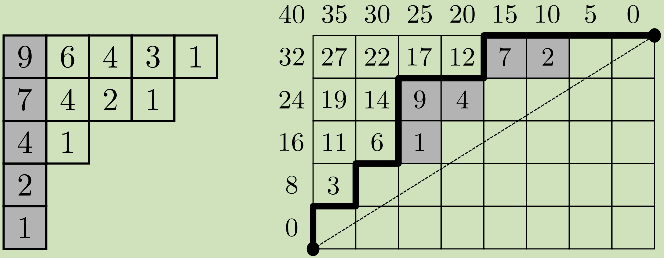
Example (Voila!)



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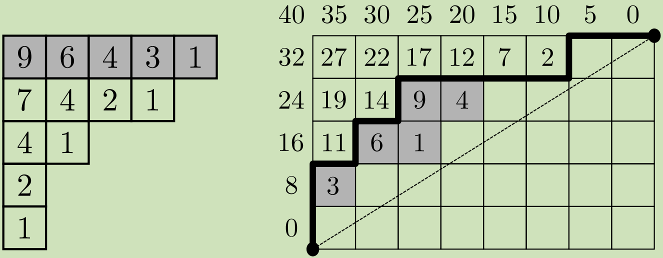
Example (Observe: Conjugation is a bit strange.)

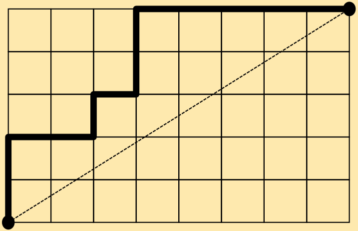


Next: Rational Parking Functions/Spaces

The Rational Parking Space

Definition

- Label the up-steps by $\{1, 2, \ldots, a\}$, increasing up columns.



- Call this a **parking function**.
- Let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote the set of parking functions.
- **Classical form** $(z_1, z_2, \ldots, z_a)$ has label z_i in column i.
- Example: $(3, 1, 4, 4, 1)$

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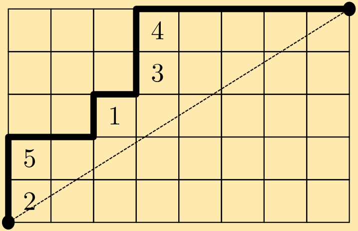

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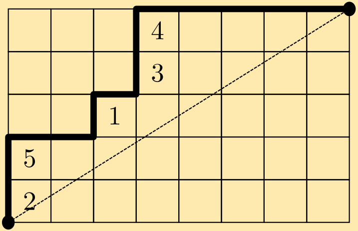

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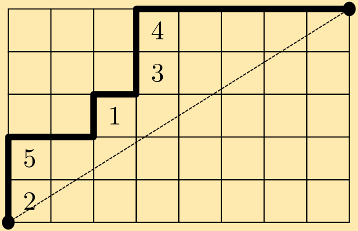

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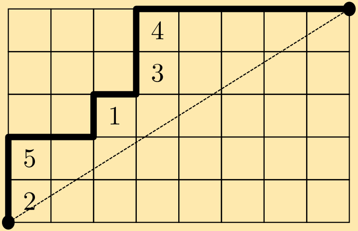

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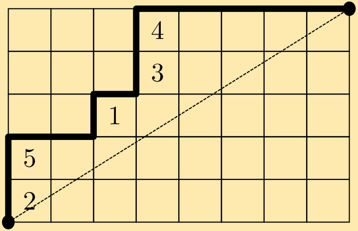
The Rational Parking Space

Definition

- The symmetric group $\mathfrak{S}_a$ acts on classical forms.



- Example: $(3, 1, 4, 4, 1)$ versus $(3, 1, 1, 4, 4)$
- By abuse, let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote this representation of $\mathfrak{S}_a$.
- Call it the **rational parking space**.

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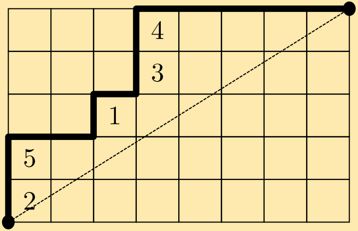

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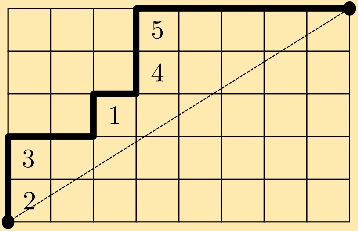

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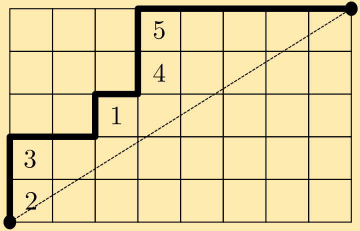
- The symmetric group $\mathfrak{S}_a$ acts on classical forms.



