Recall: Given field $\mathbb{F}$ we define the projective plane $\mathbb{FP}^2$ as follows:

$$\begin{aligned}
\mathbb{FP}^2 &= \text{lines in } \mathbb{F}^3 \text{ through origin} \\
&= (\mathbb{F}^3 - \bar{0}) / \text{scalar} \\
&= \frac{\{(x,y,z) : x,y,z \in \mathbb{F} \text{ not all zero}\}}{(x,y,z) = (\lambda x, \lambda y, \lambda z) \ \forall \lambda \neq 0.}
\end{aligned}$$

The projective plane is covered by 3 "affine charts," each "isomorphic" to the affine plane $\mathbb{F}^2$:

$$\left.\begin{aligned}
\mathcal{U}_x &= \{(1,y,z) : y,z \in \mathbb{F}\} \\
\mathcal{U}_y &= \{(x,1,z) : x,z \in \mathbb{F}\} \\
\mathcal{U}_z &= \{(x,y,1) : x,y \in \mathbb{F}\}
\end{aligned}\right\} \approx \mathbb{F}^2$$

The complements

$$\begin{aligned}
H_x &= \mathbb{FP}^2 - \mathcal{U}_x \\
H_y &= \mathbb{FP}^2 - \mathcal{U}_y \\
H_z &= \mathbb{FP}^2 - \mathcal{U}_z
\end{aligned}$$

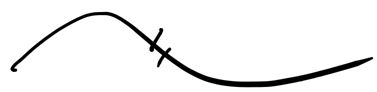
are called "coordinate lines at ∞."

Each is "isomorphic" to a projective line:

$$H_x = \{(x,y,z) \in \mathbb{FP}^2 : x = 0\}$$

$$\hookrightarrow \{(y,z) : (y,z) = (\lambda y, \lambda z)\ \forall\ \lambda \neq 0\}.$$

$$\approx \mathbb{FP}^1.$$

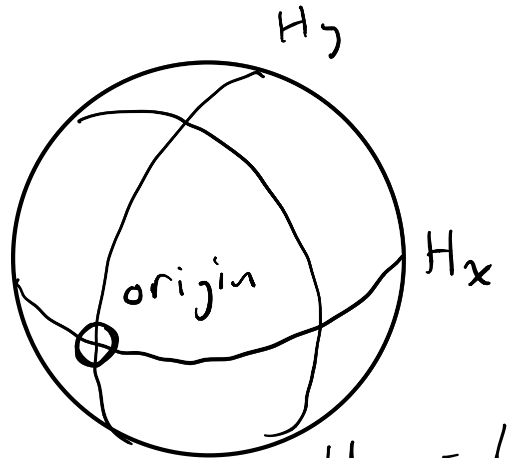


Picture of $\mathbb{RP}^2$:

$\mathbb{RP}^2$ = lines in $\mathbb{R}^3$ through $\bar{O}$.

Intersect each line with a sphere S^2 centered at $\bar{O}$. Then

points of $\mathbb{RP}^2$ = antipodal pairs of points of S^2.

