Summary of last week:

Let $C \subseteq \mathbb{CP}^2$ be an irreducible cubic.

- If C is singular, then

$$C \approx V(y^2z - x^3) \qquad \text{"cusp"}$$

or

$$C \approx V(y^2z - x^2(x+z)) \qquad \text{"node"}$$

- If C is non-singular, then

$$\begin{aligned} C &\approx V(y^2z - x^3 - axz^2 - bz^3) \\ &\approx V(y^2z - x(x-z)(x-\lambda z)) \\ &\approx V(x^3+y^3+z^3 - 3kxyz) \end{aligned}$$

These are classified up to projective equivalence by the "j-invariant":

$$J(C) = \frac{4a^3}{4a^3+27b^2} = \frac{4(\lambda^2-\lambda+1)^3}{27\lambda^2(\lambda-1)^2}$$

$$= \left(\frac{k(k^3+8)}{4(k^3-1)}\right)^3 \qquad ///$$

There is more to say about cubics

[such as: for any basepoint $\bar{o} \in C$, there exists a unique abelian group structure $(C, +, \bar{o})$ on the points of C, with the property that

$$\bar{p} + \bar{q} + \bar{r} = \text{some constant point}$$

for all collinear $\bar{p}, \bar{q}, \bar{r} \in C$.]

but I will leave it here for now!

Next Topic: The "intrinsic" structure of a curve.

So far, all curves in our course have been given as subvarieties of $\mathbb{CP}^2$, and we have defined "equivalence" of curves $C, D \subseteq \mathbb{CP}^2$

by the existence of a global automorphism $\varphi \in PGL$ sending

$$\varphi(C) = D.$$

This is the "extrinsic" view of curves.

We already have some reason to think that this is not sufficient.

Namely: $\exists$ 1D subvarieties of $\mathbb{CP}^n$ that deserve to be called "algebraic curves" but it is not clear how to relate these to curves in $\mathbb{CP}^2$.

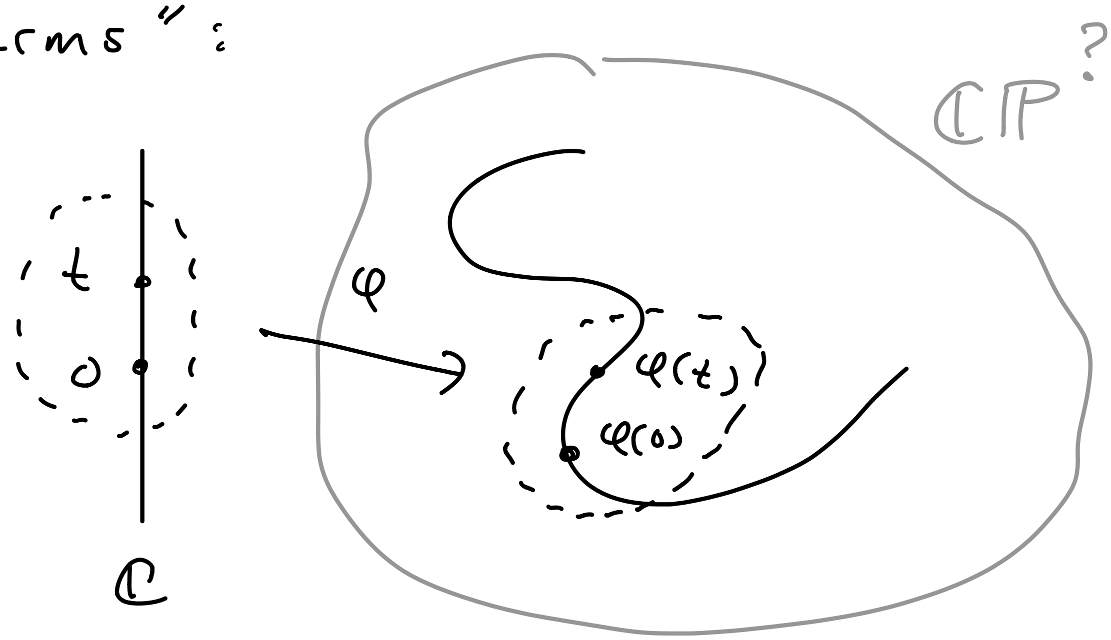
This can be partially remedied by considering "projection maps"

