We have seen that <u>some</u> singular curves $C \subseteq \mathbb{CP}^2$ arise from non-singular curves $\tilde{C} \subseteq \mathbb{CP}^n$ by projection

$$\tilde{\pi} : \tilde{C} \to C.$$

In this case we say that the pair $(\tilde{C}, \tilde{\pi})$ is a "resolution of the singularities of $C$", i.e., we view the singularities of $C$ not as essential things, but as shadows caused by a projection.

Is this always possible?

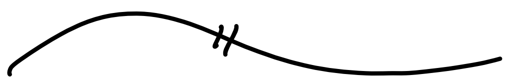
That is, given a singular curve $C \subseteq \mathbb{CP}^2$, can we always find a non-singular curve $\tilde{C} \subseteq \mathbb{CP}^n$ and a "projection map"

$$\tilde{\pi} : \tilde{C} \longrightarrow C \quad ?$$

**Important Theorem (Resolution of Singularities of Curves):**

The answer is <u>yes</u>. Moreover, the pair $(\tilde{C}, \tilde{\pi})$ is <u>in some sense</u> unique.

[Moral: Non-singular curves don't really exist. ]



Q: Unique <u>in what sense</u>?

There are messy bottom-up ways to do this (sequences of blow-ups), but there is also an elegant top-down approach that captures the big picture. This approach is due to Dedekind-Weber (1882) and Zariski (1938), and is based on Dedekind's machinery of "rings & fields" which he invented to study number theory.

[WARNING: The top-down approach works perfectly for curves, but higher-dimensional varieties are intrinsically messier. ]

General Philosophy:

an "intrinsic curve" $\equiv$ the functions that live on the curve

[ The formal statement of this philosophy is called "Yoneda's Lemma". ]

What kind of functions?

This depends on your point of view. If you want to study "curves that

can be defined via polynomials"
then you probably want to study
"functions that can be defined
via polynomials".

Definition: Let $V \subseteq \mathbb{C}^n$ be an
irreducible variety with corresponding
prime ideal $I(V) \subseteq \mathbb{C}[x_1, \dots, x_n]$,

$$I(V) := \{ f : f(\bar{p}) = 0 \ \forall \bar{p} \in V \}.$$

Then we define the <u>coordinate ring</u>

$$\mathbb{C}[V] := \mathbb{C}[x_1, \dots, x_n] / I(V).$$

We can view elements of $\mathbb{C}[V]$ as
"functions $V \to \mathbb{C}$ defined by
polynomials":

$$f(\bar{x}) + I(V) = g(\bar{x}) + I(V)$$

$$\iff f(\bar{x}) - g(\bar{x}) \in I(V)$$

$$\iff f(\bar{p}) - g(\bar{p}) = 0 \quad \forall\ \bar{p} \in V$$

$$\iff f(\bar{p}) = g(\bar{p}) \quad \forall\ \bar{p} \in V.$$

The idea is that the ring $\mathbb{C}[V]$ forgets the embedding $V \subseteq \mathbb{C}^n$, while remembering the "intrinsic structure" of the variety.

In fact, one <u>defines</u>

$$V \cong W \iff \mathbb{C}[V] \cong \mathbb{C}[W]$$

"intrinsic isomorphism" $\iff$ "isomorphic as $\mathbb{C}$-algebras"

I claim that "affine equivalence" (extrinsic isomorphism) implies "intrinsic isomorphism", but the converse is false.

<u>Proof</u>: Let $\varphi \in \mathrm{Aff}(\mathbb{C}^n)$, i.e.,

$$\varphi(\bar{x}) = A\bar{x} + \bar{t}$$

for some $A \in GL_n$ & $\bar{t} \in \mathbb{C}^n$. If we define $f^\varphi(\bar{x}) := f(\varphi^{-1}(\bar{x}))$ then

$$I(\varphi V) = I(V)^\varphi := \{ f^\varphi : f \in I(V) \}.$$

One can check that the function

$$\hat{\varphi} : \mathbb{C}[V] \longrightarrow \mathbb{C}[\varphi V]$$
$$f + I(V) \longmapsto f^\varphi + I(\varphi V)$$

is a $\mathbb{C}$-algebra isomorphism.

To see that the converse is false, we will show that the line $V(y)$ & parabola $V(y-x^2)$ in $\mathbb{C}^2$ are intrins. isomorphic. [And they are not extrins. isom. because $Aff(\mathbb{C}^2)$ preserves degree.]

