Bézout's Theorem, Part I:

If $F(x,y,z), G(x,y,z) \in \mathbb{C}[x,y,z]$ are homogeneous of degrees d, e then the resultant

$$H(x,z) = \mathrm{Res}_y(x,z) \in \mathbb{C}[x,z]$$

is homogeneous of degree de.

Proof by example $(d=3, e=2)$:

let $F = a_0 y^d + a_1 y^{d-1} + \cdots + a_d$
$G = b_0 y^e + b_1 y^{e-1} + \cdots + b_e$

with $a_k, b_k \in \mathbb{C}[x,z]$ homogeneous of degree k. We define

$$H(x,z) := \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \\ & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & & \\ & b_0 & b_1 & b_2 & \\ & & b_0 & b_1 & b_2 \end{pmatrix}.$$

Assume F, G coprime in $\mathbb{C}[x,y,z]$, hence also in $\mathbb{C}[x,z][y]$ so that $H(x,z) \not\equiv 0$. To see that H is homogeneous, substitute $(x,z) \to (\lambda x, \lambda z)$. Since

$$a_k(\lambda x, \lambda z) = \lambda^k a_k(x,z)$$
$$b_k(\lambda x, \lambda z) = \lambda^k b_k(x,z)$$

we get

$$H(\lambda x, \lambda z) = \det \begin{pmatrix} a_0 & \lambda a_1 & \lambda^2 a_2 & \lambda^3 a_3 & \\ & a_0 & \lambda a_1 & \lambda^2 a_2 & \lambda^3 a_3 \\ b_0 & \lambda b_1 & \lambda^2 b_2 & & \\ & b_0 & \lambda b_1 & \lambda^2 b_2 & \\ & & b_0 & \lambda b_1 & \lambda^2 b_2 \end{pmatrix}$$

$$= \frac{1}{1\cdot\lambda\cdot 1\cdot\lambda\cdot\lambda^2} \det \begin{pmatrix} a_0 & \lambda a_1 & \lambda^2 a_2 & \lambda^3 a_3 & \\ & \lambda a_0 & \lambda^2 a_1 & \lambda^3 a_2 & \lambda^4 a_3 \\ b_0 & \lambda b_1 & \lambda^2 b_2 & & \\ & \lambda b_0 & \lambda^2 b_1 & \lambda^3 b_2 & \\ & & \lambda^2 b_0 & \lambda^3 b_1 & \lambda^4 b_2 \end{pmatrix}$$

$$= \frac{1\cdot\lambda\cdot\lambda^2\cdot\lambda^3\cdot\lambda^4}{1\cdot\lambda\cdot 1\cdot\lambda\cdot\lambda^2} \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \\ & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & & \\ & b_0 & b_1 & b_2 & \\ & & b_0 & b_1 & b_2 \end{pmatrix}$$

$$= \frac{\lambda^{10}}{\lambda^4} H(x,y) = \lambda^6 H(x,y).$$

Hence H is homogeneous of degree 6.

$$6 = 2 \cdot 3 \quad \checkmark$$

In general:

$$\frac{1 \cdot \lambda \cdot \lambda^2 \cdots \lambda^{d+e-1}}{1 \cdot \lambda \cdots \lambda^{d-1} \cdot 1 \cdot \lambda \cdots \lambda^{e-1}} = \lambda^{de}$$

because $\binom{d+e}{2} - \binom{d}{2} - \binom{e}{2} = de$.

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Geometric Application:

If $F, G \in \mathbb{C}[x,y,z]$ are coprime of degrees d & e, then

$$1 \leq \# V(F) \cap V(G) \leq de.$$

Remark: No such thing as <u>disjoint curves</u> in $\mathbb{CP}^2$.

To prove this ....

Lemma (Study's Lemma for lines):

If $L, F \in \mathbb{C}[x,y,z]$ homogeneous with $\deg(L) = 1$, then

$$V(L) \subseteq V(F) \implies L \mid F.$$

$\# V(L) \cap V(F) = \infty$ is sufficient.