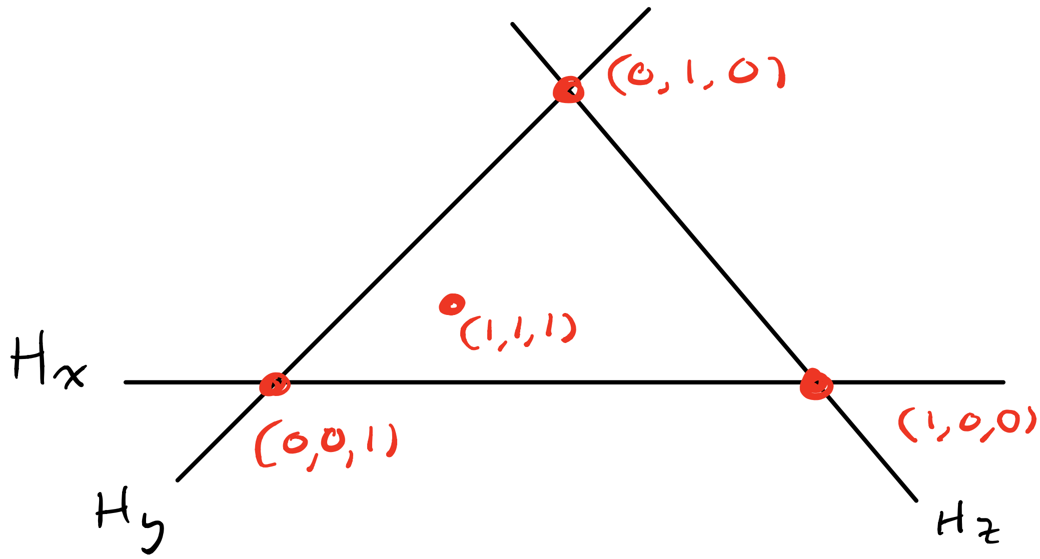


$\mathcal{U}_x, \mathcal{U}_y, \mathcal{U}_z$ are "open hemispheres"

H_x, H_y, H_z are "coordinate axes

H_z = line at ∞.

Compare to the more abstract picture from last time:



Automorphisms:

Fundamental Theorem:

$$\mathrm{Aut}(\mathbb{FP}^2) = GL_3(\mathbb{F})/\text{scalars}$$
$$= PGL_3(\mathbb{F}),$$

the group of "symmetries" of $\mathbb{FP}^2$ in any reasonable sense.

Theorem: Bijection

$$PGL_3(\mathbb{F}) \leftrightarrow \text{choices of coordinates.}$$

Proof: Group acts "simply, transitively" on sets of (4 points, no 3 collinear).

Given points $P, Q, R, S \in \mathbb{F}P^2$, no 3 collinear, want to find $\varphi \in PGL_3(\mathbb{F})$

$$P, Q, R, S \xrightarrow{\varphi} \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$$

Note: Since P, Q, R not collinear $\exists!$

$$S = aP + bQ + cR \quad (a+b+c=1)$$

[Möbius' "barycentric coordinates"]

Let A be 3×3 matrix with columns aP, bQ, cR, so that

$$A\begin{pmatrix}1\\0\\0\end{pmatrix} = aP \sim P$$
$$A\begin{pmatrix}0\\1\\0\end{pmatrix} = bQ \sim Q$$
$$A\begin{pmatrix}0\\0\\1\end{pmatrix} = cR \sim R$$

Finally:

$$A\begin{pmatrix}1\\1\\1\end{pmatrix} = A\begin{pmatrix}1\\0\\0\end{pmatrix} + A\begin{pmatrix}0\\1\\0\end{pmatrix} + A\begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$= aP + bQ + cR = S.$$

Define $\varphi \in PGL_3(\mathbb{F})$ by equivalence class of matrix A^{-1}.

What is an "algebraic curve" in $\mathbb{FP}^2$?

Since $\mathbb{FP}^2 = (S^2 \subseteq \mathbb{F}^3)$ with antipodal points identified, curve in $\mathbb{FP}^2$ should correspond to surface in $\mathbb{F}^3$ that is "centrally symmetric," i.e., closed under scalar multiplication.

Surfaces in $\mathbb{F}^3$ look like

$$V(f) = \{(a,b,c) : f(a,b,c) = 0\}$$

for some polynomial $f(x,y,z) \in \mathbb{F}[x,y,z]$.

When is this centrally symmetric?

Theorem: $V(f) \subseteq \mathbb{F}^3$ is centrally symmetric $\Longleftrightarrow$ f is "homogeneous."

hard direction uses algebraic closure.

i.e., $\exists\, d$ such that

$$f(x,y,z) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=d}} a_{ijk}\, x^i y^j z^k$$

In this case we say f is "homogeneous of degree d."

<u>Proof</u>: If f is hom. of degree d, then for any $\lambda \in \mathbb{F}$ we have

$$f(\lambda x, \lambda y, \lambda z) = \sum a_{ijk} (\lambda x)^i (\lambda y)^j (\lambda z)^k$$

$$= \underbrace{\lambda^i \lambda^j \lambda^k}\, \sum a_{ijk}\, x^i y^j z^k$$

$$= \lambda^d f(x,y,z).$$

Therefore, if $\bar{p} \in V(f)$ so $f(\bar{p}) = 0$, then for all $\lambda \in \mathbb{F}$ we have

$$f(\lambda \bar{p}) = \lambda^d f(\bar{p}) = \lambda^d \cdot 0 = 0,$$

so $\lambda \bar{p} \in V(f)$.