Indeed, by Study's Lemma we have $I(V(f)) = (f)$ for any irreducible polynomial. Thus we only need to show that $\mathbb{C}[x,y]/(y) \cong \mathbb{C}[x,y]/(y-x^2)$.

The desired isomorphism is obtained by composing the following isomorphisms:

i) $$\mathbb{C}[x] \xrightarrow{\sim} \mathbb{C}[x,y]/(y)$$
$$f(x) \longmapsto f(x) + (y)$$

&

ii) $$\mathbb{C}[x] \xrightarrow{\sim} \mathbb{C}[x,y]/(y-x^2)$$
$$f(x) \longmapsto f(x) + (y-x^2)$$

To see that these are indeed isoms:

ii) We consider divisibility in the ring $\mathbb{C}[x,y] = \mathbb{C}[x][y]$. For any $F(x,y)$ divide by $y$ to get

deg 0 in y

$$F(x,y) = q(x,y)(y-x^2) + r(x)$$

$$\Longrightarrow F(x,y) + (y-x^2) = r(x) + (y-x^2).$$

Then $f(x) + (y-x^2) = g(x) + (y-x^2)$

$$\Longleftrightarrow y-x^2 \mid \underbrace{f(x)-g(x)}_{\substack{\deg \geq 1 \text{ in } y \\ \text{or } = 0}} \Longleftrightarrow f(x) - g(x) = 0.$$

i) Proof is the same but easier. ///

I agree with you that the definition of "isomorphism of affine varieties" doesn't seem very geometric:

a line $\cong$ a parabola ?!

The problem is in the name. It should really be called "local isomorphism":

a line $\cong$ a parabola ✓

"locally isomorphic"

To capture "global structure" algebraically requires an algebraic substitute for topology. As you know, the words "algebraic topology" open a horrifying box of worms. For us, the topology of algebraic <u>curves</u> is quite difficult enough.

This begins with the notion of the "function field of a projective variety $V \subseteq \mathbb{CP}^n$".

Problem: A polynomial (even a homogeneous one) does not define a <u>function</u> $\mathbb{CP}^n \longrightarrow \mathbb{C}$.

Solution: Quotients of homogeneous polynomials of the same degree do define functions "almost everywhere".

To make this precise takes a bit of work. Let $V \subseteq \mathbb{CP}^n$ be a projective variety with homogeneous ideal $I(V) \subseteq \mathbb{C}[x_1, x_2, \ldots, x_{n+1}]$.

We note that the "homogeneous coordinate ring"

$$\mathbb{C}[V] := \mathbb{C}[\bar{x}] / I(V)$$

has a natural "grading by degree", meaning every coset $f + I(V)$ has a unique expression of the form

$$f + I(V) = \sum_{d \geq 0} f^d + I(V)$$

where $f^d \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ is homog. of degree $d$.

Proof: Existence is clear.

For uniqueness, suppose that

$$\sum f^d + I(V) = \sum g^d + I(V), \text{ i.e.,}$$

$$\sum (f^d - g^d) \in I(V)$$

where $f^d, g^d, f^d - g^d$ are hom. of deg. $d$. Since $I(V)$ is homog. a result from last semester implies that $f^d - g^d \in I(V)$, and hence

$f^d + I(V) = g^d + I(V)$, for all $d$. ///

We say that a coset $f + I(V)$ has "degree $d$" if $f + I(V) = F + I(V)$ for some (unique) homogeneous $F \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ of degree $d$.

Thus we can make the following definition.

**Definition (The function field of a projective variety):**

Let $V \subseteq \mathbb{C}\mathbb{P}^n$ be an irreducible projective variety, so that $I(V)$ is a homogeneous prime ideal and $\mathbb{C}[V] := \mathbb{C}[x_1, \dots, x_{n+1}]/I(V)$ is an integral domain graded by degree. We define the **function field** as the following subfield of the field of fractions $\operatorname{Frac} \mathbb{C}[V]$:

$$\mathbb{C}(V) := \left\{ \frac{F + I(V)}{G + I(V)} : \begin{array}{l} F, G \text{ homogeneous} \\ \text{of the same degree} \end{array} \right\}$$

Exercises:

- verify that this is indeed a field
- verify the more concrete realization:

$$\mathbb{C}(V) = \left\{ \frac{F}{G} : \begin{array}{l} F, G \text{ homogeneous} \\ \text{of the same degree} \end{array} \right\} \Big/ \sim$$

where $\dfrac{F}{G} \sim \dfrac{F'}{G'} \iff FG' - F'G \in I(V)$.

