Last time we constructed "curves" of every genus in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

That is, for an irreducible polynomial $F(\bar{x}, \bar{y}) \in \mathbb{C}[x_1, x_2, y_1, y_2]$ that is bihomogeneous of bidegree (m, n)

$$\left(\text{i.e. } \begin{aligned} F(\lambda \bar{x}, \bar{y}) &= \lambda^m F(\bar{x}, \bar{y}) \\ F(\bar{x}, \lambda \bar{y}) &= \lambda^n F(\bar{x}, \bar{y}) \end{aligned} \right),$$

we used the projection map

$$\begin{aligned} \mathbb{CP}^1 \times \mathbb{CP}^1 &\longrightarrow \mathbb{CP}^1 \\ (\bar{x}, \bar{y}) &\longmapsto \bar{x} \end{aligned}$$

to show that the set $V(F) \subseteq \mathbb{CP}^1 \times \mathbb{CP}^1$ is a 2D real surface of genus

$$g = (m-1)(n-1).$$

Example: $F = \frac{1}{2} x_1^3 y_1^2 + x_2^3 y_1 y_2 + \frac{1}{2} x_1^3 y_2^2$ is irreducible & bihomogeneous of bidegree $(3, 2)$, so defines a surface of genus $(3-1)(2-1) = 2$.

[In fact, it is a two sheeted covering of $\mathbb{CP}^1$ branched over the 6th roots of unity in $\mathbb{C} \subseteq \mathbb{CP}^1$.]

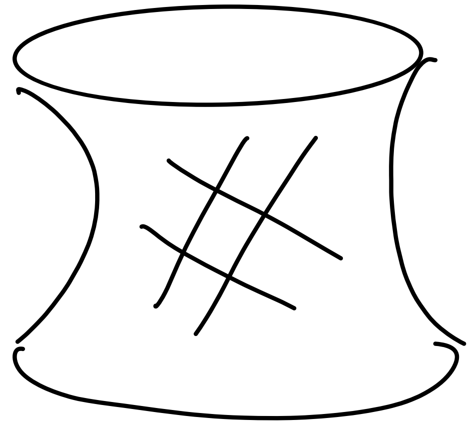
However, it is not so clear if this $V(F)$ is an "algebraic variety" since we have not defined a "Zariski topology" on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

[The obvious product topology is "wrong."]

The "correct" structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is defined via the Segre embedding

$$\sigma : \mathbb{CP}^1 \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^3$$
$$(x_1, x_2, y_1, y_2) \longmapsto (z_{11}, z_{12}, z_{21}, z_{22})$$

where $z_{ij} := x_i y_j$. We showed last time that σ is well-defined & bijective onto its image, which is the smooth quadric surface

$$Q = V(z_{11}z_{22} - z_{12}z_{21}) =$$ 

The claim is that subsets of $\mathbb{CP}^1 \times \mathbb{CP}^1$ defined by bihomog. polynomials correspond to subsets of Q defined by homogeneous polynomials in $\mathbb{CP}^3$.

Example: $F = \frac{1}{2}x_1^3 y_1^2 + x_2^3 y_1 y_2 + \frac{1}{2}x_1^3 y_2^2$

We cannot immediately write this as $F = G(z_{11}, z_{12}, z_{21}, z_{22})$

because, for any hom G of degree d the polynomial

$$F(\bar{x},\bar{y}) = G(x_1y_1, x_1y_2, x_2y_1, x_2y_2)$$

is bihomogeneous of degree (d,d).

In order to view our F, which is degree $(3,2)$ in terms of polynoms. on $\mathbb{CP}^3$ we will bump it up to degree $(3,3)$. There are two basic ways to do this:

$$G := y_1F = \frac{1}{2}x_1^3y_1^3 + x_2^3y_1^2y_2 + \frac{1}{2}x_1^3y_1y_2^2,$$
$$H := y_2F = \frac{1}{2}x_1^3y_1^2y_2 + x_2^3y_1y_2^2 + \frac{1}{2}x_1^3y_2^3.$$

Since both of these are degree $(3,3)$, they are polynomials of degree 3 in the variables $z_{11}, z_{12}, z_{21}, z_{22}$:

$$G = \frac{1}{2}z_{11}^3 + z_{21}^2z_{22} + \frac{1}{2}z_{11}z_{12}^2,$$

$$H = \frac{1}{2} z_{11}^2 z_{12} + z_{21} z_{22}^2 + \frac{1}{2} z_{12}^3 .$$

Now we observe that

$$V(F) = V(y_1 F, y_2 F) \subseteq \mathbb{CP}^1 \times \mathbb{CP}^1 .$$

Indeed, if $F = 0$ then $y_1 F = y_2 F = 0$. Conversely, if $y_1 F = y_2 F = 0$ then since $(y_1, y_2) \in \mathbb{CP}^1$ we must have $y_1 \neq 0$ or $y_2 \neq 0$, hence $F = 0$. ✓

