Proof of Weierstrass Normal Form, continued...

(1) Any irreducible cubic can be expressed as $y^2 = x^3 + ax + b$.

<u>Proof</u>: Pick a simple inflection $\bar{p} \in C$ (which must exist), change variables to get

$$\bar{p} = (0,1,0)$$
$$T_{\bar{p}} C = V(z).$$

The rest is details (previous lecture).

(2) Let

$$C: y^2 = x^3 + ax + b$$
$$C': y^2 = x^3 + Ax + B.$$

Then $C \cong C' \iff A^3 b^2 = a^3 B^2$.

projective equivalence

[We will assume one fact without proof.
If $C \cong C'$ and $\bar{p} \in C$, $\bar{q} \in C'$ are simple inflections, then $\exists\, \varphi \in PGL$,

$\varphi : C \xrightarrow{\sim} C'$ & $\varphi(\bar{p}) = \bar{q}$.]

So let's assume $C \cong C'$ and choose an isomorphism $\varphi : C \to C'$ sending

$$(0,1,0) \longrightarrow (0,1,0)$$

Then φ also preserves the line at infinity $z=0$. It follows that φ restricted to x,y plane $(z=1)$ is an affine transformation:

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

with $\alpha\delta - \beta\gamma \neq 0$.

Let
$$X(x,y) = \alpha x + \beta y + \xi$$
$$Y(x,y) = \gamma x + \delta y + \eta$$

and define

$$f(x,y) = -Y^2 + AX + B \in \mathbb{C}[x,y]$$

By assumption ($\varphi : C \xrightarrow{\sim} C'$):

$$y^2 = x^3 + ax + b$$

$$\Longrightarrow \quad f(x,y) = 0$$

Since $-y^2 + x^3 + ax + b$ is irreducible, Study's Lemma implies that

$$f(x,y) = \text{const} \cdot (-y^2 + x^3 + ax + b)$$

Expanding f and comparing coefficients gives $\beta = \gamma = \xi = \eta = 0$

$$\& \quad \alpha^3 = \delta^2 \neq 0.$$

Let $t := \delta/\alpha$ so that

$$t^2 = \delta^2/\alpha^2 = \alpha^3/\alpha^2 = \alpha,$$

$$t^3 = \delta^3/\alpha^3 = \delta^3/\delta^2 = \delta.$$

We conclude that

$$\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t^2 & 0 \\ 0 & t^3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t^2 x \\ t^3 y \end{pmatrix}.$$

Finally, plugging this in gives

$$Y^2 = X^3 + AX + B$$

$$t^6 y^2 = t^6 x^3 + t^2 A x + B$$

$$y^2 = x^3 + \underbrace{\frac{A}{t^4}}_{a} x + \underbrace{\frac{B}{t^6}}_{b}$$

$$\Rightarrow A = t^4 a, \quad B = t^6 b$$

$$\Rightarrow A^3 b^2 = t^{12} a^3 b^2 = a^3 B^2.$$

And the converse is easy. ///

(3) The curve $y^2 = x^3 + ax + b$ is non-singular $\Longleftrightarrow 4a^3 + 27b^2 \neq 0$.

Proof: More generally, let

$$f(x,y) = -y^2 + g(x)$$

where $\deg(g) = 3$. I claim that

$V(f)$ is singular $\Longleftrightarrow$ $g, g' \in \mathbb{C}[x]$ not coprime

In which case, $g(x), g'(x)$ have a common prime factor, say $x-p$, hence a common root $g(p) = g'(p) = 0$.

To see this, note that

$$\nabla f = (g'(x), -2y)$$

If g, g' are not coprime, let

$g(p) = g'(p) = 0$, for some $p \in \mathbb{C}$.

Then $(p, 0)$ is a singular point of $V(f)$ because

$$(\nabla f)_{(p,0)} = (g'(p), -2 \cdot 0)$$
$$= (0, 0).$$

Conversely let (p, q) be a singular point of $V(f)$ so that

- $q^2 = g(p)$
- $(g'(p), -2q) = (0,0)$.

It follows that $g(p) = g'(p) = 0$, and hence $g(x)$, $g'(x)$ have the common factor $x-p$. ✓

Conclusion: $y^2 = x^3 + ax + b$ is singular if and only if the discriminant of $g(x) = x^3 + ax + b$ is zero:

$(g'(x) = 3x^2 + a)$

$$
\begin{aligned}
0 &= \mathrm{Disc}(g) \\
&= \mathrm{Res}(g, g') \\
&= \det \begin{pmatrix} 1 & 0 & a & b & \\ & 1 & 0 & a & b \\ 3 & 0 & a & & \\ & 3 & 0 & a & \\ & & 3 & 0 & a \end{pmatrix} \\
&= 4a^3 + 27b^2
\end{aligned}
$$

✓

So if $C: y^2 = x^3 + ax + b$ is non-singular then we can define the "j-invariant"

$$J(C) = \frac{4a^3}{4a^3 + 27b^2} \in \mathbb{C}$$

(4) Let $C: y^2 = x^3 + ax + b$

$C': y^2 = x^3 + Ax + B$.

Then we have

$$C \cong C' \iff J(C) = J(C').$$

Proof: Let $J(C) = J(C')$.

