Language of "affine varieties" continued . . .

Zariski Topology on $\mathbb{F}^n$:

Two maps

$$V : \begin{matrix} \text{ideals of} \\ \mathbb{F}[x_1, \dots, x_n] \end{matrix} \rightleftarrows \begin{matrix} \text{subsets} \\ \text{of } \mathbb{F}^n \end{matrix} : I$$

NOT INVERSE, but $VI$ & $IV$ are so-called "closure operators." Closed subsets of $\mathbb{F}^n$ ( $S = VI(S)$ ) are called "Zariski closed."

This is a topology on $\mathbb{F}^n$.

Proof: Most properties are automatic. Only one property refers to polynomials or ideals — finite union of closed sets is closed. This follows from the identity

$$V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$$

Exercise: Prove this identity. ///

Nullstellensatz: The maps $V, I$ restrict to a bijection

$$\begin{array}{c} \text{varieties} \\ \text{in } \mathbb{F}^n \end{array} \longleftrightarrow \begin{array}{c} \text{radical ideals} \\ \text{of } \mathbb{F}[x_1, \dots, x_n] \end{array}$$

Prop: Prime ideals are radical.

Proof: Let $P$ be prime. Always have $P \subseteq \sqrt{P}$. Conversely, suppose that

$g \in \sqrt{P}$. By definition this means that $g^k \in P$ for some $k \geq 1$. Since $P$ is prime, we have $g$ or $g^{k-1}$ is in $P$.

But $g^{k-1} \in P \Rightarrow g$ or $g^{k-2} \in P$.

By induction, $g \in P$. ///

Question:

prime ideals of $\mathbb{F}[x_1, \dots, x_n]$ $\longleftrightarrow$ what kind of varieties ?

Definition: Say variety is reducible if $\exists\ V = V_1 \cup V_2$ with $V_1, V_2 \neq V$.

Proposition: Given pair $V \leftrightarrow I$ we have

$$V \text{ irreducible} \iff I \text{ prime}.$$

Proof: Exercise. ///

Furthermore, we have "unique factorization" theorems:

- variety has a unique expression is union of irreducible varieties.
- radical ideal has unique expression as intersection of prime ideals.

Proof: Purely combinatorial. Exercise. ///

Min & Max primes/varieties:

Prop: $R$ UFD $\Rightarrow$ prime ideal is minimal if and only if principal.

Proof: Exercise. ///

[Remark: If $R$ Noetherian then converse also holds. Harder to prove. Related to non-singularity of varieties.]

Prop: If $\mathbb{F}$ is algebraically closed then

max (prime) ideals of $\mathbb{F}[x_1, \dots, x_n]$ $\longleftrightarrow$ $M_{\bar{a}}$ for $\bar{a} \in \mathbb{F}^n$

Where $M_{\bar{a}}$ = ideal of the point $\bar{a} \in \mathbb{F}^n$

$= \{ f : f(\bar{a}) = 0 \}$

$= (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$.

Proof: Last time we proved that $M_{\bar{a}}$ are maximal ideals. Conversely, let $M \subseteq \mathbb{F}[x_1, \dots, x_n]$ be any maximal ideal, hence prime, hence radical.

Claim: $V(M) \neq \emptyset$.

Follows from NSS because

- $V$ injective on radical ideals
- $\mathbb{F}[\bar{x}] \subseteq \mathbb{F}[\bar{x}]$ is radical
- $V(\mathbb{F}[\bar{x}]) = \emptyset$
- $M \subsetneq \mathbb{F}[\bar{x}]$ by definition of "maximal ideal." ///

Therefore, $\exists$ point $\bar{a} \in V(M)$.