$$\mathbb{CP}^N \twoheadrightarrow \mathbb{CP}^n$$

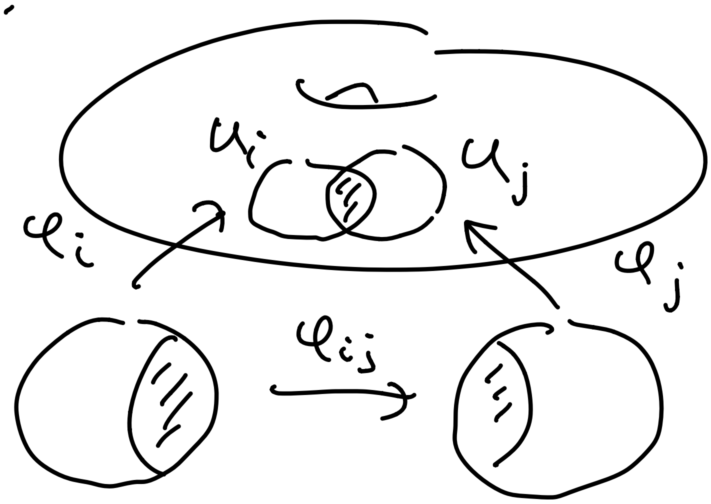
where we think of $\mathbb{CP}^n$ as a projective subspace of $\mathbb{CP}^N$.

But the modern approach is more radical!

Since holomorphic functions have locally convergent Taylor series, we can think of an "abstract curve" C as being locally parametrized by "germs":



Better Picture: Think of $\mathbb{C}$ as a real 2D plane:



We only require that the germs overlap in a holomorphic way, i.e., that the transfer functions

$$\varphi_{ij} : \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$$

are holomorphic. Then an "abstract curve" is an equivalence class of local parametrizations with good overlap. In modern terms:

"a 1D complex manifold."

[The concept of a manifold ("Mannigfaltigkeit", or "many-foldness") was introduced by Riemann (1850s) and formalized by Weyl (1910s).]

In this course we will follow Riemann's <u>intuitive approach</u>.

The key idea is to think of a 1D complex "curve" in $\mathbb{CP}^2$ as a 2D real "surface" in $\mathbb{RP}^4$ [or in $\mathbb{RP}^2 \times \mathbb{RP}^2$; doesn't matter.]

Let $f(x,y) \in \mathbb{C}[x,y]$ be irreducible & non-singular of degree d. After changing coordinates we may assume that

$$f(x,y) = y^d + \sum_{k=1}^{d} \varphi_k(x)\, y^{d-k},$$

coefficient of y^d does not involve x.

where $\varphi_k(x) \in \mathbb{C}[x]$ has degree $\leq k$. Let $C = V(f) \subseteq \mathbb{C}^2$ and consider the projection map

$$\begin{aligned} \pi : C &\longrightarrow \mathbb{C} \\ (x,y) &\longmapsto x \end{aligned}$$

Since $\mathbb{C}$ is algebraically closed and since the coeff of y^d does not involve x, it follows that each point $x \in \mathbb{C}$ has d preimages under π, counted with multiplicity. Call them

$$\overline{p_1}, \overline{p_2}, \ldots, \overline{p_d} \in C$$

where $\overline{p_i} = (x, y_i)$ & $f(x, y_i) = 0$.