Conversely, suppose that $V(f)$ satisfies $\bar{p} \in V(f) \Rightarrow \lambda \bar{p} \in V(f)$.

Write f as a unique sum of homogeneous polynomials:

$$f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$$

From above remarks,

$$f(\lambda \bar{x}) = \sum \lambda^k f^{(k)}(\bar{x})$$

[From now on we will assume that $\mathbb{F}$ is algebraically closed, hence also infinite.]

Now suppose $f(\bar{p}) = 0$ so that also $f(\lambda \bar{p}) = 0 \ \forall \lambda$:

$$0 = f(\lambda \bar{p}) = \sum \lambda^k f^{(k)}(\bar{p}).$$

Define $g_{\bar{p}}(y) := \sum y^k f^{(k)}(\bar{p}) \in \mathbb{F}[y]$.

Since $g_{\bar{p}}(y)$ has infinitely many roots $\lambda \in \mathbb{F}$ we must have $g_{\bar{p}}(y) \equiv 0$, i.e., each coefficient is zero:

$$f(\bar{p}) = 0 \Rightarrow f^{(k)}(\bar{p}) = 0 \ \forall k.$$

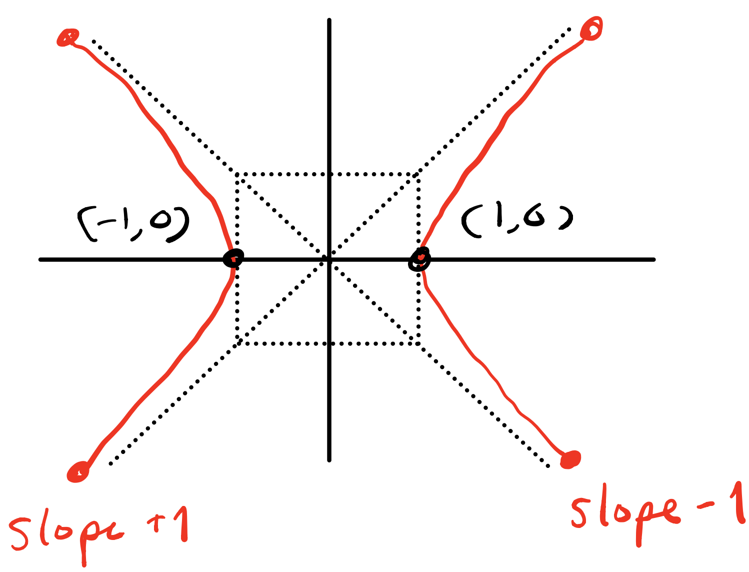
Since $\mathbb{F}$ is alg closed, Study's Lemma tells us that $f \mid f^{(k)}$ in ring $\mathbb{F}[x,y,z]$ for all k. Because $\deg(f) = d$ & $\deg(f^{(k)}) = k \leq d$, this implies that

$$f^{(0)}, f^{(1)}, \ldots, f^{(d-1)} \equiv 0.$$

Conclude that $f = f^{(d)}$, hence f is homogeneous. ///

Example: $f(x,y) = x^2 - y^2 - 1$.

$V(f) \subseteq \mathbb{F}^2$ is a hyperbola:



two points at infinity, corresponding to "slope ± 1"

Algebraically? Need a homogeneous polynomial $F(x,y,z) \in \mathbb{R}[x,y,z]$ satisfying $F(x,y,1) = f(x,y)$

[restriction to affine patch $\mathcal{U}_z$ is our hyperbola.]

Any guesses? $F(x,y,z) = x^2 - y^2 - z^2$ will work. Also,

$$F(x,y,z) = (x^2 - y^2 - z^2)\, z^k$$

for any $k \geq 0$ will work. Choose the smallest such k.