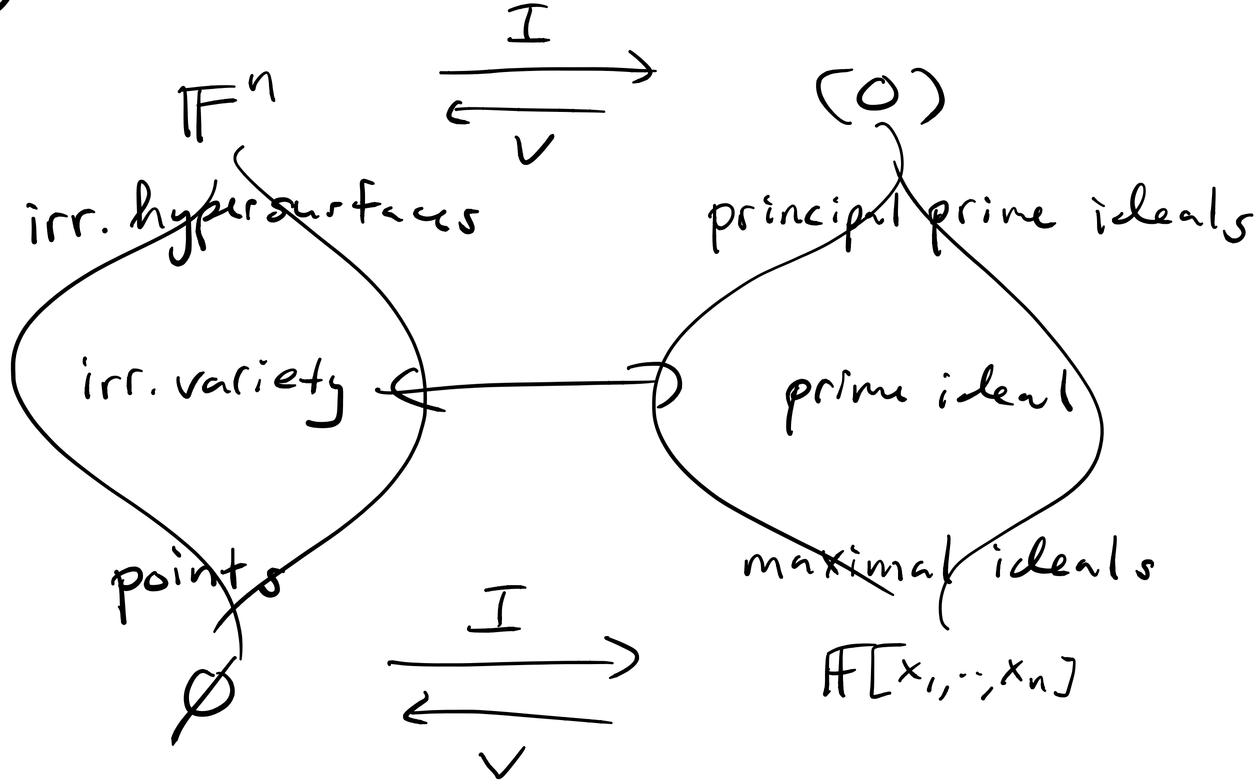
Apply (order reversing) map $I$:

$$M_{\bar{a}} = I(\{\bar{a}\}) \supseteq IV(M) = M$$

Finally, since $M$ is maximal, have

$$M = M_{\bar{a}}. \qquad ///$$

## Big Picture:




**Definition**: The <u>dimension</u> of an irreducible variety $V$ is min $d$ such that there exist varieties

$$\emptyset \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V.$$

Makes Sense!

**Theorem**: $\dim(\mathbb{F}^n) = n$.

Surprisingly hard to prove!

See Arrondo: Geometric introduction

to Commutative Algebra for a nice treatment. (Prop 6.18, pg 62)

Remark: "Dimension theory" is difficult & technical. Likely we will only consider this in the case of "one dimensional" varieties.

Examples ($n=1$): Varieties $\subseteq \mathbb{F}^1$ are

- $\emptyset$
- finite sets of points
- all of $\mathbb{F}^1$

($n=2$): Varieties $\subseteq \mathbb{F}^2$ (alg closed!)