<u>Proof</u>: Choose $\varphi \in PGL$ so that $L^{\varphi}(x,y,z) = x$. Note that

$$L^{\varphi} \mid F^{\varphi} \iff L \mid F.$$

So it suffices to prove

$$V(x) \subseteq V(F^{\varphi}) \implies x \mid F^{\varphi}.$$

So write $F^{\varphi}(x,y,z) = \sum_{k \geq 0} F_k(y,z)\, x^k$

If $(a,b,c) \in V(x)$, i.e., if $a=0$ then $V(x) \subseteq V(F^q)$ implies

$$0 = F^q(0,b,c) = \sum F_k^q(b,c)\,0^k$$

$$= F_0^q(b,c).$$

alg. closure not necessary

Since $\mathbb{C}$ is infinite, this implies that $F_0^q(y,z) \equiv 0$, hence

$$x \mid F^q(x,y,z).$$

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Corollary: If $F, G \in \mathbb{C}[x,y,z]$ homogeneous & coprime then

$$1 \leq \# V(F) \cap V(G) < \infty$$

Proof: Let $C = V(F)$, $D = V(G)$.

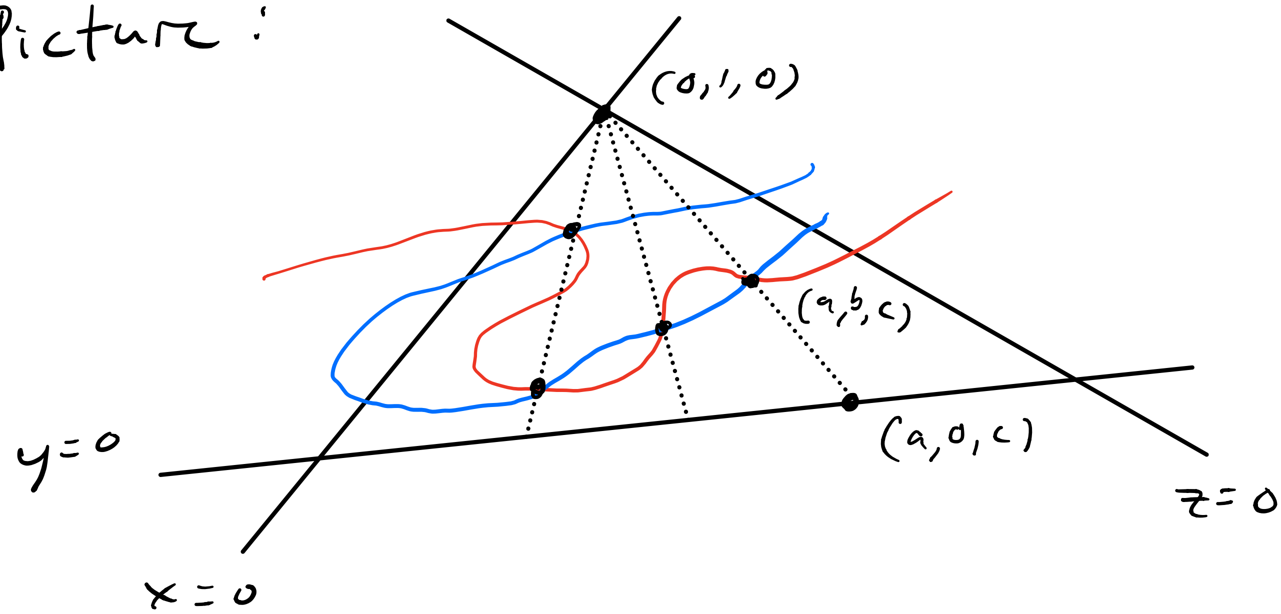
Let $H(x,z) = \mathrm{Res}_y(F,G) \in \mathbb{C}[x,z]$, which is nonzero and homogeneous of degree de (Bézout I)

After some $\varphi \in PGL$ we may assume that $(0,1,0) \notin C \cup D$. Now we observe:

$(a,b,c) \in C \cap D$

$\iff$ $f(y) = F(a,y,b)$ $\in \mathbb{C}[y]$
$g(y) = G(a,y,b)$
have common root $y = b$.

Picture:



Given (a,c), such b exists

$\iff$ $f(y), g(y)$ not coprime

$\iff$ $H(a,c) = 0$.

