We showed last time: If C is irreducible curve of degree d, then

$$\#\text{ singular points} \leq \frac{(d-1)(d-2)}{2}.$$

Not surprisingly, we can refine this by including multiplicities.

Theorem: Let C be curve of degree d with singular points $\overline{p}_1, \ldots, \overline{p}_k$ of multiplicity $m_i := \mathrm{mult}_{\overline{p}_i}(C) \geq 2$.

(1) $$\sum m_i(m_i - 1) \leq d(d-1)$$

(2) If C is irreducible then

$$\sum m_i(m_i-1) \leq (d-1)(d-2).$$

And both bounds are sharp. ///

Proof Sketch (see Walker, pg. 65):
Let $C = V(F)$, F square-free.

(1) If $\text{mult}_{\bar{p}_i}(C) = m_i$ then

$$\text{mult}_{\bar{p}_i}(V(F_x)) \geq m_i - 1$$

So from Bézout,

$$\deg(F)\deg(F_x) = \sum I_{\bar{p}_i}(F, F_x)$$

$$d(d-1) \geq \sum \text{mult}_{\bar{p}_i}(F)\,\text{mult}_{\bar{p}_i}(F_x)$$

$$\geq \sum m_i(m_i - 1).$$

(2) Let C be irreducible. From (1):

$$\sum \frac{m_i(m_i-1)}{2} \leq \frac{d(d-1)}{2}$$

$$\leq \frac{(d+2)(d-1)}{2} = \frac{(d-1)(d-1+3)}{2}$$

Hence $\exists$ curve D of degree $d-1$ having $\bar{p}_i$ as a point of mult m_i and passing through any further

$$\frac{(d-1)(d+2)}{2} - \sum \frac{m_i(m_i-1)}{2}$$

given non-singular points of C.

Then from Bézout:

$$d(d-1) = \deg(C)\deg(D)$$

$$\geq \sum m_i(m_i-1) + \frac{(d-1)(d+2)}{2} - \sum \frac{m_i(m_i-1)}{2}$$

miracle!

$$\frac{(d-1)(d-2)}{2} \geq \sum \frac{m_i(m_i-1)}{2}$$

///

Exercise: To see that this is sharp, show that $F(x,y,z) = x^d - y^{d-1}z$ is irreducible. Since it has a point of multiplicity $d-1$ at $(0,0,1)$ we get $\sum m_i(m_i-1) = (d-1)(d-2)$.

There are no other singular points

Interesting Remark:

We assumed char $\mathbb{F} = 0$.

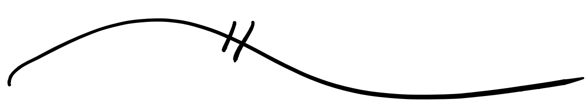
Given irreducible curve C of degree d, we define

$$g(C) := \frac{(d-1)(d-2)}{2} - \sum \frac{m_i(m_i-1)}{2} \geq 0.$$

We will see later that this g is the topological genus of the complex curve, thought of as a (connected, compact, orientable) 2D surface living in $\mathbb{RP}^4$.

In particular, if C is smooth then

$$g(C) = \frac{(d-1)(d-2)}{2}.$$



Examples:

- $\deg(C) = 2$

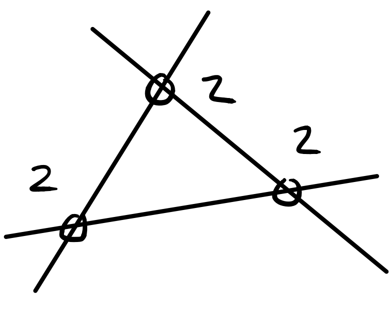
Reducible: $\sum m_i(m_i - 1) \leq 2(2-1)$.

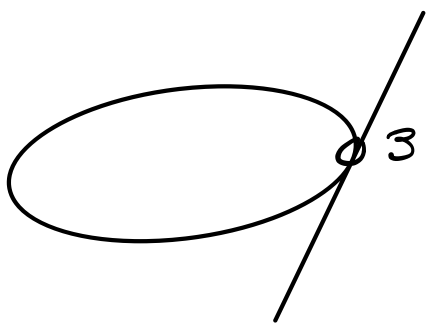
Irreducible: $\sum m_i(m_i - 1) \leq 1(0) = 0$.

$\Rightarrow$ no singularities.

