Topic: Bézout's Theorem.

[Abhyankar (Historical Ramblings in Algebraic Geometry): Three parallel tracks in Alg. Geom:

- High School Alg.
- College Alg.
- University Alg.]

Recall def of intersection multiplicity:

$$I_{\bar{p}}(C, L).$$

Parametrize the line, $L : \bar{p} + t\bar{q}$.
Consider the polynomial

$$\varphi(t) = F(\bar{p} + t\bar{q})$$
$$= F^{(0)}(\bar{q}) + t\, F^{(1)}(\bar{q}) + t^2 F^{(2)}(\bar{q}) + \cdots$$

Then we define

$I_{\bar{p}}(C, L)$ = mult of t as a root of $\varphi(t)$.

Note: Replacing $\bar{q}$ by $\lambda\bar{q}$ replaces

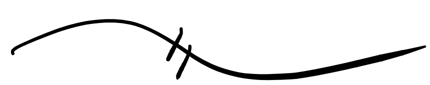
$$F^{(k)}(\bar{q}) \text{ by } F^{(k)}(\lambda\bar{q}) = \lambda^k F^{(k)}(\bar{q}),$$

so $I_{\bar{p}}(C,L)$ is invariant under re-parametrization of L.

Define the multiplicity of $\bar{p} \in C$:

$$m_{\bar{p}}(C) := I_{\bar{p}}(C,L)$$

for any line $L \ni \bar{p}$ <u>not tangent</u> to C.



An "intersection theory" for curves in the plane should have the following properties:

[Temporarily write f to denote the unique curve defined by square-free polynomial f.]

1) $I_{\bar{p}}(f,g) = 0 \iff \bar{p} \notin f \cap g$.

2) If $\bar{p} \in f \cap g$, then

- if $h \mid f, g$ and $\bar{p} \in h$ then $I_{\bar{p}}(f,g) = \infty$
- otherwise $I_{\bar{p}}(f,g) \in \{1, 2, 3, \dots\}$.

3) For affine $\varphi: \mathbb{F}^2 \to \mathbb{F}^2$ then

$$I_{\bar{p}}(f,g) = I_{\varphi(\bar{p})}(f^{\varphi}, g^{\varphi}).$$

4) $I_{\bar{p}}(f,g) = I_{\bar{p}}(g,f)$

5) $I_{\bar{p}}(f,g) \geq m_{\bar{p}}(f)\, m_{\bar{p}}(g)$

with equality $\iff$ f, g do not share a tangent line at $\bar{p}$.

6) If $f = \prod_i f_i^{r_i}$ & $g = \prod g_j^{s_j}$ then

$$I_{\bar{p}}(f,g) = \sum_{i,j} r_i s_j I_{\bar{p}}(f_i, g_j)$$

7) $I_{\bar{p}}(f,g) = I_{\bar{p}}(f, g + hf) \;\forall h$.

[Meaning of 7): $I_{\bar{p}}(f,g)$ depends only on the image of g in the quotient ring $\mathbb{F}[x,y]/(f)$, i.e., on the restriction of the "function g" to the "curve f." Think: substitute a "parametrization for f" into the equation "$g=0$."]

Theorem (Fulton Alg. Curves, pg 37): Such $I_{\bar{p}}$ (if it exists) is unique.

Proof: Given ① – ⑦ we will say how to compute the numbers $I_{\bar{p}}(f,g)$.

By ①, ②, ③ Assume $\bar{p} = \bar{0} \in f \cap g$ with f, g coprime. Thus we have

$$I_{\bar{p}}(f,g) = n > 0 \text{ for some } n.$$

Assume for induction we can compute $I_{\bar{p}}(A,B)$ when $I_{\bar{p}}(A,B) < n$.

Now consider

$$\varphi(x) = f(x,0) \quad \text{of degree } r$$
$$\gamma(x) = g(x,0) \quad \text{of degree } s.$$

Polynomials $\varphi(x), \gamma(x)$ not both zero because y is not common factor of f & g. By (4) can assume $r \leq s$.

