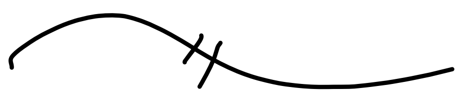
Last time we proved that a smooth plane curve $C \subseteq \mathbb{CP}^2$ of degree $d$ has genus

$$g = \frac{(d-1)(d-2)}{2}.$$

Today we will discuss this formula.



Corollary: Harnack's Theorem.

Let $F(x,y,z) \in \mathbb{R}[x,y,z]$ be homogeneous of degree $d$, irreducible & non-singular in $\mathbb{C}[x,y,z]$.

Consider the real & complex projective curves:

$$\begin{array}{ccc} \mathbb{RP}^2 & \subseteq & \mathbb{CP}^2 \\ \cup\!\!\mid & & \cup\!\!\mid \\ C(\mathbb{R}) & \subseteq & C(\mathbb{C}) \end{array}$$

By assumption, $C(\mathbb{R}) \subseteq \mathbb{RP}^2$ is smooth and compact, so consists of some finite number of closed loops. I claim that

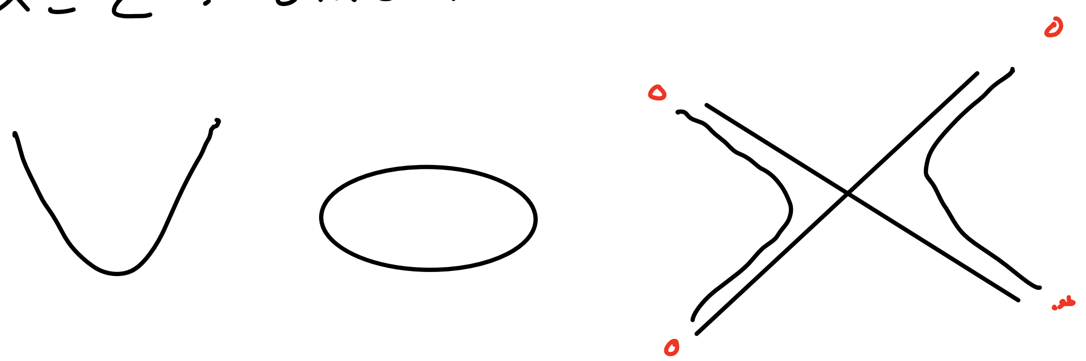
$$\begin{array}{l}\#\text{ loops} \\ \text{of } C(\mathbb{R})\end{array} \leq \frac{(d-1)(d-2)}{2} + 1.$$

//

[In fact this is true for <u>any</u> homogeneous polynomial in $\mathbb{R}[x,y,z]$ of degree $d$, but the proof will follow from the generic case above.]

Examples:

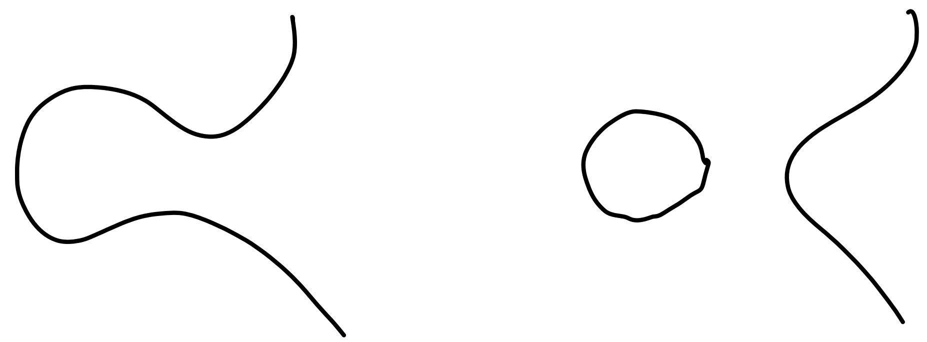
- $d=2$: smooth conics



Each of these has 1 loop in $\mathbb{RP}^2$.

- $d = 3$: smooth cubic curves.

Now we can get 1 or 2 loops:



1 Loop 2 loops.

because $\frac{(d-1)(d-2)}{2} + 1 = 2$.

Proof: We know that $C(\mathbb{C}) \subseteq \mathbb{CP}^2$ is a compact orientable real 2D surface of genus

$$g = \frac{(d-1)(d-2)}{2}.$$

Since the coefficients of the defining polynomial $F(x,y,z)$ are real, we obtain a continuous involution on points of $C(\mathbb{C})$, defined by complex conjugation

$$\begin{aligned} \mathrm{conj} : C(\mathbb{C}) &\to C(\mathbb{C}) \\ (x,y,z) &\mapsto (\bar{x},\bar{y},\bar{z}), \end{aligned}$$

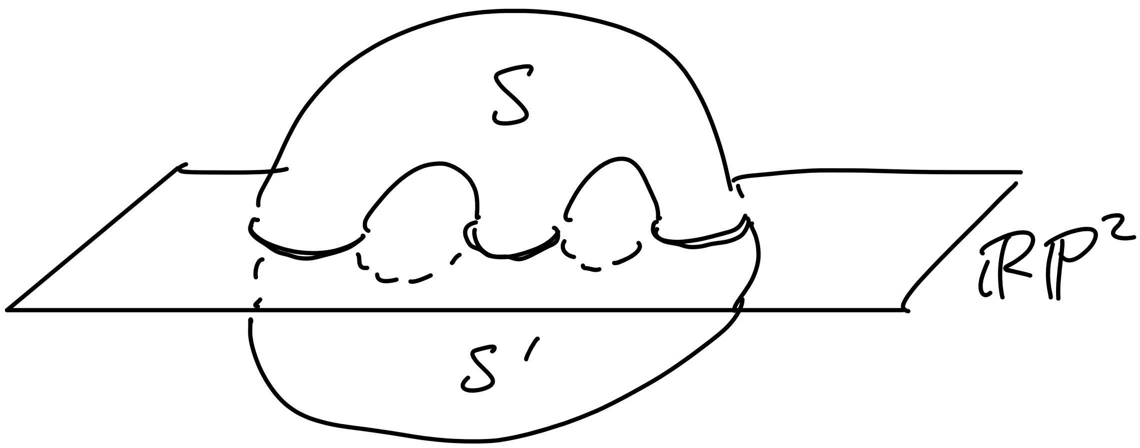
whose fixed points are $C(\mathbb{R})$.