Since $H(x,y)$ is homogeneous and nonzero, there are finitely many such $(a,c) \in \mathbb{CP}^1$. Since $\mathbb{C}$ is algebraically closed,

$$H(x,z) = \prod_i (c_i x - a_i y).$$

For each (a_i, c_i) I claim that

$$f_i(y) = F(a_i, y, c_i) \ \& \ g_i(y) = G(a_i, y, c_i)$$

are nonzero. Indeed, if $f_i(y) \equiv 0$, then F vanishes at ∞ many points of the line L connecting $(a_i, 0, c_i)$ & $(0,1,0)$. From Study's Lemma for lines, this implies that $L \subseteq C$ hence $(0,1,0) \in L \subseteq C$.

Contradiction. Hence $f_i(y) \not\equiv 0$.

Since $f_i(y)$ & $g_i(y)$ are nonzero, they have finitely many common roots $y = b_i$. ///

Corollary (Study's Lemma for Curves):

Let $F, G \in \mathbb{C}[x,y,z]$ be homogeneous with F irreducible. Then

$$V(F) \subseteq V(G) \implies F \mid G.$$

Proof: First note that $V(F)$ has infinitely many points.

[Let $F = \sum c_k(x,z)\, y^k$. Since $F \not\equiv 0$ $\exists\, k$ such that $c_k(x,z) \not\equiv 0$.

Since $\mathbb{C}$ is infinite, $\exists$ infinitely many (a_i, c_i) such that $c_k(a_i, c_i) \neq 0$.

For each such (a_i, c_i) we have

$$F(a_i, y, c_i) = \sum a_k(a_i, c_i)\, y^k \not\equiv 0 .$$

Since $\mathbb{C}$ is alg. closed, $\exists$ at least one b_i such that $F(a_i, b_i, c_i) = 0$.]

Now suppose for contradiction that $V(F) \subseteq V(G)$ with F irreducible & $F \nmid G$.

Since $\# V(F) = \infty$ & $V(F) \subseteq V(G)$,

$$\# V(F) \cap V(G) = \infty .$$

On the other hand, since F is irr. and $F \nmid G$, we see that F, G are coprime, so that

$$\# V(F) \cap V(G) < \infty .$$

Bézout's Theorem, Part II:

More specifically, let $\bar{\eta} \in C \cap D$.
There is a notion of multiplicity

$$I_{\bar{\eta}}(C,D) \in \{1,2,\ldots\}$$

such that

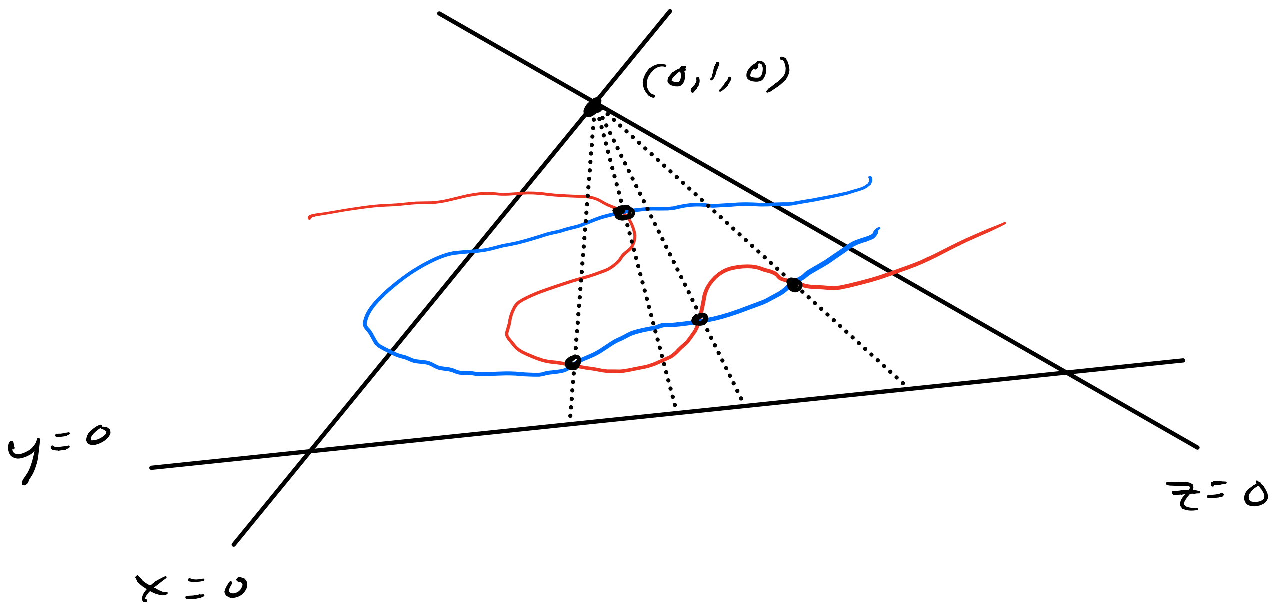
$$\sum_{\bar{\eta} \in C \cap D} I_{\bar{\eta}}(C,D) = \deg(C) \cdot \deg(D).$$