Two Cases:

i) If $\varphi(x) \equiv 0$ $(r = -\infty)$

then $y \mid f$ so $f(x,y) = y\, h(x,y)$.

Then from (6):

$$I_{\bar{p}}(f,g) = I_{\bar{p}}(y,g) + I_{\bar{p}}(h,g).$$

Let $m = m_{\bar{p}}(g)$ so that

$$\gamma(x) = x^m \mu(x), \quad \mu(0) \neq 0, \quad m > 0.$$

Claim: $I_{\bar{p}}(y,g) = m$, so that

$$I_{\bar{p}}(h,g) < I_{\bar{p}}(f,g)$$

and we are done by induction.

To see this: let $g(x,y) = \gamma(x) + y\,\pi(x,y)$.

$$I_{\bar{p}}(y, g) = I_{\bar{p}}(y, \gamma + y\pi)$$

$$= I_{\bar{p}}(y, \gamma) \qquad (7)$$

$$= I_{\bar{p}}(y, x^m \mu(x))$$

$$= I_{\bar{p}}(y, x^m) + \underbrace{\cancel{I_{\bar{p}}(y, \mu(x))}}_{0 \ (\mu(0) \neq 0)} \qquad (1)$$

$$= m \, \underbrace{\cancel{I_{\bar{p}}(y, x)}}^{1} \qquad (6)$$

$$= m \qquad (5)$$

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ii) $\varphi(x), \gamma(x)$ both nonzero. So define

$$h(x,y) = g(x,y) - x^{s-r} f(x,y)$$

with $\deg(h) < \deg(g)$ and

$$I_{\bar{p}}(f, g) = I_{\bar{p}}(f, h).$$

Repeat until we get back to case i).

QED.

So... does the function $I_{\bar{p}}(f,g)$ actally EXIST ??

There are at least 3 ways to define it:

1. determinants of polynomials
2. power series expansions (theory of valuations)
3. localization of rings.

We will pursue 1 first.

"Elimination Theory"

Euler $\rightsquigarrow$ Sylvester $\rightsquigarrow$ Macaulay.

Theorem: Let R be a UFD and consider $f(x), g(x) \in R[x]$.

Then TFAE:

- f, g are <u>not</u> coprime in $R[x]$
- $\exists\, \varphi, \gamma \in R[x]$ satisfying
  - $\deg \varphi < \deg f$
  - $\deg \gamma < \deg g$
  - $f\gamma = g\varphi$.
- The "resultant" is nonzero:

Let $f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_d$
$g(x) = b_0 x^e + b_1 x^{e-1} + \cdots + b_e$.

$$\operatorname{Res}(f,g) := \det \underbrace{\begin{pmatrix} a_0 & \cdots & a_d & & \\ & \ddots & & \ddots & \\ & & a_0 & \cdots & a_d \\ \hline b_0 & \cdots & b_e & & \\ & \ddots & & \ddots & \\ & & b_0 & \cdots & b_e \end{pmatrix}}_{(d+e)\times(d+e)} \neq 0$$

Various properties of the resultant:

- $\operatorname{Res}(f,g) = \pm \operatorname{Res}(g,f)$
- $\exists\, \tilde{f}, \tilde{g} \in R[x]$ such that

$$f(x)\tilde{f}(x) + g(x)\tilde{g}(x) = \mathrm{Res}(f,g) \in R.$$

- If $f(x) = \prod (x-\lambda_i)$

  $g(x) = \prod (x-\mu_j)$ then

$$\mathrm{Res}(f,g) = \prod_{i,j} (\lambda_i - \mu_j)$$

- If $f(x) = \prod (x-\lambda_i)$ and $f'(x)$ is the derivative wrt x, then

$$\mathrm{Res}(f,f') = \pm \prod_{i<j} (\lambda_i - \lambda_j)$$

Called the "discriminant" of $f(x)$.