Def: If $f(x,y)$ has deg d, then define the "homogenization":

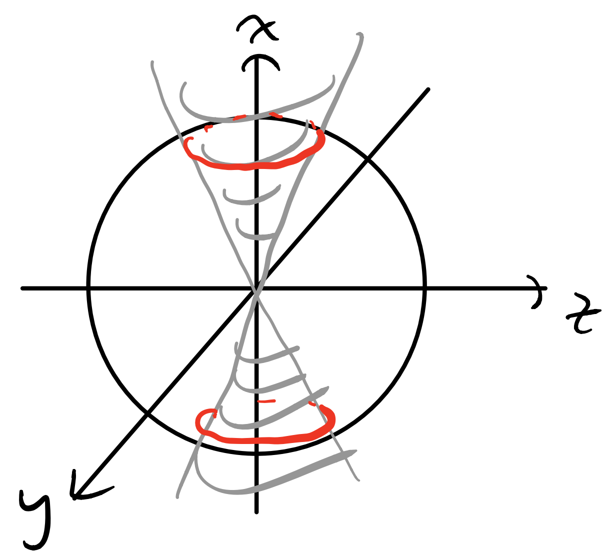
$$F(x,y,z) := z^d f\left(\frac{x}{z}, \frac{y}{z}\right).$$

The "projectivization" $V(F) \subseteq \mathbb{RP}^2$ of the hyperbola can be viewed as intersection of sphere S^2 with the centrally symmetric surface

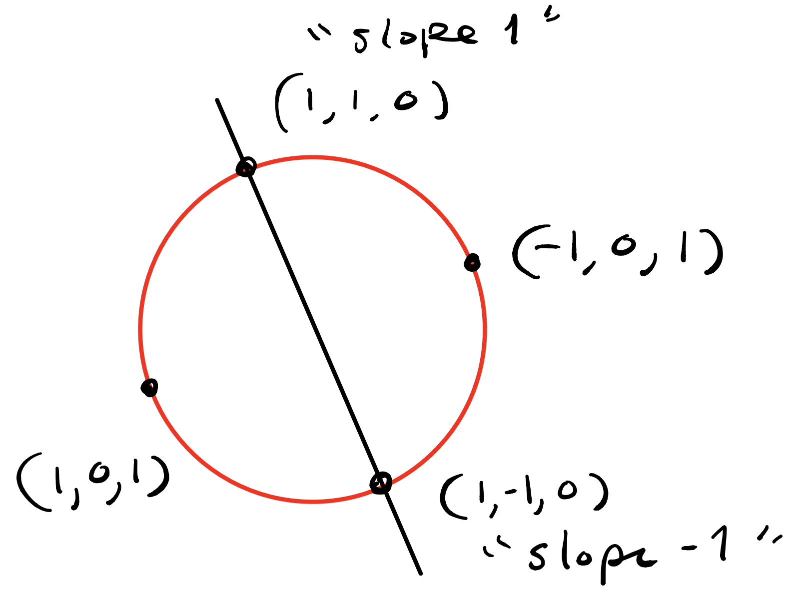
$$F(x,y,z) = 0$$

$$x^2 - y^2 - z^2 = 0,$$

which is a circular cone:



Curve is a pair of antipodal "circles" on the sphere, passing through "line at infinity" $H_z = \{(x,y,0)\}$ in two points.



Any line in $\mathbb{FP}^2$ can equally well be the line at infinity.

Automorphisms of $\mathbb{FP}^2$ act on curves and polynomials.

Given homogeneous polynomial

$$F(x,y,z) = F(\bar{x})$$

and automorphism $\varphi \in PGL_3(\mathbb{F})$, define the polynomial

$$F^{\varphi}(\bar{x}) := F(\varphi^{-1}(\bar{x})).$$

Claim: $\varphi(V(F)) = V(F^{\varphi})$

Proof: suppose $F(\bar{a}) = 0$, so $\varphi(\bar{a}) \in \varphi(V(F))$. Then

$$\begin{aligned} F^{\varphi}(\varphi(\bar{a})) &= F(\varphi^{-1}(\varphi(\bar{a}))) \\ &= F(\bar{a}) = 0, \end{aligned}$$

so that $\varphi(\bar{a}) \in V(F^{\varphi})$.

Conversely, if $\bar{a} \in V(F^{\varphi})$ then

$$F^{\varphi}(\bar{a}) = 0$$
$$F(\varphi^{-1}(\bar{a})) = 0$$
$$\varphi^{-1}(\bar{a}) \in V(F)$$
$$\bar{a} \in \varphi(V(F)).$$

///

Two curves / hypersurface that differ by an element of PGL are called "projectively equivalent"

Fact: Any two lines in $\mathbb{F}\mathbb{P}^2$ are projectively equivalent.

Meaning: Any line in $\mathbb{F}\mathbb{P}^2$ can be the line at infinity.