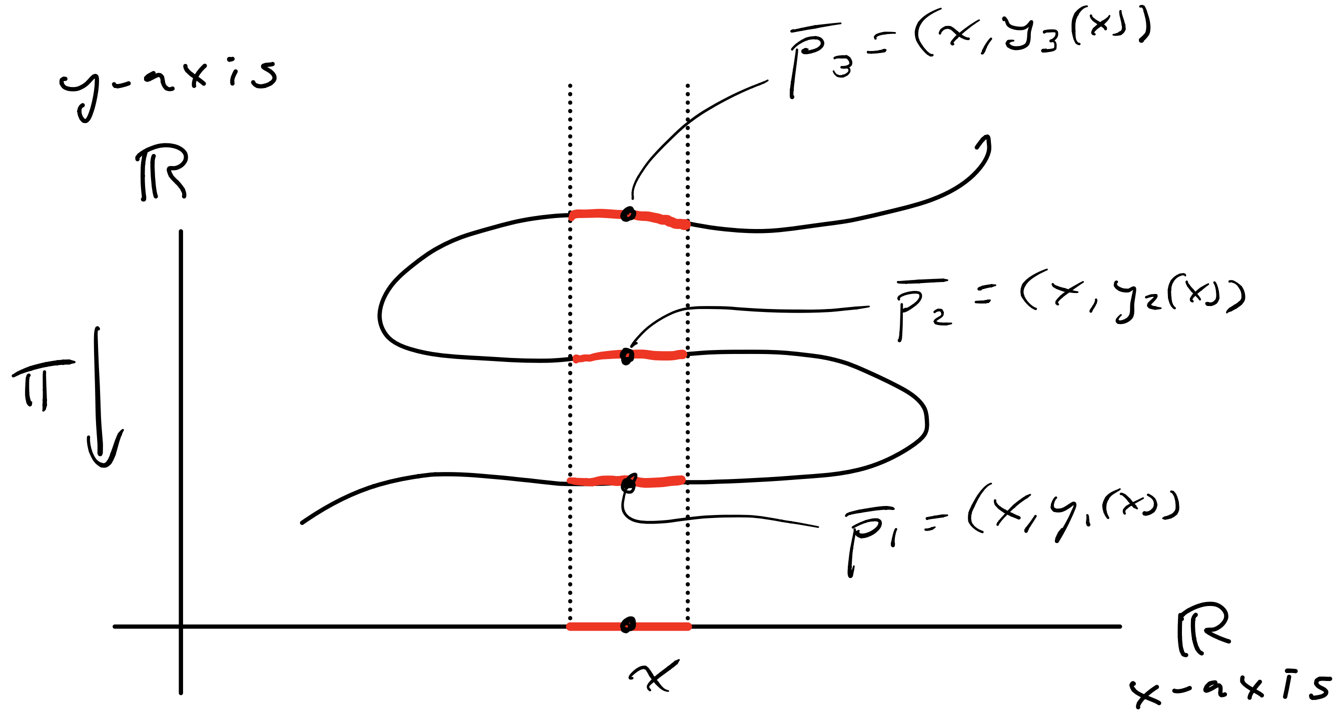
By implicit function theorem, we can think of each $y_i = y_i(x)$ as a locally defined holomorphic function (i.e. a "germ")

