

Last time I defined the **function field** of a projective variety $V \subseteq \mathbb{C}P^n$:

$$\mathbb{C}(V) := \left\{ \frac{F}{G} : \begin{array}{l} F, G \text{ are homogeneous} \\ \text{of the same degree} \end{array} \right\} / \sim$$

with equivalence relation

$$\frac{F}{G} \sim \frac{F'}{G'} \iff (FG' - F'G)(\bar{p}) = 0 \text{ for all } \bar{p} \in V$$

This has a much nicer, coordinate-free description:

$$\mathbb{C}(V) \cong \text{Frac } \mathbb{C}[V'],$$

where $\mathbb{C}[V'] = \mathbb{C}[x_1, \dots, x_n] / I(V')$ is the coordinate ring on any affine chart $V' = V \cap \mathbb{C}^n \subseteq \mathbb{C}^n \subseteq \mathbb{C}P^n$.

[Assume V is irreducible, so $I(V')$ is prime & $\mathbb{C}[V']$ is a domain.]



So what? Our goal is to formalize the concept of an "intrinsic projective variety", independent of any embedding. The first and most important such formalization uses the function field.

Definition (Birational Equivalence):

We say that projective varieties $V \subseteq \mathbb{CP}^n$ & $W \subseteq \mathbb{CP}^m$ are "birationally equivalent" (or just "birational") if there exists an isomorphism of function fields:

$$\varphi: \mathbb{C}(V) \xrightarrow{\sim} \mathbb{C}(W) : \psi \quad \equiv \equiv \equiv$$

Projective equivalence implies birational equivalence, but the converse is not true. Indeed,

We saw last time that the coordinate rings of $V(y)$, $V(y-x^2) \subseteq \mathbb{C}^2$ are both isomorphic to $\mathbb{C}[x]$, hence their projective completions $V(y)$, $V(yz-x^2) \subseteq \mathbb{CP}^2$ are birational. But they are certainly not projectively equivalent because they have different degrees.



So what is the geometric meaning of birationality?

Definition of Dimension: Given an irreducible projective variety $V \subseteq \mathbb{CP}^n$ we define the dimension of V as

$$\dim V := \text{tr. deg}_{\mathbb{C}} \mathbb{C}(V).$$

[For any field extension K/L , Steinitz (1910) showed any two maximal sets of algebraically independent elements of K/L have the same size, called the "transcendence degree" $\text{tr. deg } L \subset K$.

Note that

$$\text{tr. deg } L \subset K \leq \dim_L K$$

↑ transcendence degree ↑ vector space dimension

It turns out that this definition of dimension is equivalent to any other reasonable definition.

Here is a brief justification:

If $\dim V = d$ then we have a transcendence basis

$$\mathbb{C} \subseteq \mathbb{C}(x_1, \dots, x_d) \subseteq \mathbb{C}(V)$$

where the last inclusion is algebraic.

By Galois' "Primitive Element Theorem" there exists a single element y such that $\mathbb{C}(V) = \mathbb{C}(x_1, \dots, x_d, y)$. Since y is algebraic over \bar{x} there exists an irreducible polynomial equation

$$f(\bar{x}, y) = \sum y^k a_k(\bar{x}) = 0$$

where $a_k(\bar{x}) \in \mathbb{C}(x_1, \dots, x_d)$. By clearing denominators we may assume that $a_k(\bar{x}) \in \mathbb{C}[x_1, \dots, x_d]$, hence $f(\bar{x}, y)$ is an irreducible polynomial in $d+1$ variables.

It follows that V is birational to the affine hypersurface

$V(f) \subseteq \mathbb{C}^{d+1}$, which certainly should have "dimension d ".

[Exercise : Verify the isomorphism

$$\mathbb{C}(V) \cong \text{Frac } \mathbb{C}[x_1, \dots, x_d, y] / (f).]$$



In modern terms, the birational geometry of projective varieties is expressed as an "equivalence of categories":

projective varieties and "surjective rational maps"	\cong	finitely generated field extensions / \mathbb{C} and \mathbb{C} -algebra homomorphisms
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Let me sketch how this should work in the case of curves: Let

$V \subseteq \mathbb{C}P^n$, $W \subseteq \mathbb{C}P^m$ be projective

varieties and consider any

\mathbb{C} -algebra homomorphism

$$\varphi: \mathbb{C}(W) \rightarrow \mathbb{C}(V).$$

We note that φ must be injective because $\ker \varphi \subseteq \mathbb{C}(W)$ is an ideal of a field, hence $\ker \varphi = 0$.

The idea is that φ should correspond to some kind of "unique surjective map"

$$\varphi^*: V \rightarrow W$$

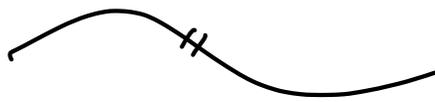
in the opposite direction, with the property that " $\varphi(\Phi) = \Phi \circ \varphi^*$ "

as "functions $V \rightarrow \mathbb{C}$ ". The

difficulty is that the terms

in quotes cannot be interpreted

literally. [Locally they can, but not globally.]



Next time we will discuss some
examples, without which the
general theory

MAKES NO SENSE !