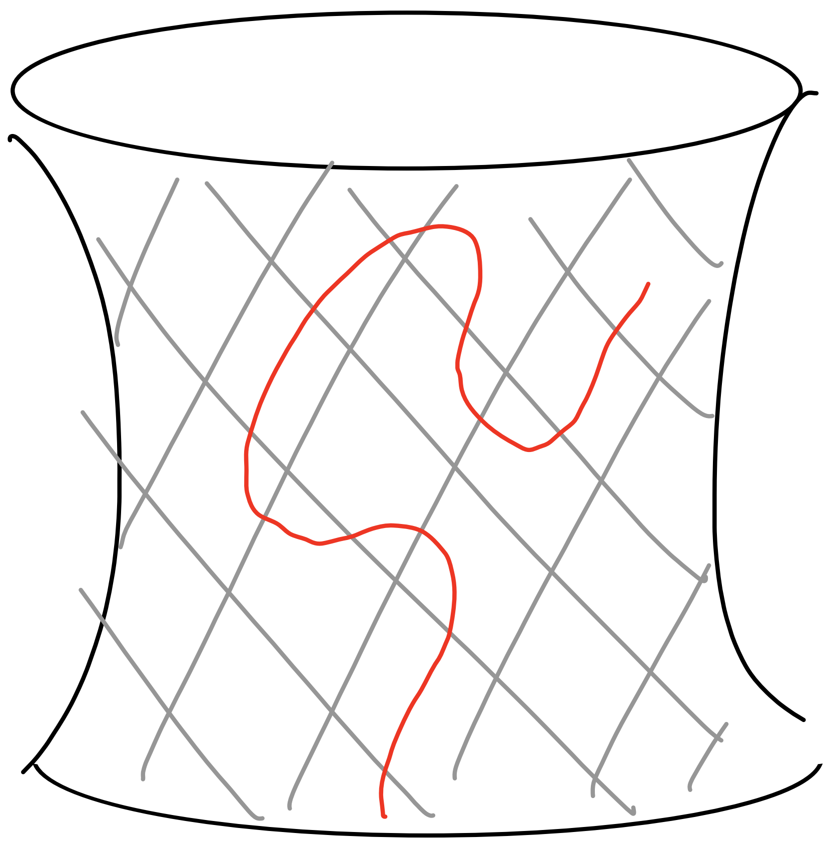
In conclusion, the Segre map restricts to a bijection

$$V(F) \longleftrightarrow \underbrace{V(Q, G, H)}_{\substack{\text{defined by 3} \\ \text{homogeneous polynomials.}}} \subseteq \mathbb{CP}^3$$

[Remark: I checked using Maple that all three polynomials

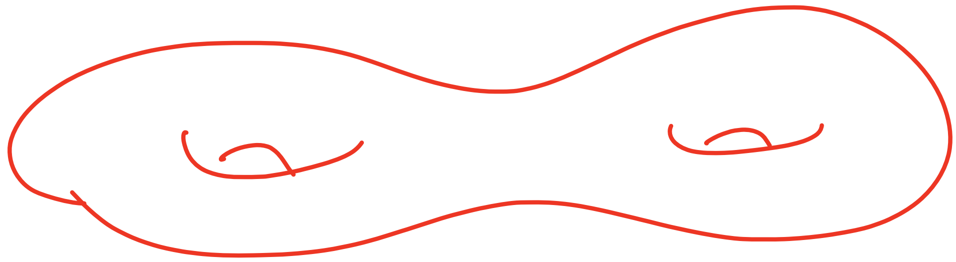
Q, G, H are necessary here.
It is also true that the ideal
$(Q, G, H) \subseteq \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}]$
cannot be generated by two
polynomials, but we do not have
the technology to prove this.]

Very rough picture:

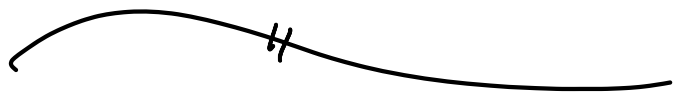


Crosses one parallel family twice,
the other 3 times.

