Let S_d = the set of degree d homogeneous polynomials

$$F(x,y,z) = \sum_{i+j+k=d} a_{ijk}\, x^i y^j z^k$$

$$= (a_{ijk})_{i+j+k=d}$$

Two such collections of coefficients (a_{ijk}) define the same curve when they are the same up to scaling.

$$S_d \approx \text{set of degree } d \text{ curves in } \mathbb{CP}^2$$

$$\approx \mathbb{CP}^{\binom{2+d}{2}-1}$$

$$= \mathbb{CP}^{d(d+3)/2}.$$

Natural geometric structure on curves of a given degree.

If we require a curve to pass through a point (p,q,r) this gives a linear restriction on the coefficients:

$$\sum a_{ijk}\, p^i q^j r^k = 0$$

Hence

dim (curves of degree d going through a given point)

$$= \frac{d(d+3)}{2} - 1 .$$

More generally, if we require a curve to have a given point of multiplity $\geq m$, we need all $(m-1)^{th}$ order derivatives to vanish at this point. There are $m(m+1)/2$ such derivatives, leading to this many linear restrictions on the coefficients. Hence:

dim (space of deg d curves having mult $\geq m_i$ at given points $\overline{p_i}$)

$$\geq \frac{d(d+3)}{2} - \sum \frac{m_i(m_i+1)}{2} .$$

Conclusion: If

$$\sum \frac{m_i(m_i+1)}{2} \le \frac{d(d+3)}{2}$$

then there exists such a curve.

If $\sum \frac{m_i(m_i+1)}{2} > \frac{d(d+3)}{2}$ then there may or may not exist such a curve.

Illustration: Cramer - Euler Paradox.

A degree d curve is generally uniquely determined by $d(d+3)/2$ distinct points.

Cramer: $9 = \frac{3(3+3)}{2}$ distinct points generally determine a unique cubic curve. But two cubic curves generally intersect in $9 = 3 \cdot 3$ distinct points.

Paradox?

Euler (1750): It depends what you mean by "general."

e.g. The 9 points of intersection of two cubics are not in "3-general position."

Definition: Say points in $\mathbb{CP}^2$ are "d-generic" or "in d-general position" if their images under Veronese

$$\mathbb{CP}^2 \longrightarrow \mathbb{CP}^{d(d+3)/2}$$

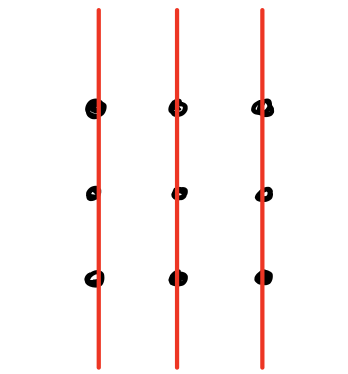
are linearly independent. Equivalently, passing though these points imposes independent linear restrictions on the space of degree d curves.

Example: Consider the polynomials

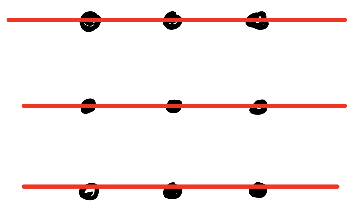
$$F_\lambda(x,y,z) = x(x+z)(x-z) + \lambda y(y+z)(y-z)$$

For any $\lambda \neq \mu$, the cubic curves F_λ & F_μ intersect in 9 distinct points: $(a,b,1)$, $a,b \in \{0,1,-1\}$.

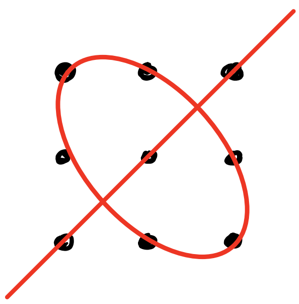
Pictures: For $\lambda \in \{0, \pm 1, \infty\}$ we get degenerate cubics:



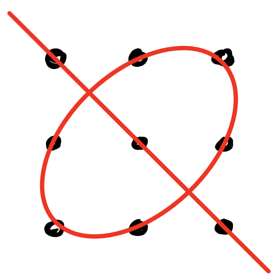
$\lambda = 0$: $x(x-1)(x+1)$



$\lambda = \infty$: $y(y+1)(y-1)$



$\lambda = 1$: $(x+y)(x^2 - xy + y^2 - 1)$



$\lambda = -1$: $(x-y)(x^2 + xy + y^2 - 1)$

$V(F_\lambda)$ is irreducible for $\lambda \notin \{0, \pm 1, \infty\}$.

In fact these are <u>all</u> of the cubic curves passing through these 9 points.

$\dim(\text{cubics through these 9 points}) = 1$.

Any projective 1D space of curves is called a "pencil".