$$y_i : \mathbb{C} \to \mathbb{C}$$

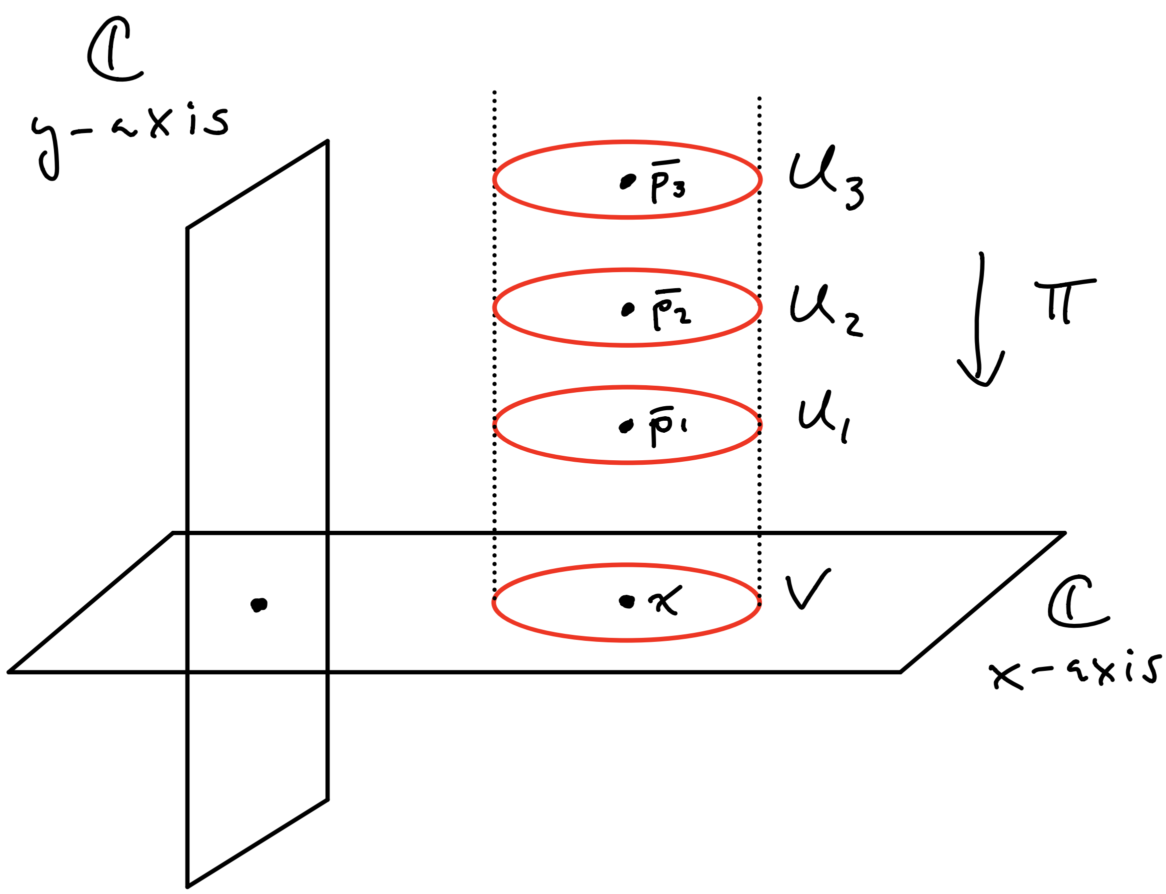
defined in a neighborhood of $x \in \mathbb{C}$.

If the $y_i(x)$ are <u>distinct</u> then we can visualize them as follows.

Real Picture ($d=3$):



Complex Local Picture ($d=3$):



Remarks :

- In $\mathbb{R}^2$ we can view the preimages $\bar{p}_1, \bar{p}_2, \bar{p}_3$ as ordered by "height."
  In $\mathbb{C}^2$ this is no longer possible, so we must regard $\bar{p}_1, \bar{p}_2, \bar{p}_3$ as "unordered" (the labeling is arbitrary)

- The global complex picture is "real 4 dimensional" so we can not draw it.

When the $y_1(x), \ldots, y_d(x) \in \mathbb{C}$ are *not* distinct, it is not so clear how to visualize the situation. Luckily this only happens for *finitely many* values of $x \in \mathbb{C}$.

Proof: As before, let $f(x,y) \in \mathbb{C}[x,y]$ be irr. & non-singular of degree d.

By changing coords we may assume

$$f(x,y) = y^d + \varphi_1(x) y^{d-1} + \dots + \varphi_d(x)$$

with $\varphi_k(x) \in \mathbb{C}[x]$ of degree $\leq k$.

We are looking for $a \in \mathbb{C}$ such that

$$f^a(y) := f(a,y) \in \mathbb{C}[y]$$

has a multiple root $y_i(a) \in \mathbb{C}$.

I claim that there are finitely many such $a \in \mathbb{C}$. To see this we consider the resultant polynomial

$$r(x) = \mathrm{Res}_y(f, f_y) \in \mathbb{C}[x].$$

Since $f(x,y)$ is irreducible in $\mathbb{C}[x,y]$ we know that f, f_y are coprime in $\mathbb{C}[x,y]$ because $\deg_y f_y < \deg_y f$.

