Extrinsic: Curves embedded in projective space (originally $\mathbb{CP}^2$).

Intrinsic: Curves as 1D complex manifolds & holomorphic maps between them.

Extrinsic $\rightsquigarrow$ Intrinsic:

Riemann surface of a projection.

Intrinsic $\rightsquigarrow$ Extrinsic:

Theory of line bundles ...

Today: The genus of a smooth curve.

Let M, N be 1D complex manifolds. A holomorphic map $\varphi: M \to N$ is locally described by convergent power series. A holomorphic function on M is a holomorphic map

$$f: M \to \mathbb{C}$$

A meromorphic function on M is a holomorphic map

$$f : M \to \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$$

By treating $\infty \in \mathbb{CP}^1$ as a number, we obtain a field of meromorphic functions on M:

$$\mathbb{C}(M) = \{ f : M \to \mathbb{CP}^1 \}.$$

Fundamental Example:

Let $M, L \subseteq \mathbb{CP}^2$ be algebraic curve of degree $d \geq 2$ and a line (i.e. a curve of degree 1). Let $\bar{p} \in \mathbb{CP}^2$ be any point not on M

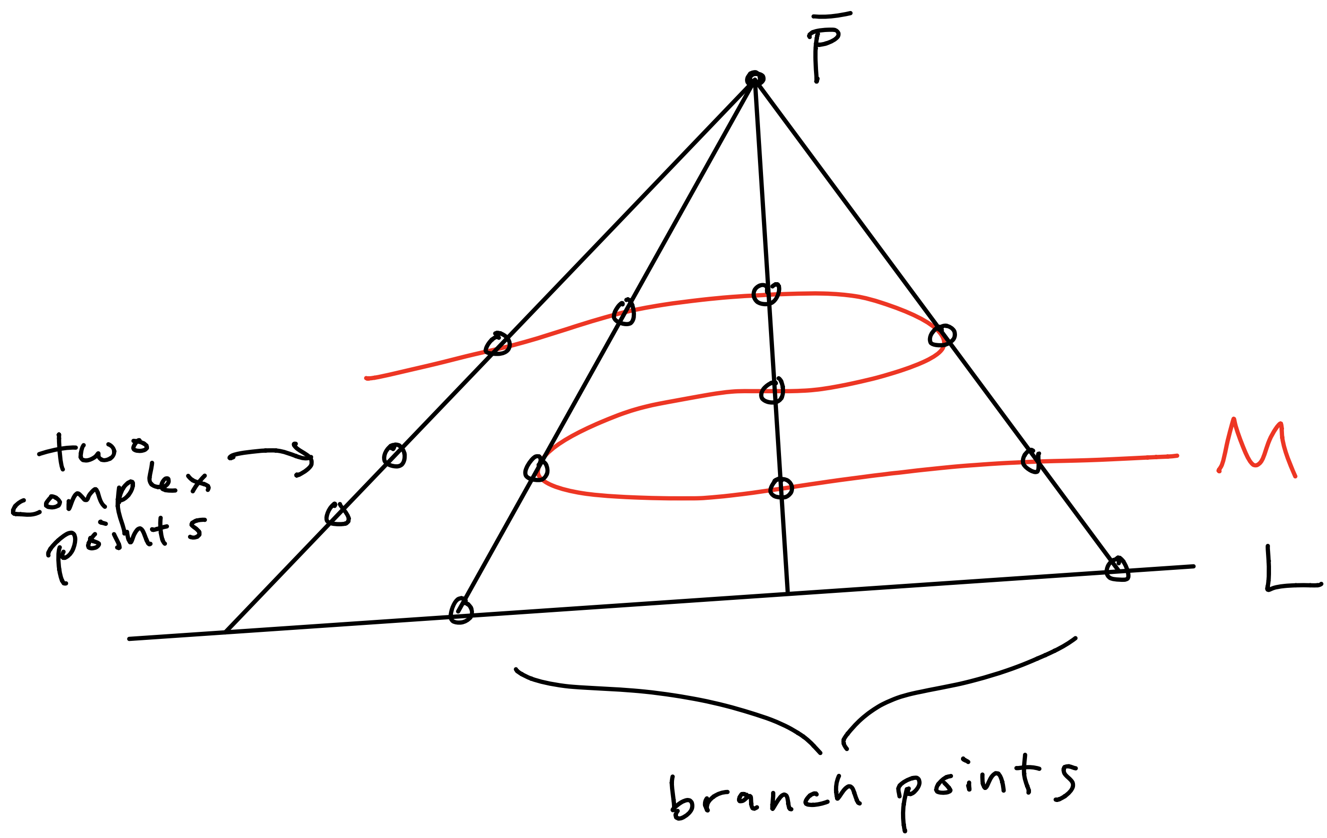
Then the <u>projection</u>

$$\pi_{\bar{p}} : M \to L \ (\cong \mathbb{CP}^1)$$

is a meromorphic function.

The projection is generically d-to-1.

Ramification happens when the line of projection is tangent to M:



Topologically, $\pi_{\bar{p}} : M \to L$ is a d-sheeted ramified covering of the Riemann sphere $\mathbb{CP}^1 \cong S^2$.

Riemann originally used the projection from $\bar{p} = (0, 1, 0)$

onto the axis $y = 0$:

$$\pi : M \longrightarrow \mathbb{CP}^1$$
$$(x,y,z) \longmapsto (x,0,z)$$

Note that we require $(0,1,0) \notin M$, equivalently that $M = V(F)$ where F is <u>monic</u> in y:

$$F(x,y,z) = y^d + \text{lower terms}.$$