- Example: $(3, 1, {\color{red}4}, 4, {\color{red}1})$ versus $(3, 1, {\color{red}1}, 4, {\color{red}4})$
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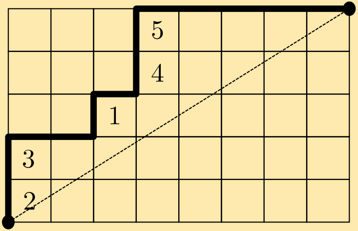

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A Few Facts

Theorems (with N. Loehr and N. Williams)

- The dimension of $\mathrm{PF}(a,b)$ is b^{a-1}.
- The **complete homogeneous expansion** is

$$\mathrm{PF}(a,b) = \sum_{\mathbf{r} \vdash a} \frac{1}{b} \binom{b}{r_0, r_1, \ldots, r_a} h_{\mathbf{r}},$$

  where the sum is over $\mathbf{r} = 0^{r_0} 1^{r_1} \cdots a^{r_a} \vdash a$ with $\sum_i r_i = b$.
- That is: $\mathrm{PF}(a,b)$ is the coefficient of t^a in $\frac{1}{b} H(t)^b$, where

$$H(t) = h_0 + h_1 t + h_2 t^2 + \cdots.$$

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Observation/Definition

The multiplicities of the **hook Schur functions** $s[k+1, 1^{a-k-1}]$ in $\mathrm{PF}(a,b)$ are given by the **Schröder numbers**

$$\mathrm{Schrö}(a,b;k) := \frac{1}{b} s_{[k+1,1^{a-k-1}]}(1^b) = \frac{1}{b}\binom{a-1}{k}\binom{b+k}{a}.$$

Special Cases:

- Trivial character: $\mathrm{Schrö}(a,b;a-1) = \mathrm{Cat}(a,b)$.
- Smallest k that occurs is $k = \max\{0, a-b\}$, in which case

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Rational Duality

Problem

Given a, b coprime we have an $\mathfrak{S}_a$-module $\mathrm{PF}(a, b)$ of dimension b^{a-1} and an $\mathfrak{S}_b$-module $\mathrm{PF}(b, a)$ of dimension a^{b-1}.

- What is the relationship between $\mathrm{PF}(a, b)$ and $\mathrm{PF}(b, a)$?
- Note that hook multiplicities are the same:

$$\mathrm{Schrö}(a, b; k) = \mathrm{Schrö}(b, a; k + b - a).$$

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Summary of Catalan Refinements

- The Kirkman/Narayana/Schröder numbers are equivalent. They contain information about rank. $(1 < k < a-1)$

$$\left.\begin{aligned} \mathrm{Kirk}(x;k) &= \tfrac{1}{a}\tbinom{a}{k}\tbinom{b+k-1}{k-1} \\[2ex] \mathrm{Nar}(x;k) &= \tfrac{1}{a}\tbinom{a}{k}\tbinom{b-1}{k-1} \\[2ex] \mathrm{Schr\ddot{o}}(x;k) &= \tfrac{1}{b}\tbinom{a-1}{k}\tbinom{b+k}{a} \end{aligned}\right\} \quad \begin{aligned} &f\text{-vector} \\[2ex] &h\text{-vector} \\[2ex] &\text{“dual” } f\text{-vector} \end{aligned}$$

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Finally: How about q and t?