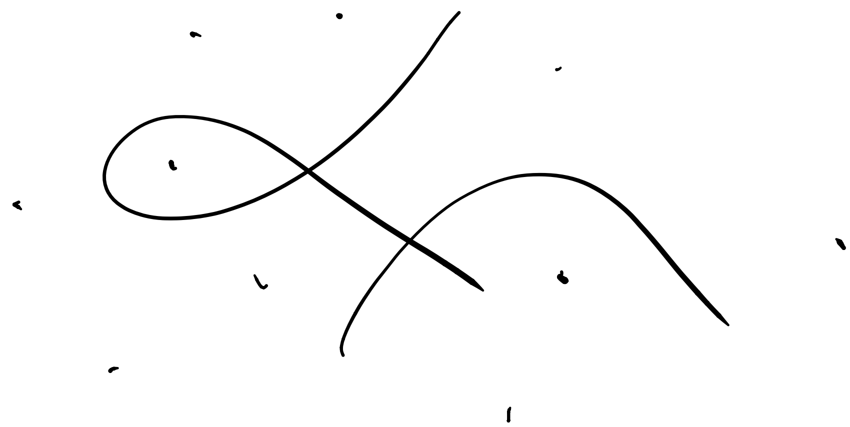
- $\emptyset$
- points
- irreducible curves

  } finite unions of these
- all of $\mathbb{F}^2$

The proof that there is nothing else

is nontrivial. We'll see it later.

"Picture" of closed subset of $\mathbb{F}^2$:



(Two irreducible curves and some points. Note that irreducible components of a curve <u>need not</u> be disjoint. In fact they <u>never</u> will be in $\mathbb{CP}^2$.)

Of course this is just a picture in $\mathbb{R}^2$. The true "picture" is in $\mathbb{C}^2$ or $\mathbb{CP}^2$, so we can't really "see" it.

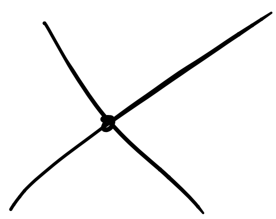
Intersections with $\mathbb{R}^2$ can be deceptive, e.g., might even have the wrong dimension.

Example: $f(x,y) = x^2+y^2 = (x-iy)(x+iy)$

The curve in $\mathbb{C}^2$ is a pair of intersecting lines

$$V(f) = V(x^2+y^2)$$
$$= V(x-iy) \cup V(x+iy).$$

"Picture":



But the real picture is just a single point $\{(0,0)\} = V(x^2+y^2) \subseteq \mathbb{R}^2$, which has the "wrong dimension."

This causes algebraic problems:

Study's Lemma: Given $f, g \in \mathbb{F}[\bar{x}]$,

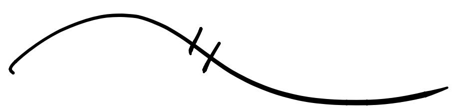
$$\begin{array}{l} f \text{ irreducible} \\ \& \ V(f) \subseteq V(g) \end{array} \Rightarrow f \mid g \text{ in } \mathbb{F}[\bar{x}]$$

But if $\mathbb{F} = \mathbb{R}$, $f = x^2+y^2$, $g = x$, then $f$ is irreducible in $\mathbb{R}[x,y]$,

$$V(f) \subseteq V(g)$$

point $(0,0)$      $y$-axis

and $f \nmid g$ $(x^2+y^2 \nmid x)$ because $\deg(pq) = \deg(p) + \deg(q)$ for all polynomials over a domain.



$(n=3)$: Closed sets in $\mathbb{F}^3$ are

- $\emptyset$
- points      $(x-a, y-b, z-c)$
- curves      ?
- surfaces      $(f(x,y,z))$
- all of $\mathbb{F}^3$

Example of a curve in $\mathbb{F}^3$:

$$C = \{(t, t^2, t^3) : t \in \mathbb{F}\}$$

"Twisted cubic curve"

Theorem: $C$ is an irreducible variety with prime ideal

$$I(C) = (x^2 - y, x^3 - z).$$

Proof is surprisingly difficult!
(see last semester's notes)