Let's use projection (i.e. a meromorphic function) to compute the genus of a smooth curve.

Recall Euler's Formula: Let M be a compact closed real 2D surface, i.e., a g-holed torus. If we triangulate the surface using v_M, e_M, f_M vertices, edges, faces (i.e. triangles) then the Euler characteristic

$$\chi(M) := v_M - e_M + f_M$$

is independent of the triangulation and satisfies

$$\chi(M) = 2-2g. \qquad ///$$

This leads immediately to the

Riemann-Hurwitz Theorem:

Let $\pi : M \to N$ be a d-sheeted ramified covering of 2D real surfaces.

(e.g. a holomorphic map of algebraic curves).

Let $\Delta = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_r\} \subseteq N$ be the branch locus:

$$\Delta = \{\bar{b} \in N : \#\pi^{-1}(\bar{b}) < d\}.$$

And let

$$m_i = \# \pi^{-1}(\bar{b}_i).$$

Then

$$\chi(M) = d\left(\chi(N) - r\right) + \sum_{i=1}^{r} m_i .$$

Proof: Take a sufficiently fine triangulation (v_N, e_N, f_N) of N so the branch points Δ are included among the vertices. This lifts to a triangulation (v_M, e_M, f_M) of M satisfying:

$$f_M = d f_N$$
$$e_M = d e_N$$
$$v_M = d(v_N - r) + \sum_{i=1}^{r} m_i .$$

Then the result follows.

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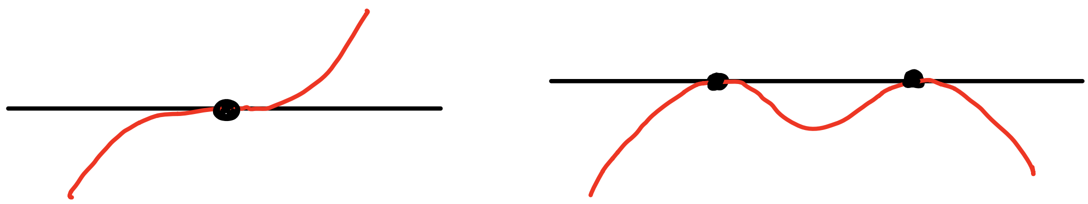
Application of Riemann-Hurwitz: Any smooth plane curve $M \subseteq \mathbb{CP}^2$ of degree d has genus

$$g = \frac{(d-1)(d-2)}{2}.$$

Proof: We know that a curve of degree $d \geq 2$ has finitely many inflection points. I claim that it also has finitely many bitangents:
or tritangents, etc.



inflectional tangent

bitangent

[Proof: Exercise.] Hence we can choose some point $\bar{p} \in \mathbb{CP}^2 - M$ not on any of these lines.