We want a "Shuffle Conjecture"

Define a quasisymmetric function with coefficients in $\mathbb{N}[q,t]$ by

$$\mathrm{PF}_{q,t}(a,b) := \sum_{P} q^{\mathrm{qstat(P)}} t^{\mathrm{tstat(P)}} F_{\mathrm{iDes(P)}}.$$

- Sum over (a,b)-parking functions P.
- F is a fundamental (Gessel) quasisymmetric function.
  — *natural refinement of Schur functions*
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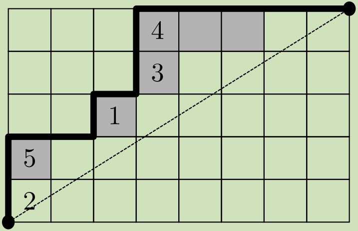
qstat is easy

Definition

- Let qstat := area := # boxes between the path and diagonal.
- Note: Maximum value of area is $(a-1)(b-1)/2$. (Frobenius) — *see Beck and Robins, Chapter 1*

Example

- This $(5,8)$-parking function has area $= 6$.



iDes is reasonable

Definition

- Read labels by increasing "height" to get permutation $\sigma \in \mathfrak{S}_a$.
- iDes := the descent set of σ^{-1}.

Example

- Remember the "height"?

40	35	30	25	20	15	10	5	0
32	27	22	17	12	7	2	-3	-8
24	19	14	9	4	-1	-6	-11	-16
16	11	6	1	-4	-9	-14	-19	-24
8	3	-2	-7	-12	-17	-22	-27	-32
0	-5	-10	-15	-20	-25	-30	-35	-40

- iDes = $\{1, 4\}$

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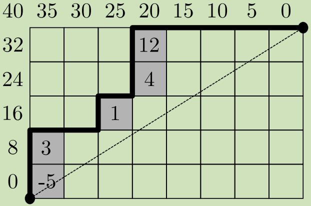
Example

- Look at the heights of the vertical step boxes.



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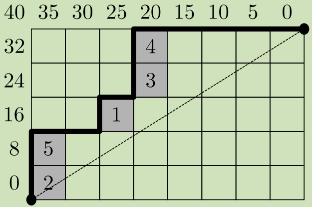
Example

- Remember the labels we had before.



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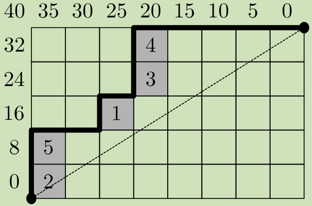
Example

- Read them by increasing height to get $\sigma = 2\bar{1}53\bar{4} \in \mathfrak{S}_5$.



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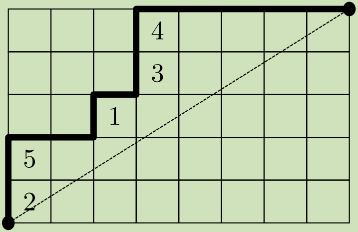
tstat is hard (as usual)

Definition

- "Blow up" the (a, b)-parking function.
- Compute "dinv" of the blowup.

Example

- Recall our favorite the $(5, 8)$-parking function.



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Definition

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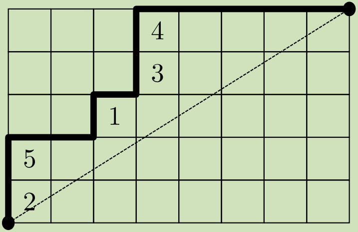
- Compute "dinv" of the blowup.

Example

- Since $2 \cdot 8 - 3 \cdot 5 = 1$ we "blow up" by 2 horiz. and 3 vert....



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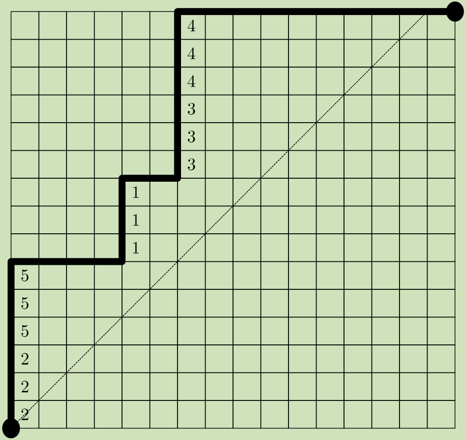
- To get this!