Since $I_{\bar{\eta}}(C,D) \geq 1$ it follows that

$$\# C \cap D \leq \deg(C) \cdot \deg(D).$$

<u>Proof</u>: Since $\# C \cap D < \infty$ there are finitely many lines connecting pairs of points in $C \cap D$. After some $\varphi \in PGL$, we can assume that $(0,1,0)$ is on none of these lines.

Picture:



As before, $\exists$ point $(a_i, b_i, c_i) \in C \cap D$ $\iff$ $H(a_i, c_i) = 0$. Since $H(x,z) \in \mathbb{C}[x,z]$ is homogeneous of degree $\deg(C) \cdot \deg(D)$ we have

$$H(x,z) = \prod (c_i x - a_i z)^{m_i}$$

where $\sum m_i = \deg(C) \cdot \deg(D)$.

By choice of φ, $\exists$ exactly one $b_i \in \mathbb{C}$ for each (a_i, c_i) such that $\bar{\eta}_i = (a_i, b_i, c_i) \in C \cap D$.

We define

$$I_{\bar{\eta}_i}(C, D) := m_i .$$

It is not obvious that this

$$I_{\bar{\eta}}(C, D)$$

is well-defined, or that it satisfies Fulton's axioms 1 — 7.

We get a better definition if we allow the points

$$\bar{p} = (a, 0, c) \ \& \ \bar{q} = (0, 1, 0)$$

to be completely arbitrary:

$$\bar{p} = (p_1, p_2, p_3) \ \& \ \bar{q} = (q_1, q_2, q_3).$$

Definition of Intersection Multiplicity (van der Waerden, Intro. to Algebraic Geometry, Chapter 3)

Consider two curves $C = V(F)$, $D = V(G)$ of degrees d, e and a generic line

$$L : s\bar{p} + t\bar{q} .$$

Define the homogeneous polynomial

$$R(\bar{p}, \bar{q}) := \mathrm{Res}_{s,t}\left(F(s\bar{p} + t\bar{q}), G(s\bar{p} + t\bar{q})\right)$$

This is a bihomogeneous polynomial in p_1, p_2, p_3 & q_1, q_2, q_3 of bidegree de, de.

Observe that $R(\bar{p}, \bar{q}) = 0$

$\Longleftrightarrow$ the line L passes through an intersection point $\bar{\eta} \in C \cap D$.

On the other hand, we have $\bar{\eta} \in L$ if and only if the following determinant vanishes:

$$[\bar{p},\bar{q},\bar{\eta}] := \det\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix} = 0.$$

bihomogeneous in $\bar{p}, \bar{q}$ of bidegree 1,1.

By Study's Lemma, it follows that

$$R(\bar{p},\bar{q}) = \prod_{\bar{\eta}\in C\cap D} [\bar{p},\bar{q},\bar{\eta}]^{m_{\bar{\eta}}}.$$

with $\sum_{\bar{\eta}\in C\cap D} m_{\bar{\eta}} = \deg(R) = de$,

and we <u>define</u>

$$I_{\bar{\eta}}(F,G) := m_{\bar{\eta}}.$$

It is easier to show that this definition satisfies Fulton's 1 — 7, but we still won't.