Furthermore, conjugation reverses the natural orientation on $C(\mathbb{C})$.

By smoothness assumptions, each component of $C(\mathbb{R})$ is a closed loop on the surface $C(\mathbb{C})$.

I claim that $C(\mathbb{C}) \setminus C(\mathbb{R})$ has exactly two connected components.

Picture :




Indeed, conjugation sends connected components to connected components of $C(\mathbb{C}) \setminus C(\mathbb{R})$, and if $S$ is such a connected component then

$$S \cup \bar{S}$$

are glued along their boundary.

Observe that $S = \bar{S}$ is never possible because then $S$ is not orientable!


orientation flips when crossing boundary

Therefore $C(\mathbb{C}) \setminus C(\mathbb{R})$ has an even # of connected components ;

$$C(\mathbb{C}) = \coprod_{i=1}^{r} \left( S_i \cup \overline{S_i} \right)$$

glued along their common bdry.

But we also know that $C(\mathbb{C})$ is <u>connnected</u>, hence $r = 1$ and thus $C(\mathbb{C}) \setminus C(\mathbb{R})$ has 2 components.

Finally, by the <u>definition</u> of the genus, any $g$ closed loops will not disconnect the surface $C(\mathbb{C})$, and any $g+k$ closed loops will decompose the surface into $\geq k+1$ pieces.

So assume for contradiction that the real curve $C(\mathbb{R})$ consists of $\geq g+2$ disjoint loops.

Then the set $C(\mathbb{C}) \setminus C(\mathbb{R})$ has at least 3 connected components.

Contradiction! //

Remark: There is a higher-dimensional analogue. Let $V \subseteq \mathbb{CP}^n$ be any smooth projective variety of complex dimension $d$. If $V$ is defined by real polynomials then the real points $V(\mathbb{R}) \subseteq V$ are the fixed points of conjugation. We obtain two real manifolds of dimensions $d$ & $2d$:

$$\dim_{\mathbb{R}} V(\mathbb{R}) = d$$
$$\dim_{\mathbb{R}} V = 2d.$$

Corresponding to any real manifold $M$ we have a sequence of topological invariants $b_0(M), b_1(M), \ldots, b_n(M) \in \mathbb{N}$ called the Betti numbers.

Example:

- single 1D loop $\rightsquigarrow (b_0, b_1) = (1,1)$
- $k$ disjoint loops $\rightsquigarrow (b_0, b_1) = (k, k)$
- $g$-holed torus $\rightsquigarrow (b_0, b_1, b_2) = (1, 2g, 1)$.

The multidimensional Smith inequality says that

$$\sum_{i=1}^{d} b_i(V(\mathbb{R})) \leq \sum_{i=1}^{2d} b_i(V).$$

Example: If $C(\mathbb{R})$ consists of $k$ disjoint loops and $C(\mathbb{C})$ is a $g$-holed torus, then we obtain :

$$k + k \leq 1 + 2g + 1$$

$$2k \leq 2g + 2$$

$$k \leq g + 1.$$

Which genera can occur?

If we solve $g = (d-1)(d-2)/2$ for $d$ then we obtain

$$d = \frac{3 + \sqrt{1+8g}}{2}.$$

So a smooth plane curve of genus $g$ can only exist when $1 + 8g$ is square.

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $g$ | 0 | 0 | 1 | 3 | 6 | 10 |

In particular, there do <u>not</u> exist smooth plane curves $\subseteq \mathbb{CP}^2$ of genera $2, 4, 5, 7, 8, 9, \ldots$

However, smooth curves in $\mathbb{CP}^n$ also have a genus & satisfy the Riemann-Hurwitz equation by considering projections within $\mathbb{CP}^n$.

To be precise, let $\underline{\Lambda} \subseteq \mathbb{CP}^n$ be any $(n-2)$-dimensional subspace. Then for any line $L \subseteq \mathbb{CP}^n$ we have a projection map:

$$\begin{aligned} \pi_{\underline{\Lambda}} : \mathbb{CP}^n \setminus \underline{\Lambda} &\longrightarrow L \\ \bar{p} &\longmapsto \overline{\underline{\Lambda} \cup \{\bar{p}\}} \cap L \end{aligned}$$

where $\overline{\underline{\Lambda} \cup \{\bar{p}\}}$ is the unique hyperplane containing $\underline{\Lambda} \cup \{\bar{p}\}$, which intersects the line $L$ in a unique point $\pi_\Lambda(\bar{p}) \in L$.

Example: $\Lambda \subseteq \mathbb{CP}^2$ a point. ///

We will see that there exist smooth curves in $\mathbb{CP}^3$ of every genus.

And, moreover, any smooth abstract curve can be embedded in $\mathbb{CP}^3$

[not necessarily in $\mathbb{CP}^2$].