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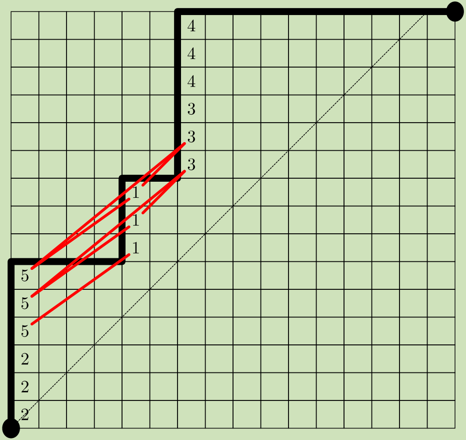
Example

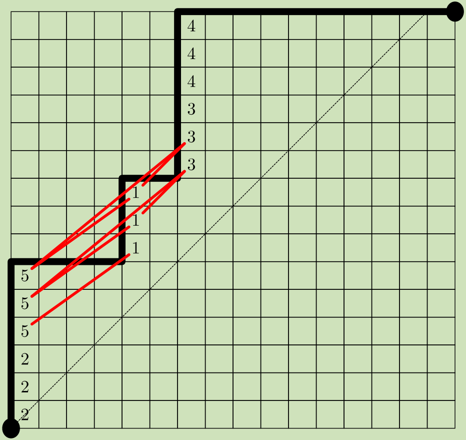
- To get this! Now compute dinv $= 7$.



tstat is hard (as usual)

Example

- (There's a scaling factor *depending on the path*, so tstat $= 3$.)



All Together

Example

- So our favorite $(5,8)$-parking function contributes $q^6 t^3 F_{\{1,4\}}$.
- Proof of Concept: The coefficient of $s[2,2,1]$ in $\mathrm{PF}_{q,t}(5,8)$ is

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- $\mathrm{PF}_{1,1}(a, b) = \mathrm{PF}(a, b)$.
- $\mathrm{PF}_{q,t}(a, b)$ is symmetric and Schur-positive with coeffs $\in \mathbb{N}[q, t]$.
  — via LLT polynomials (HHLRU Lemma 6.4.1)
- **Experimentally:** $\mathrm{PF}_{q,t}(a, b) = \mathrm{PF}_{t,q}(a, b)$.
  — this will be "impossible" to prove (see Loehr's Maxim)
- **Definition:** The coefficient of the hook $s[k + 1, 1^{a-k-1}]$ is the q, t-Schröder number $\mathrm{Schrö}_{q,t}(a, b; k)$.
- **Experimentally:** Specialization $t = 1/q$ gives

$$\mathrm{Schrö}_{q,\frac{1}{q}}(a, b; k) = \frac{1}{[b]_q} \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \begin{bmatrix} b+k \\ a \end{bmatrix}_q \qquad \text{(centered)}$$

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Epilogue: Lie Theory

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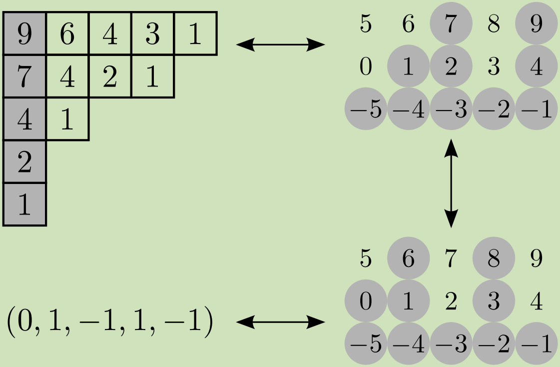
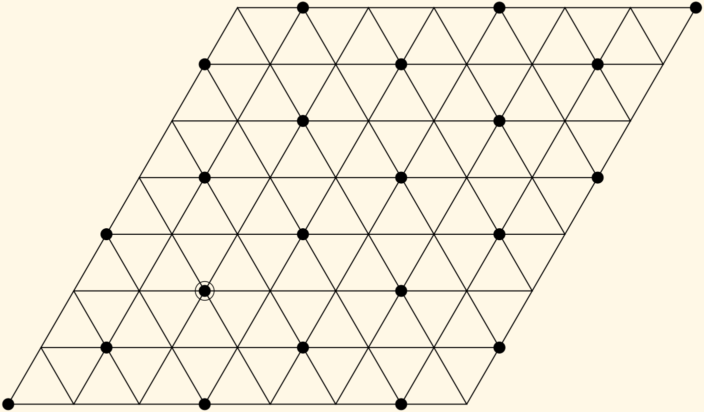
The James-Kerber Bijection