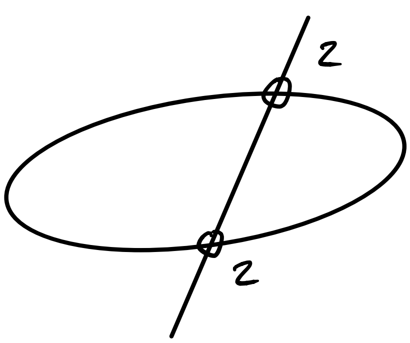
- $\deg(C) = 3$

Reducible: $\sum m_i(m_i - 1) \leq 3 \cdot 2 = 6$





$3 \cdot 2(2-1) = 6$ $\qquad$ $1 \cdot 3(3-1) = 6$

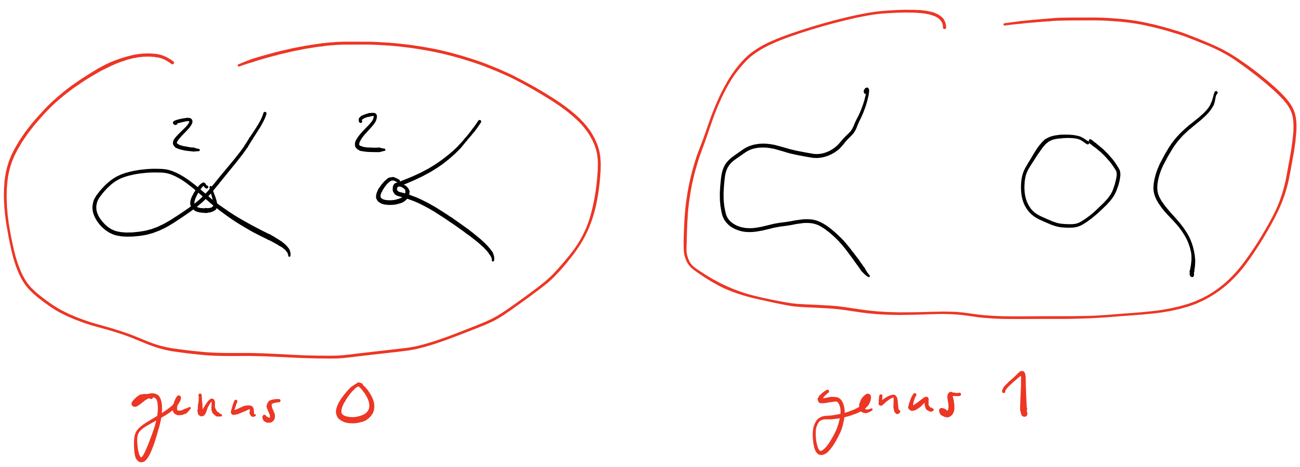


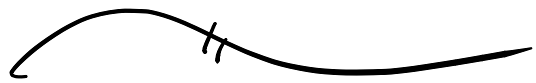
$2 \cdot 2(2-1) = 4 \leq 6$

✓

Irreducible : $\sum m_i(m_i - 1) \leq 2 \cdot 1 = 2$.

At most one singular point, of multiplicity 2.





Inflection points & Classification of Cubic Curves.

Recall that a point $\bar{p}$ is an inflection point of curve $C = V(F) \iff$

$$F(\bar{p}) = 0$$

$$(\nabla F)_{\bar{p}} \neq (0,0,0)$$

$$\det(HF)_{\bar{p}} = 0.$$

We can use Bézout's Theorem to count inflection points.

Define the Hessian curve of $C = V(F)$ as $V(H)$ where

$$H(x,y,z) = \det \begin{pmatrix} F_{xx} & F_{xy} & F_{zz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}$$

Claim: If $\deg(C) = d$ then either $H \equiv 0$ or H is homogeneous of degree $3(d-2)$.