Change coordinates so $\bar{p} = (0,1,0)$ and consider the projection

$$\pi : M \to \mathbb{CP}^1$$
$$(x,y,z) \mapsto (x,0,z)$$

Note that $\bar{b} = (b, 0, c) \in \mathbb{CP}^1$ is a branch point

$\iff$ polynomials $F(b, y, c)$, $F_y(b, y, c) \in \mathbb{C}[y]$ have a common factor

$\iff$ $H(x, z) = \mathrm{Res}_y(F, F_y) \in \mathbb{C}[x, z]$ has a root $(x, z) = (b, c)$.

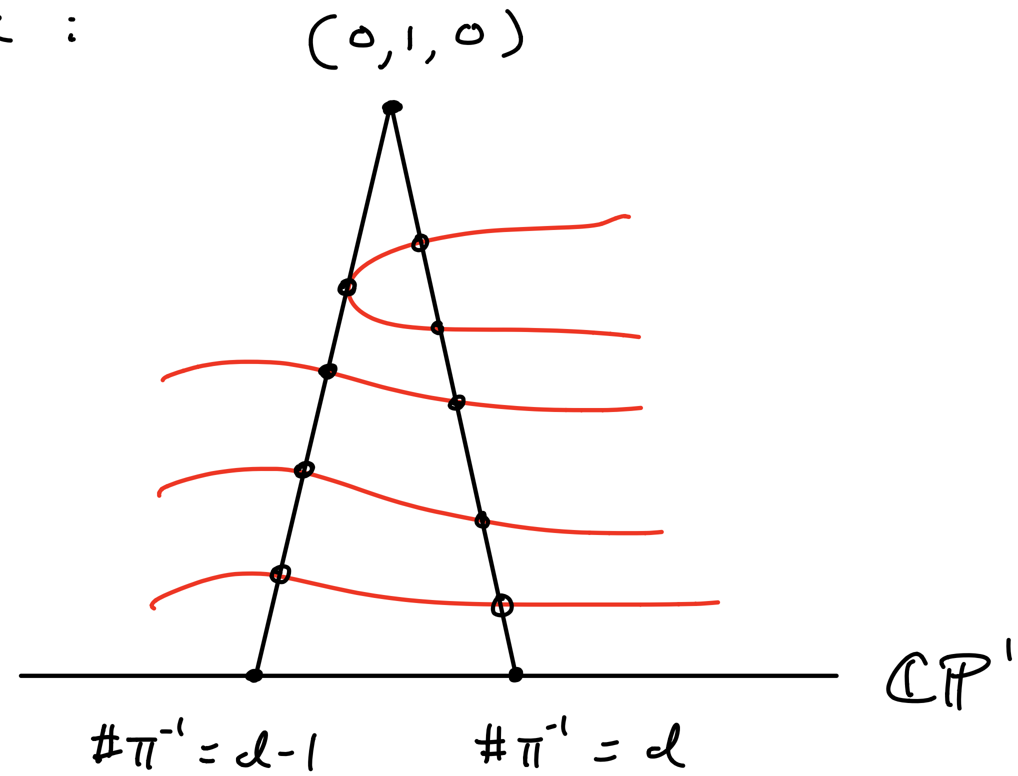
Since [Bézout] H is homogeneous of degree $\deg(F)\deg(F_y) = d(d-1)$, we conclude that there are $d(d-1)$ brach points (counted with multiplicity). Our assumptions on $\bar{p}$ guarantee that the points are distinct,

$$\Delta = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{d(d-1)}\} \subseteq \mathbb{CP}^1$$

and that the multiplicities are

$$m_i = \pi^{-1}(\bar{b}_i) = d-1 \quad \forall\, i.$$

Picture:



It follows from Riemann Hurwitz that

$$\chi(M) = d\left(\chi(\mathbb{CP}^1) - r\right) + \sum_{i=1}^{r} m_i$$

$$= d\left(2 - d(d-1)\right) + d(d-1)(d-1)$$

$$\vdots$$

$$= 3d - d^2$$

and hence

$$\chi(M) = 2 - 2g_M$$

$$3d - d^2 = 2 - 2g_M$$

$$2g_M = d^2 - 3d + 2$$

$$2g_M = (d-1)(d-2)$$

$$g_M = \frac{(d-1)(d-2)}{2}.$$

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[Remark on the Proof: For a smooth curve $M \subseteq \mathbb{CP}^2$ of degree and a generic point $\bar{p} \in \mathbb{CP}^2 - M$ there exist $d(d-1)$ lines through $\bar{p}$ that are tangent to M.