- between a-cores and the root lattice of the Weyl group $\mathfrak{S}_a$



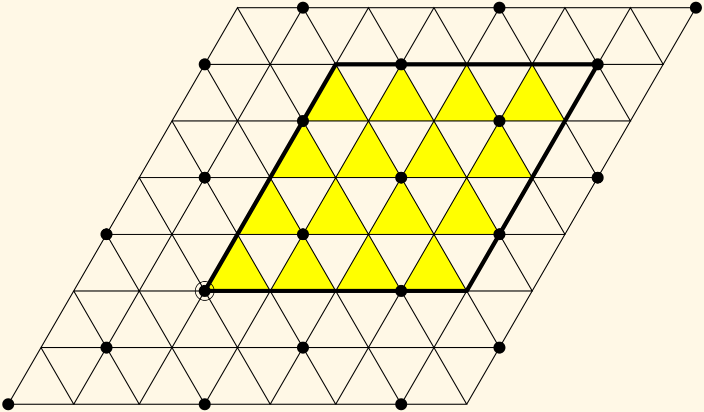
Epilogue: Lie Theory

- These are the root and weight lattices $Q \subseteq \Lambda$ of $\mathfrak{S}_a$.



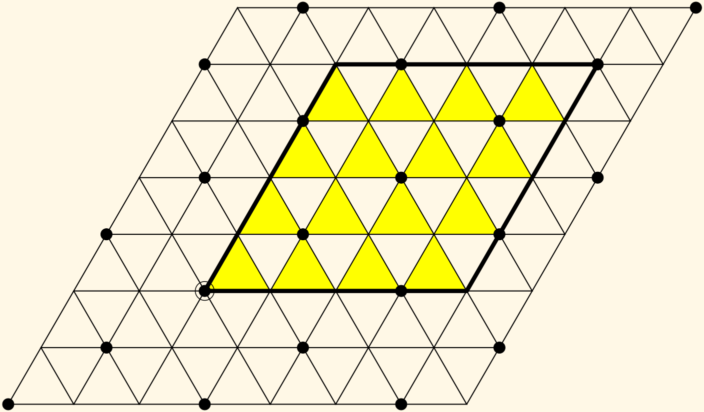
Epilogue: Lie Theory

- Here is a fundamental parallelepiped for $\Lambda / b\Lambda$.



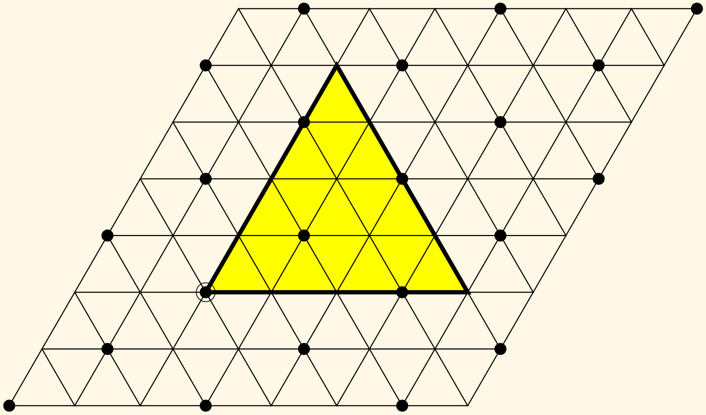
Epilogue: Lie Theory

- It contains b^{a-1} elements (these are the "parking functions").



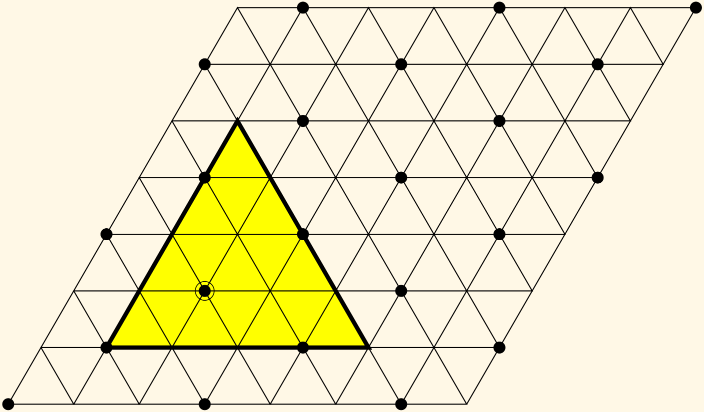
Epilogue: Lie Theory

- But they look better as a simplex...