<u>Proof</u>: For any i, j the second derivative F_{ij} is homogeneous of degree $d-2$, so that

$$F_{ij}(\lambda \bar{x}) = \lambda^{d-2} F_{ij}(\bar{x}).$$

It follows that

$$H(\lambda \bar{x}) = (\lambda^{d-2})^3 H(\bar{x})$$

$$= \lambda^{3(d-2)} H(\bar{x}) \qquad ///$$

Exercise: Use Euler's identity

$$xF_x + yF_y + zF_z = dF$$

to show that

$$H = \frac{d-1}{z} \det \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_x & F_y & F_z \end{pmatrix} \qquad (*)$$

$$= \frac{(d-1)^2}{z^2} \det \begin{pmatrix} F_{xx} & F_{xy} & F_x \\ F_{yx} & F_{yy} & F_y \\ F_x & F_y & dF/d-1 \end{pmatrix} \qquad (**)$$

It follows from $(*)$ that

$$(\nabla F)_{\bar{p}} = (F_x(\bar{p}), F_y(\bar{p}), F_z(\bar{p})) = (0,0,0)$$

implies $H(\bar{p}) = 0$.

In other words, every singular point of $C = V(F)$ is on the Hessian curve:

$$V(F) \cap V(H) = \left\{ \begin{matrix} \text{singular} \\ \text{points} \end{matrix} \right\} \cup \left\{ \begin{matrix} \text{inflection} \\ \text{points} \end{matrix} \right\}$$

And if C is non-singular then

$$V(F) \cap V(H) = \{ \text{inflections} \}.$$

Theorem: Let $C = V(F)$ be non-singular (hence irreducible) of degree $d \geq 3$. Then we have

$$1 \leq \# \text{inflections} \leq 3d(d-2).$$

Proof: Let $V(H)$ be the Hessian curve. If we can show that

$$H \not\equiv 0 \text{ and } F, H \text{ coprime}$$

then the result will follow from Bézout:

$$1 \leq \# V(F) \cap V(H) \leq \deg(F)\deg(H)$$
$$= d\left(3(d-2)\right).$$

QED.

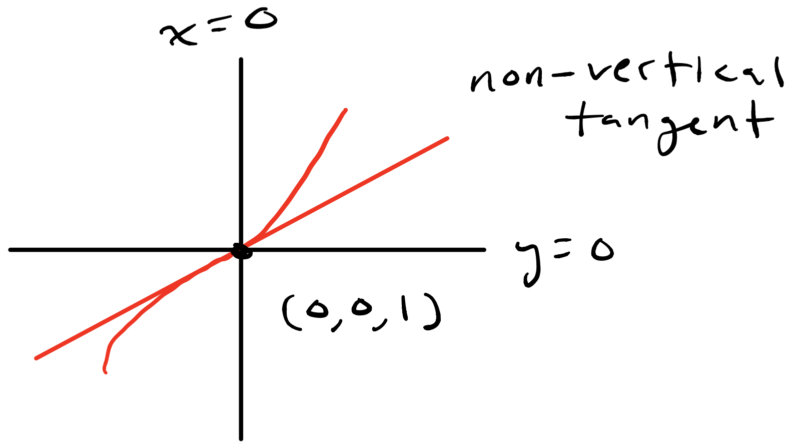
We still need to prove a lemma.

Lemma: Let F be irreducible. Then $F \mid H \iff \deg F = 1$.

i.e., every point of $V(F)$ is an inflection (or singular) if and only if $V(F)$ is a line.

Proof: One direction is clear. For the other direction suppose $F \mid H$. By choosing $\varphi \in PGL$ we can assume that $F(0,0,1) = 0$,
$$F_y(0,0,1) \neq 0.$$

Picture:



Implicit Function Theorem: We can regard y locally as a holomorphic function of x, satisfying

$$F(x,y,1) = 0$$

By implicit differentiation:

$$0 = dF = F_x\,dx + F_y\,dy$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$\Rightarrow \quad F_x + \frac{dy}{dx} F_y = 0.$$

Differentiate again, apply formula (**) from exercise, and a lot of manipulations to show that

$$\frac{d^2y}{dx^2} = \frac{H(x,y,1)}{(d-1)^2 (F_y)^3}$$

Since $F \mid H$ we get $H(0,0,1) = 0$,

hence $\frac{d^2y}{dx^2} = 0$.

Since $y(x)$ is holomorphic (determined by its Taylor series) this implies that $y(x) = \lambda x$ for some $\lambda \in \mathbb{C}$.

For all x, y near 0 we have

$$F(x,y,1) = F(x,\lambda x,1) = 0.$$

But $F(x,\lambda x,1) = 0$ is a polynomial. Since it's zero for so many values of x we get

$$F(x,\lambda x,1) \equiv 0$$

$$\Rightarrow \quad y - \lambda x \mid F(x,y,1)$$

Finally, since F is irreducible this implies $\deg(F) = 1$.

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