Sadly I can't draw them because some are imaginary.]

Q: But *why* do we have

$$g = \binom{d-1}{2} = \text{"} d-1 \text{ choose } 2 \text{"} ?$$

Here is a more intuitive / less rigorous proof.

Recall that the set of projective plane curves of degree d can be viewed as a projective space:

$$S_d \cong \mathbb{CP}^{d(d+3)/2}.$$

The property of being singular is determined by algebraic constraints on the coefficients, so the singular curves form a proper subvariety:

$$\underset{\text{singular curves}}{X} \subsetneq \underset{\text{all curves}}{S_d}$$

Topologically we have

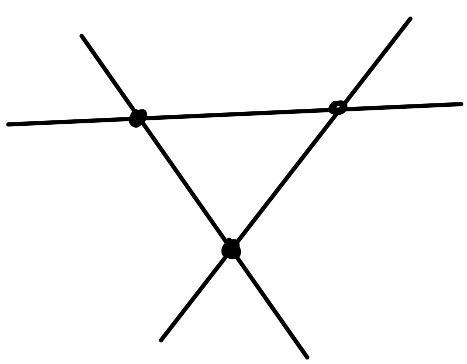
$$\operatorname{codim}_{\mathbb{R}}(X) \geq 2,$$

so the set of non-singular curves $S_d - X$ is connected in the usual topology. Since the genus varies continuously with the coefficients, we conclude that all non-singular curves of degree d have the same genus.

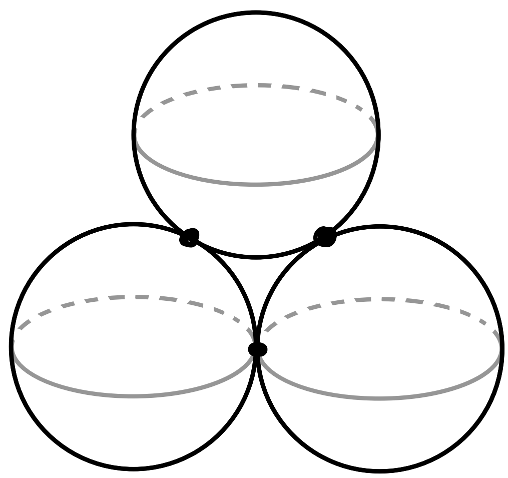
What is it ?

Let's pick a very simple-minded curve of degree d : a union of d lines in general position.

$d = 3$:



picture in $\mathbb{R}^2$



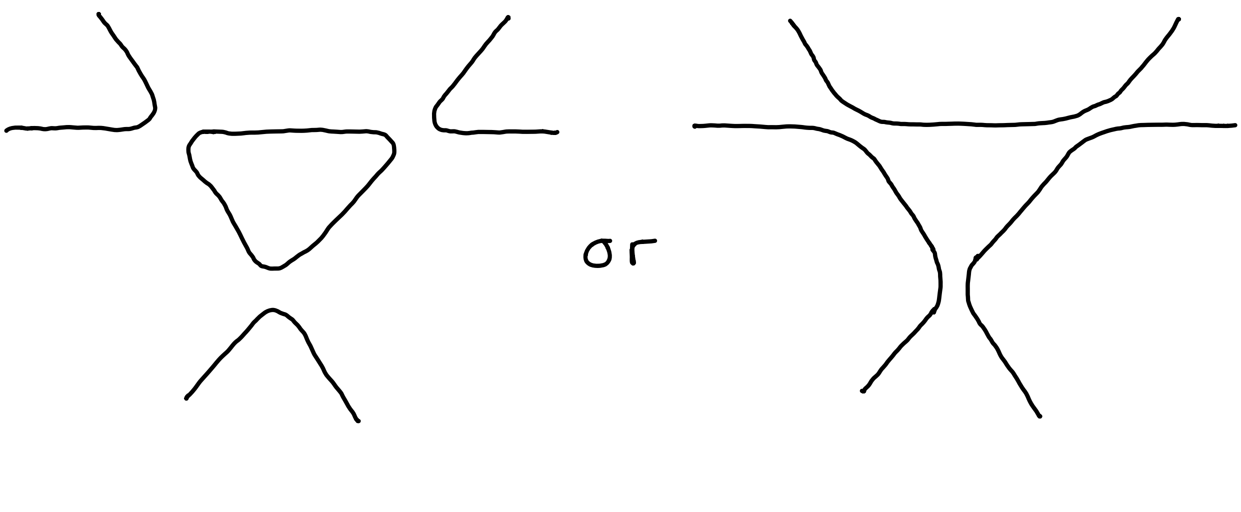
picture in $\mathbb{CP}^2$

Topologically, this is a collection of d Riemann spheres, each two meeting at a point. The equation of this (singular) curve is

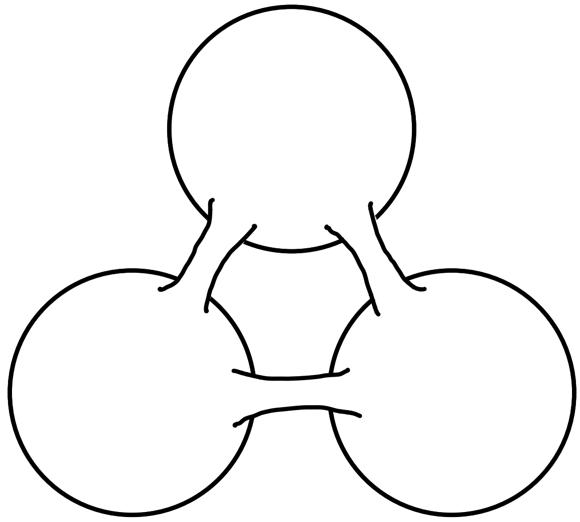
$$F = L_1 L_2 \cdots L_d = 0,$$

where L_i are the equations of the lines. Now consider the slightly deformed curve $L_1 L_2 \cdots L_d = \varepsilon$ for some small $\varepsilon \neq 0$, which is non-singular.

$d = 3$: The picture in $\mathbb{R}^2$ is



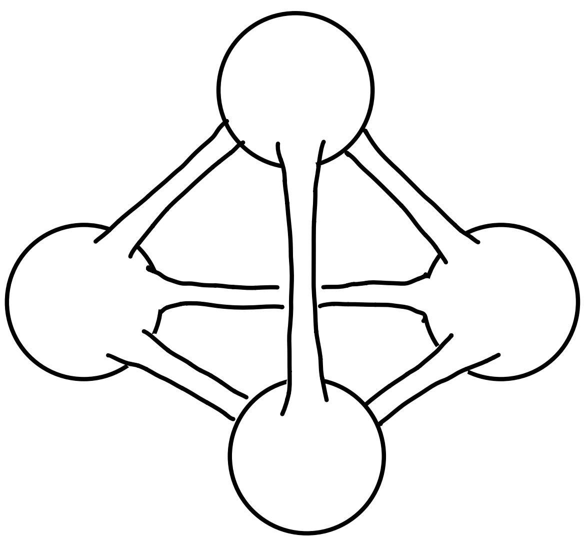
The picture in $\mathbb{CP}^2$ is



which clearly has genus 1.

The general picture in $\mathbb{CP}^2$ is d copies of $\mathbb{CP}^1$, connected pairwise by little tubes.

$d=4$:



We observe that the curve of

degree d is obtained from the curve of degree $d-1$ by adding a new sphere and connecting it to each of the previous $d-1$ spheres by tubes, which amounts to adding

$d-2$ new handles.

By induction, the total number of handles (i.e., the genus) is

$$1 + 2 + 3 + \cdots + (d-2) = \frac{(d-1)(d-2)}{2}.$$