Application to Geometry:

Note that $\mathbb{C}[x,z]$ is a UFD, so that for any

$$F, G \in \mathbb{C}[x,y,z] = \mathbb{C}[x,z][y]$$

we may consider the resultant

$$H = Res_y(F, G) \in \mathbb{C}[x, z].$$

[Say we have "eliminated" the variable y from equations $F = 0$ & $G = 0$.]

Say F, G are homogeneous of degrees d, e and consider the projective curves

$$C = V(F)$$
$$D = V(G).$$

Note that $(a, b, c) \in C \cap D$

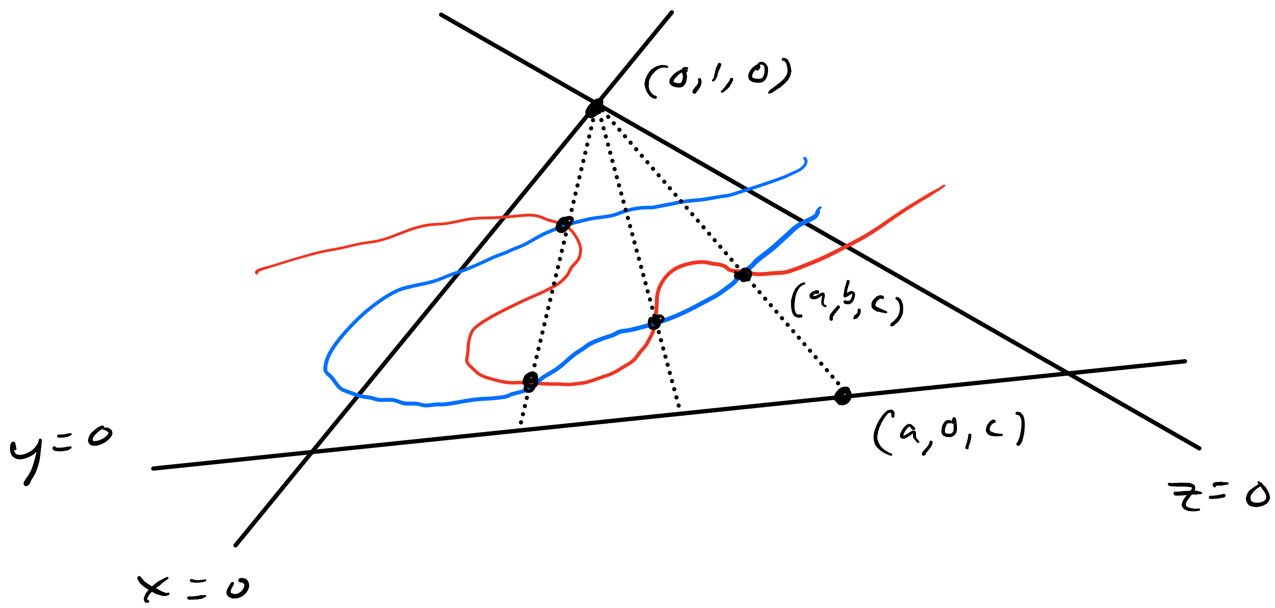
$\iff$ $F(a, y, c), G(a, y, c) \in \mathbb{C}[y]$ have common root $y = b$

$\iff$ $F(a, y, c), G(a, y, c)$ have common factor $(y - b)$.

To find such (a,b,c), assume that

$(0,1,0) \notin C$ or D.

Picture:



Consider the family of "vertical" lines.

$\exists$ point $(a,b,c) \in C \cap D$

$\iff$ $F(a,y,c), G(a,y,c) \in \mathbb{C}[y]$

have common factor

$\iff$ $H = \mathrm{Res}_y(F,G) \in \mathbb{C}[x,z]$

satisfies $H(a,c) = 0$.

How many solutions (a,c)?

Bézout's Theorem, Part I:

If $F, G \in \mathbb{C}[x,y,z]$ are homogeneous of degrees d, e then

$$H(x,z) = \mathrm{Res}_y(F, G)$$

is homogeneous of degree de.

Proof next time.

Corollary: If C, D have no common component, then they have finitely many (in fact, $\leq de$) points of intersection.

Corollary: Study's Lemma.

Proof next time.