Another very rough picture:



"smooth of genus 2"



This example showed us a new layer of subtlety in the study of algebraic curves, i.e., the plane $\mathbb{CP}^2$ is not big enough to hold every algebraic curve. Luckily, $\mathbb{CP}^3$ <u>is</u> big enough.

<u>Theorem</u>: Every projective algebraic curve $C \subseteq \mathbb{CP}^n$ $(n \geq 3)$

can be "embedded" in $\mathbb{CP}^3$.

Sketchy Proof by Induction:

Consider the set of secant (and tangent) lines to the curve

$$\mathcal{L} = \{ L_{p,q} : p, q \in C \},$$

where we write $L_{p,p}$ for the tangent line at $p \in C \subseteq \mathbb{CP}^n$. Now define the set

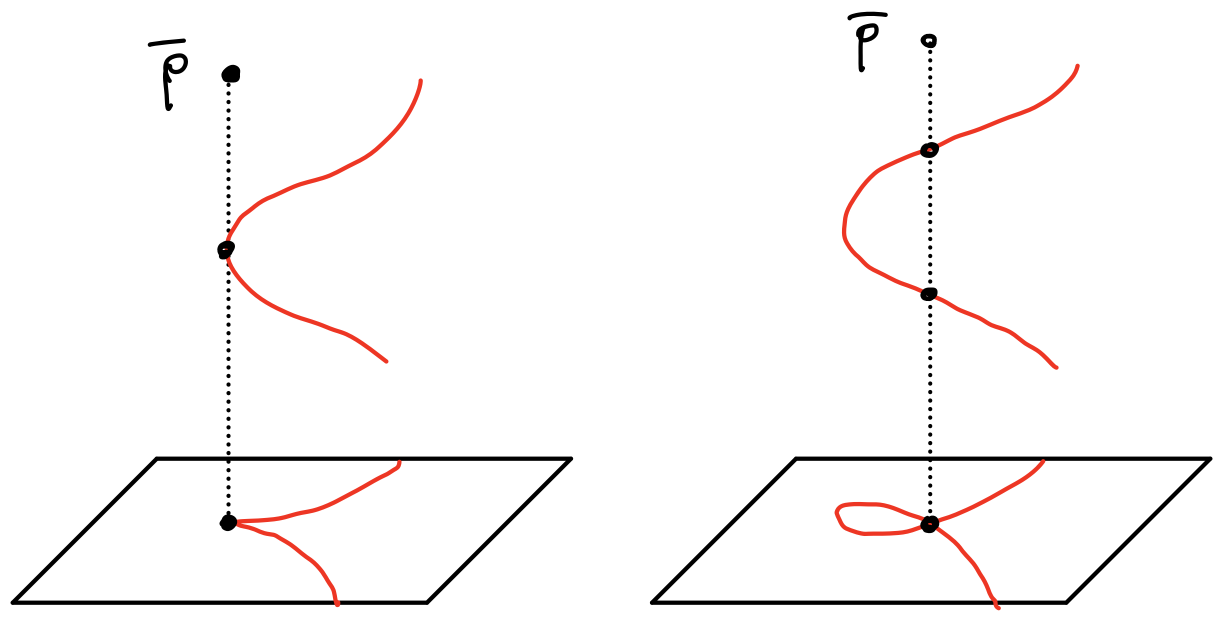
$$S := \bigcup_{L \in \mathcal{L}} L \subseteq \mathbb{CP}^n$$

One can show that this is a projective variety, called the "secant variety" of the curve.

If $n \geq 4$ then codim $S \geq 1$, so there exists a point $p \in \mathbb{CP}^n \setminus S$.

Finally, the projection from p onto a general copy of $\mathbb{CP}^{n-1} \subseteq \mathbb{CP}^n$ maps C biholomorphically onto a curve $C' \subseteq \mathbb{CP}^{n-1}$. ///

Idea of the proof: The "shadow" of curve under such a projection doesn't acquire any singularities:



We know that no smooth curve in $\mathbb{CP}^2$ can have genus 2 (or 4, 5, 7, 8, 9, 11, ...). Therefore our example curve $C = V(Q, G, H)$ must have secant variety filling up all of $\mathbb{CP}^3$.

It turns out that any smooth curve of genus 2 can be mapped into $\mathbb{CP}^2$ with a single cusp.

To be specific, any smooth curve of genus 2 maps onto

$$y^2 = f(x)$$

where $f(x) \in \mathbb{C}[x]$ has degree 5 or 6 and has no multiple roots.

The unique singularity, which is a cusp, occurs at the vertical point at ∞: $(0,1,0)$.