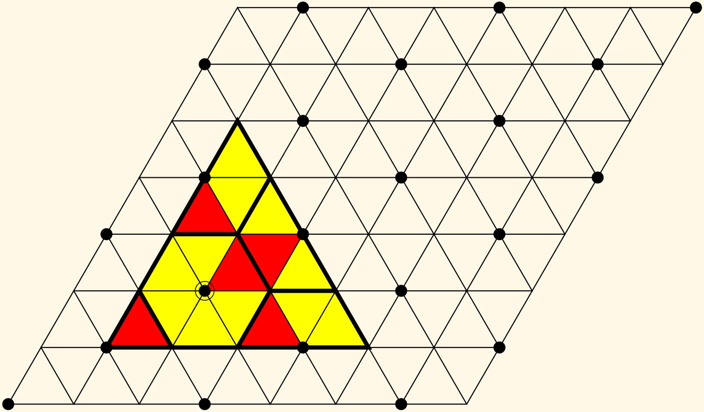
Epilogue: Lie Theory

- ...which is congruent to a nicer simplex.



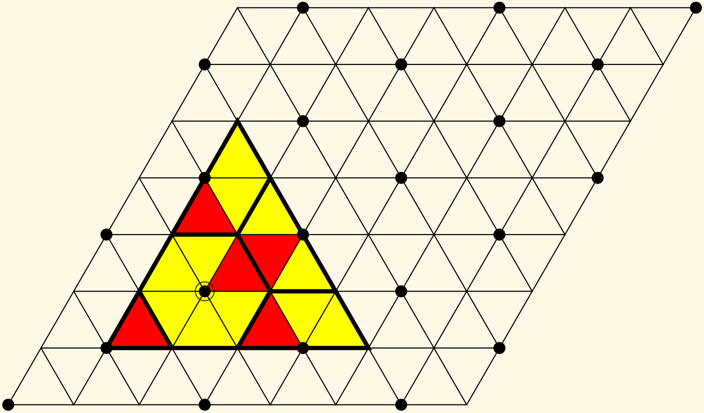
Epilogue: Lie Theory

- There are $\mathrm{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a,b}$ elements of the root lattice inside.



Epilogue: Lie Theory

- These are the (a, b)-Dyck paths (via Anderson, James-Kerber).



Other Weyl Groups?

Definition

Consider a Weyl group W with Coxeter number h and let $p \in \mathbb{N}$ be **coprime** to h. We define the **Catalan number**

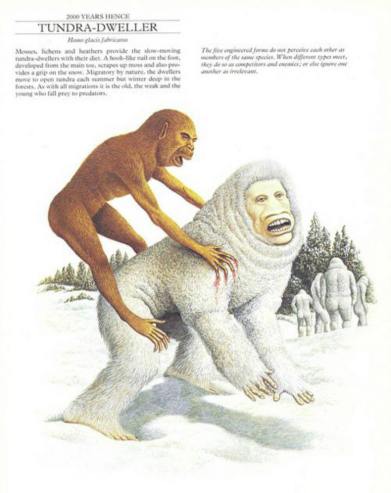
$$\mathrm{Cat}_q(W,p) := \prod_j \frac{[p+m_j]_q}{[1+m_j]_q}$$

where $e^{2\pi i m_j/h}$ are the eigenvalues of a Coxeter element.

Observation

$$\mathrm{Cat}_q(\mathfrak{S}_a, b) = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q$$

Here's to a Productive Workshop



2000 YEARS HENCE

TUNDRA-DWELLER

Homo glacis fabricatus

Mosses, lichens and heathers provide the slow-moving tundra-dwellers with their diet. A hook-like nail on the foot, developed from the main toe, scrapes up moss and also provides a grip on the snow. Migratory by nature, the dwellers move to open tundra each summer but winter deep in the forests. As with all migrations it is the old, the weak and the young who fall prey to predators.

The five engineered forms do not perceive each other as members of the same species. When different types meet, they do so as competitors and enemies; or else ignore one another as irrelevant.

10,000 YEARS HENCE

SYMBIONT CARRIER

Baiulus moderatorum

Two species form a single unit of value to both – symbiosis. The woodland-dwellers have skills that their carriers lack. The hunting ability of the swift forest-dweller provides enough food both for itself and its slow-moving carrier. The tundra-dweller, in turn, provides both with general movement and protection against the cold.

Lacking thick fur and insulating layers of fat, Moderator baiuli *can only hunt in short bursts before needing to return to the body heat of its carrier. Communication is by touch.*