Hence $r(x) \not\equiv 0$. We also know that

$$r(x) = \Phi(x,y) f(x,y) + \Psi(x,y) f_y(x,y).$$

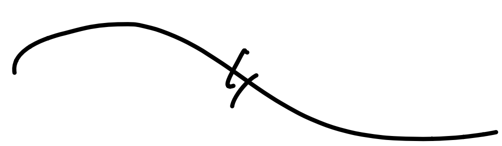
If $r(a) \neq 0$ then the equation

$$1 = \frac{\Phi(a,y)}{r(a)} f^a(y) + \frac{\Psi(a,y)}{r(a)} f_y^a(y)$$

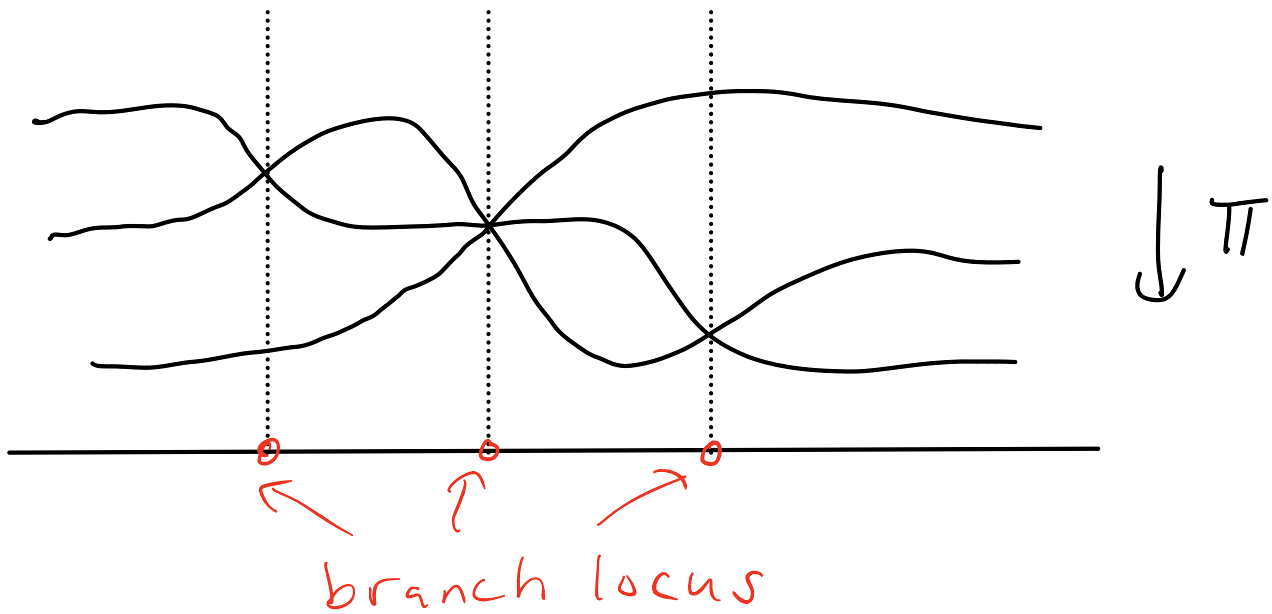
shows that $f^a(y)$ & $f_y^a(y)$ are coprime in $\mathbb{C}[y]$, hence $f^a(y)$ does not have a multiple root. Conversely, if $f^a(y)$ <u>has</u> a multiple root then $a \in \mathbb{C}$ is a root of the non-zero polynomial $r(x) \in \mathbb{C}[x]$.

We call this finite set the "branch locus" (or "ramification locus") of the projection $\pi : C \longrightarrow \mathbb{C}$:

$$\Delta(\pi) = \{ x \in \mathbb{C} : \# \pi^{-1}(x) < d \}$$



Complex picture ("from the side"):



WARNING!! This picture suggests that the curve has self-intersections, but this need not be true (in fact, won't be true when C is non-singular). The self-intersections are artifacts of cramming a real 4D situation into a real 2D picture. [Even a real 3D picture is not ample enough to avoid this issue.]

Slogan: Ramification is a property of the map $\pi : C \to \mathbb{C}$, not of the curve C.