If $J(C) = 0$ then $a = A = 0$

$\Rightarrow A^3 b^2 = a^3 b^2 \Rightarrow C \cong C'$. ✓

If $J(C) \neq 0$ then $a, A \neq 0$

$$\Rightarrow J(C) = 4 / \left(4 + 27\left(\frac{b^2}{a^3}\right)\right)$$

$$J(C') = 4 / \left(4 + 27\left(\frac{B^2}{A^3}\right)\right)$$

Then $J(C) = J(C')$

$$\Rightarrow \frac{b^2}{a^3} = \frac{B^2}{A^3}$$

$$\Rightarrow A^3 b^2 = a^3 B^2$$

$$\Rightarrow C \cong C' \quad \checkmark$$

Conversely, let $C \cong C'$ so that $A^3 b^2 = a^3 B^2$. Since $4a^3 + 27b^2 \neq 0$ we know that a, b are not both zero.

If $a = 0$ then since $A^3 b^2 = 0$ and $b \neq 0$ we must have $A = 0$ and hence $J(C) = 0 = J(C')$.

A similar argument shows that

$$A = 0 \Rightarrow J(C) = J(C')$$

Therefore we may assume that $a, A \neq 0$. Then we have $b^2/a^3 = B^2/A^3$ and hence

$$J(C) = \frac{4}{4 + 27\frac{b^2}{a^3}} = \frac{4}{4 + 27\frac{B^2}{A^3}} = J(C').$$

///

Corollary: There exist infinitely many non-equivalent cubic curves.

$$\left\{ \begin{array}{l} \text{equivalence classes} \\ \text{of non-singular} \\ \text{cubic curves} \end{array} \right\} \longleftrightarrow \mathbb{C}.$$

Loose ends: We assumed without proof that $Aut(C) \subseteq PGL$ acts transitively on the inflection points.

To see this it's better to work with the "Hesse Normal Form":

$$x^3 + y^3 + z^3 = 3k\,xyz.$$

[See Milnor's Article.]

To find the inflection points, compute the Hessian:

$$H(x,y,z) = 216\,xyz$$
$$-72k^2(x^3+y^3+z^3).$$

If (p,q,r) is on the curve, this becomes

$$H(p,q,r) = 216pqr - 216k^3pqr$$
$$= 216(1-k^3)\,pqr.$$

How Nice!

If $k^3 \neq 1$ then we get exactly 9 inflection points:

$$(0,1,-\omega)$$
$$(-\omega,0,1) \qquad \omega^3 = 1.$$
$$(1,-\omega,0)$$

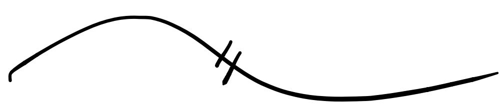
Conclusion: Consider $\varphi, \psi \in PGL$:

$$\varphi(x,y,z) = (y,z,x)$$
$$\psi(x,y,z) = (x,\omega y,\omega^2 z).$$

We observe that

- φ, ψ send points of the curve to points of the cure. i.e., $\varphi, \psi \in \mathrm{Aut}(C)$.

- φ, ψ act transitively on the set of 9 inflections.



One more remark: A non-singular cubic is often written as

$$C : y^2 = x(x-1)(x-\lambda)$$

with $\lambda \neq 0, 1$. [Exercise: Any non-singular cubic can be written in this form.] By sending $x \mapsto x + \frac{\lambda+1}{3}$

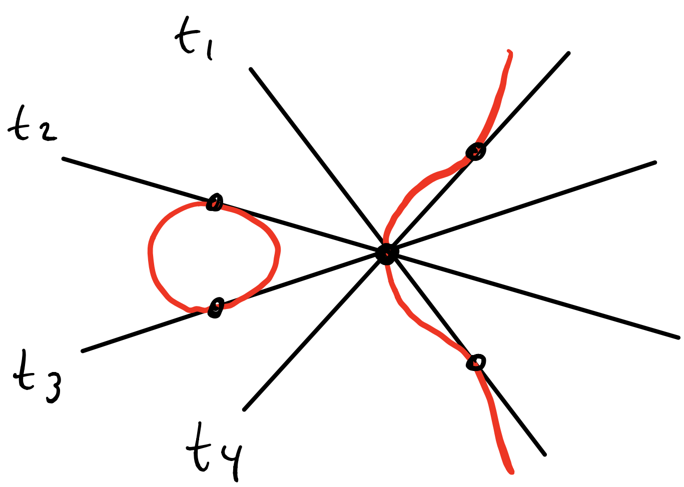
we get $y^2 = x^3 + ax + b$, where

$$a = \frac{-1+\lambda-\lambda^2}{3}, \quad b = -\frac{(\lambda+1)(\lambda-2)(2\lambda-1)}{27}$$

And we find that

$$J(C) = \frac{4a^3}{4a^3+27b^2} = \frac{4(\lambda^2-\lambda+1)^3}{27\lambda^2(1-\lambda)^2}.$$

This λ has a geometric interpretation as a "cross-ratio": Through a given point $\bar{p} \in C$ there are 5 lines tangent to C. Excluding the tangent at $\bar{p}$, let the other 4 lines have slopes t_1, t_2, t_3, t_4. Then



$$\lambda = \frac{(t_3-t_1)(t_4-t_2)}{(t_3-t_2)(t_4-t_1)}$$