Last time I defined the function field of a projective variety $V \subseteq \mathbb{CP}^n$:

$$\mathbb{C}(V) := \left\{ \frac{F}{G} : \begin{array}{l} F, G \text{ are homogeneous} \\ \text{of the same degree} \end{array} \right\} / \sim$$

with equivalence relation

$$\frac{F}{G} \sim \frac{F'}{G'} \iff (FG' - F'G)(\bar{p}) = 0 \text{ for all } \bar{p} \in V$$

This has a much nicer, coordinate-free description:

$$\mathbb{C}(V) \cong \operatorname{Frac} \mathbb{C}[V'],$$

where $\mathbb{C}[V'] = \mathbb{C}[x_1, .., x_n] / I(V')$ is the coordinate ring on any affine chart $V' = V \cap \mathbb{C}^n \subseteq \mathbb{C}^n \subseteq \mathbb{CP}^n$.

[Assume V is irreducible, so $I(V')$ is prime & $\mathbb{C}[V']$ is a domain.]

So what? Our goal is to formalize the concept of an "intrinsic projective variety", independent of any embedding.

The first and most important such formalization uses the function field.

Definition (Birational Equivalence):
We say that projective varieties $V \subseteq \mathbb{CP}^n$ & $W \subseteq \mathbb{CP}^m$ are "birationally equivalent" (or just "birational") if there exists an isomorphism of function fields:

$$\varphi : \mathbb{C}(V) \overset{\sim}{\longleftrightarrow} \mathbb{C}(W) : \psi$$

///

Projective equivalence implies birational equivalence, but the converse is not true. Indeed, we saw last time that the coordinate rings of $V(y), V(y-x^2) \subseteq \mathbb{C}^2$ are both isomorphic to $\mathbb{C}[x]$, hence their projective completions $V(y), V(yz-x^2) \subseteq \mathbb{CP}^2$ are birational. But they are certainly not projectively equivalent because they have different degrees.

So what <u>is</u> the geometric meaning of birationality?

Definition of Dimension: Given an irreducible projective variety $V \subseteq \mathbb{CP}^n$ we define the dimension of V as

$$\dim V := \operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(V).$$

[For any field extension K/L, Steinitz (1910) showed any two maximal sets of algebraically independent elements of K/L have the same size, called the "transcendence degree" $\text{tr.deg}_L K$. Note that

$$\text{tr.deg}_L K \leq \dim_L K$$

transcendence degree

vector space dimension]

It turns out that this definition of dimension is equivalent to any other reasonable definition.

Here is a brief justification:

If $\dim V = d$ then we have a transcendence basis

$$\mathbb{C} \subseteq \mathbb{C}(x_1, \dots, x_d) \subseteq \mathbb{C}(V)$$

where the last inclusion is algebraic.
By Galois' "Primitive Element Theorem" there exists a single element y such that $\mathbb{C}(V) = \mathbb{C}(x_1, \cdots, x_d, y)$. Since y is algebraic over $\bar{x}$ there exists an irreducible polynomial equation

$$f(\bar{x}, y) = \sum y^k a_k(\bar{x}) = 0$$

where $a_k(\bar{x}) \in \mathbb{C}(x_1, \cdots, x_d)$. By clearing denominators we may assume that $a_k(\bar{x}) \in \mathbb{C}[x_1, \cdots, x_d]$, hence $f(\bar{x}, y)$ is an irreducible polynomial in $d+1$ variables.
It follows that V is birational to the affine hypersurface $V(f) \subseteq \mathbb{C}^{d+1}$, which certainly should have "dimension d".

[Exercise: Verify the isomorphism
$\mathbb{C}(V) \cong \operatorname{Frac} \mathbb{C}[x_1, \dots, x_d, y]/(f)$.]

In modern terms, the birational geometry of projective varieties is expressed as an "equivalence of categories":

projective varieties and "surjective rational maps"	$\equiv$	finitely generated field extensions $/\mathbb{C}$ and $\mathbb{C}$-algebra homomorphisms

Let me sketch how this should work in the case of curves: Let $V \subseteq \mathbb{CP}^n$, $W \subseteq \mathbb{CP}^m$ be projective varieties and consider any $\mathbb{C}$-algebra homomorphism

$$\varphi : \mathbb{C}(W) \to \mathbb{C}(V).$$

We note that φ must be injective because $\ker \varphi \subseteq \mathbb{C}(W)$ is an ideal of a field, hence $\ker \varphi = 0$. The idea is that φ should correspond to some kind of "unique surjective map"

$$\varphi^* : V \to W$$

in the opposite direction, with the property that "$\varphi(\Phi) = \Phi \circ \varphi^*$" as "functions $V \to \mathbb{C}$". The difficulty is that the terms in quotes cannot be interpreted literally. [Locally they can, but not globally.]

Next time we will discuss some examples, without which the general theory

MAKES NO SENSE!