Example: Consider the smooth conic $C = V(y^2 - x)$ and the projection

$$\pi : C \to \mathbb{C}$$
$$(x,y) \mapsto x.$$

To find $\Delta(\pi)$ we compute the resultant

$$\mathrm{Res}_y(y^2 - x, 2y) = \det \begin{pmatrix} 1 & 0 & -x \\ 2 & 0 & \\ & 2 & 0 \end{pmatrix}$$

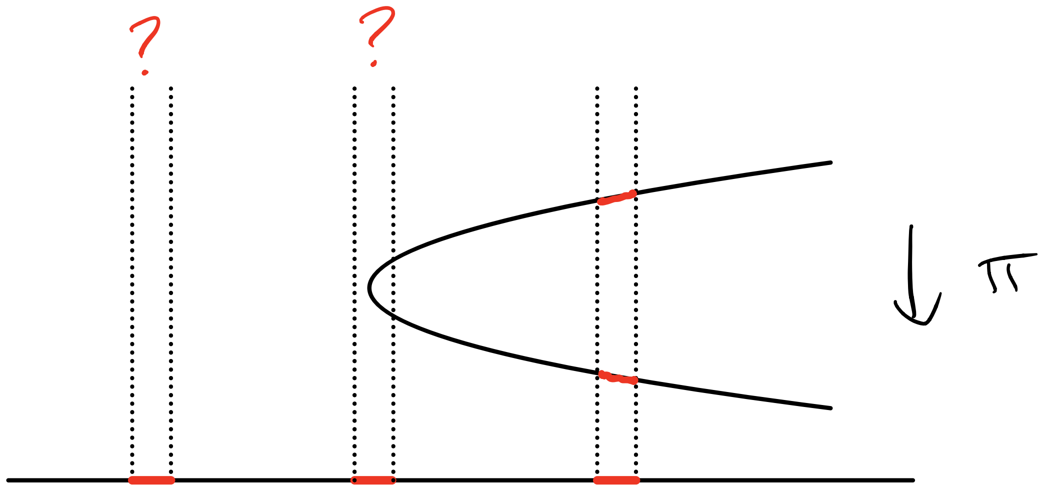
$$= -4x,$$

so that $\Delta(\pi) = \{0\} \subseteq \mathbb{C}$.

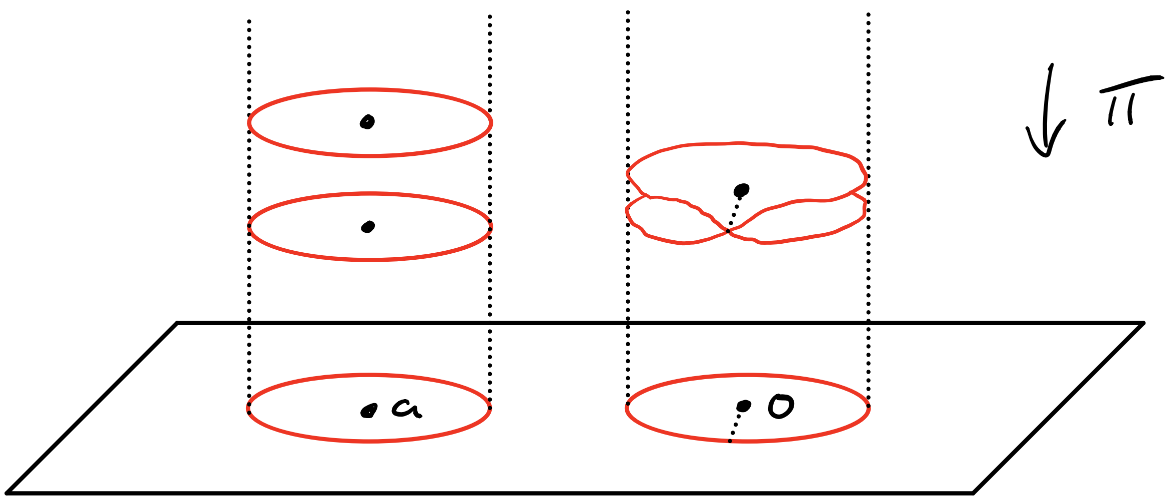
Indeed, for all $a \neq 0$ we have two preimages $\pi^{-1}(a) = \{(a, \pm\sqrt{a})\}$, but the fiber over 0 is just a single point:

$$\pi^{-1}(0) = \{(0,0)\}.$$

Unfortunately, the real picture is not big enough to show this:



The complex picture is slightly better:



These are the two local situations.

The self intersection of the neighborhood above O is again an artifact of embedding the surface into 3D, when it naturally lives in 4D.