Last time we proved: For nonsingular curve $\deg(C) = d \geq 3$ we have

$$1 \leq \# \mathrm{Flex}(C) \leq 3d(d-2).$$

In fact we can be more precise.
Let $C = V(F)$ be irreducible of degree $d \geq 3$ (possibly singular).
Let $H(x,y,z)$ be the Hessian determinant.
We have seen:

F irreducible of $\deg \geq 2$

$\Rightarrow F \nmid H$, so that

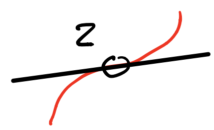
$$3d(d-2) = \deg(F)\deg(H)$$

$$= \sum_{p \in F \cap H} I_p(F,H)$$

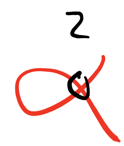
$$= \sum_{\substack{p \text{ singular} \\ \text{point of } F}} I_p(F,H) + \sum_{\substack{p \text{ is an} \\ \text{inflection of } F}} I_p(F,H)$$

By some tedious computation, one can show that

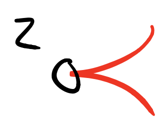
$I_p(F,H) = 1$ for simple inflections



$= 6$ for simple node



$= 8$ for simple cusp



If these are the <u>only</u> kinds of singularities & inflections (in which case we say C is a "Plücker curve") then we obtain a "Plücker formula":

$$\# \mathrm{Flex}(C) = 3d(d-2) - 6(\#\text{nodes}) - 8(\#\text{cusps})$$

Example: Every irreducible cubic curve is a Plücker curve, so that

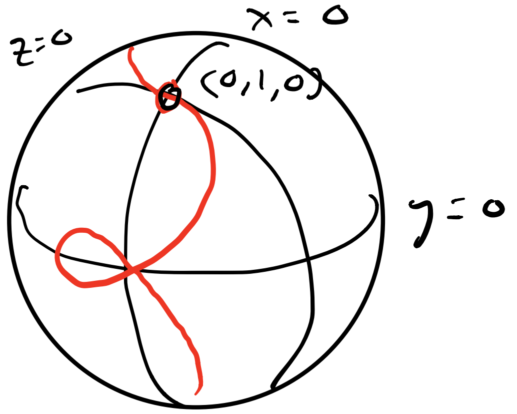
$$\# \mathrm{Flex}(C) = 9 - 6(\#\text{nodes}) - 8(\#\text{cusps}).$$

There are only three possibilities:

- 1 node, 3 inflections,
- 1 cusp, 1 inflection,
- nonsingular, 9 inflections.

WARNING: The real picture may not show all of the inflections. In fact, at most 3 inflections can be real!

e.g.



The nodal cubic has 3 inflections in $\mathbb{CP}^2$ but only 1 inflection in $\mathbb{RP}^2$.

We will use this idea to sketch the classification of cubic curves. For more details see Milnor, "On real & complex cubic curves."

Newton classified cubics in $\mathbb{R}^2$, which are rather complicated.
Much easier to classify cubics in $\mathbb{CP}^2$.

Theorem ("Weierstrass Normal Form"):
Every irreducible cubic C is projectively equivalent to

$$y^2 = x^3 + ax + b$$

for some $a, b \in \mathbb{C}$. Two such curves are projectively equivalent iff

$$A^3 b^2 = a^3 B^2. \quad (y^2 = x^3 + Ax + B)$$

Such a curve is nonsingular iff

$$4a^3 + 27b^2 \neq 0,$$

in which case we define the "j-invariant"

$$J(C) = \frac{4a^3}{4a^3+27b^2} \in \mathbb{C}.$$

Two nonsingular cubics are projectively equivalent iff they have same j-invariant.

Note that every value of $J(C)$ occurs, so

$$\left\{\begin{array}{l}\text{projective equiv} \\ \text{classes of} \\ \text{non-sing. cubics}\end{array}\right\} \longleftrightarrow \mathbb{C}$$

Remark: very different from conics!

[Notation: The standard "j-invariant" is $j(C) = 1728\, J(C)$. The 1728 is related to char $\mathbb{F} = 2, 3$.]

To prove this, first observe that

$$y^2 z = x^3 + axz^2 + bz^3$$

has a simple inflection at $(0,1,0)$ with tangent line $z=0$. Indeed, in the chart $y \neq 0$ (i.e. $y=1$) have

$$z = x^3 + axz^2 + bz^3$$

$$0 = -z + 0 + \underbrace{x^3 + axz^2 + bz^3}_{\substack{\text{not divisible} \\ \text{by } z \ \checkmark}}$$

nonsingular with tangent $z=0$

divisible by z ✓

The idea of the proof is to let $\bar{p} \in C$ be any simple inflection point (which exists by previous remarks). Change coords so that

$$\bar{p} = (0,1,0),$$
$$T_{\bar{p}}C = (z=0).$$

In these coords, our curve is defined by

$$0 = -z + \underbrace{z\varphi(x,z)}_{\text{div by } z} + \underbrace{x^3 + z\Phi(x,z)}_{\text{not div by } z}.$$

where φ, Φ are hom of degrees $1, 2$.

Homogenize:

$$F = -zy^2 + z\varphi(x,y,z) + x^3 + z\Phi(x,z)$$

Now let $z=1$:

$$f = -y^2 + \varphi(x,y) + x^3 + \Phi(x)$$

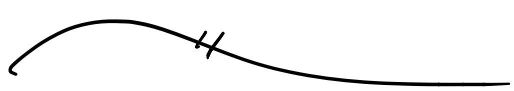
$$= -y^2 + y(sx+t) + x^3 + px^2 + qx + r$$

Replace y by $y + (sx+t)/2$:

$$f = -y^2 + x^3 + p'x^2 + q'x + r'$$

Replace x by $x - p'/3$:

$$f = -y^2 + x^3 + ax + b.$$



Proof continued next time ...