///

We had to consider a proper subfield of Frac $\mathbb{C}[V]$ because some elements of the full field cannot be viewed as functions. At least the elements of $\mathbb{C}(V)$ can be viewed as "partially-defined functions" $V \rightsquigarrow \mathbb{C}$.

Indeed, if $F, G$ are both homog. of degree $d$ then

$$\frac{F(\lambda \bar{x})}{G(\lambda \bar{x})} = \frac{\lambda^d F(\bar{x})}{\lambda^d G(\bar{x})} = \frac{F(\bar{x})}{G(\bar{x})},$$

So we have a function outside the hypersurface $V(G) \subseteq \mathbb{CP}^n$:

$$\frac{F}{G} : \mathbb{CP}^n \setminus V(G) \to \mathbb{C}.$$

I agree that these definitions look ugly at first. But the following theorem shows that they are not so bad, and helps to explain the intrinsic meaning.

Theorem: Let $V \subseteq \mathbb{CP}^n$ be an irreducible projective variety.

① For any $A \in PGL_{n+1}$ we have

$$\mathbb{C}(V) \cong \mathbb{C}(AV).$$

② For any hyperplane $H \subseteq \mathbb{CP}^n$ not equal to $V$, let $\varphi : \mathbb{C}^n \hookrightarrow\!\!\!\!\rightarrow \mathbb{CP}^n \setminus H$

be the corresponding affine chart and let $V' = V \setminus H \subseteq \mathbb{CP}^n \setminus H$.

Then

$$\mathbb{C}(V) \cong \operatorname{Frac} \mathbb{C}[\varphi^{-1}V']$$

[This is the true meaning of the function field. It stitches together the polynomial functions on all of the affine charts. And it does this without ever mentioning "sheaves".]

Proof:

(1) Recall that $A \in PGL$ acts on homogeneous polynomials, preserving the degree:

$$F^A(\bar{x}) := F(A^{-1}\bar{x}).$$

And recall that $I(AV) = I(V)^A = \{F^A : F \in I(V)\}$, which is also a homogeneous ideal of $\mathbb{C}[x_1, \ldots, x_{n+1}]$.

One can check that $\frac{F}{G} \mapsto \frac{F^A}{G^A}$ is

a well-defined isomorphism from

$\mathbb{C}(V)$ to $\mathbb{C}(AV)$. [Exercise: Check it.]

(2) From part (1) we may assume

that $H = V(x_{n+1})$ so the affine

chart is $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$.

If $I(V) = (F_1, \ldots, F_m) \subseteq \mathbb{C}[x_1, \ldots, x_{n+1}]$

is the homogeneous ideal of $V \subseteq \mathbb{CP}^n$,

we proved last semester that

$$I(V') = (F_1', \ldots, F_m') \subseteq \mathbb{C}[x_1, \ldots, x_n],$$

where $F_i'(x_1, \ldots, x_n) := F_i(x_1, \ldots, x_n, 1)$

are possibly non-homogeneous.

I claim that the desired isomorphism

isomorphism $\mathbb{C}(V) \xrightarrow{\sim} \operatorname{Frac} \mathbb{C}[V']$

is given by

$$\frac{G + I(V)}{H + I(V)} \longmapsto \frac{G' + I(V')}{H' + I(V')}$$

It is a homomorphism because $G \mapsto G'$ is a homomorphism.

- Surjective: For any $g, h$ in $\mathbb{C}[x_1, \dots, x_n]$ with $d \geq \max\{\deg(g), \deg(h)\}$ we have $g = G'$, $h = H'$ where

$$G = x_{n+1}^d \, g\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right)$$

$$H = x_{n+1}^d \, h\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right)$$

are homogeneous of degree $d$.

- Injective & Well-Defined:

$$G_1 H_2 - G_2 H_1 \in I(V)$$

$$\iff (G_1 H_2 - G_2 H_1)' \in I(V')$$

$$\iff G_1' H_2' - G_2' H_1' \in I(V').$$

///