Let $f(x,y) \in \mathbb{C}[x,y]$ be irreducible & non-singular and think of $C = V(f) \subseteq \mathbb{C}^2 = \mathbb{R}^4$.

We proved last time that C is a real 2D surface that is connected in the Euclidean topology on $\mathbb{R}^4$.

Furthermore, I claim that this surface is orientable. There are two ways to prove this:

- Intuitively: The projection $\pi: C \to \mathbb{C}$, $(x,y) \mapsto x$ provides a consistent notion of "up".

- More precisely: We can think of $f: \mathbb{C}^2 \to \mathbb{C}$ as a function $\mathbb{R}^4 \to \mathbb{R}^2$:

$$f(x_1, y_1, x_2, y_2) := f(x_1 + iy_1, x_2 + iy_2)$$

$$= u(x_1, y_1, x_2, y_1) + i\, v(x_1, y_1, x_2, y_2).$$

$$f(\bar{x}) = (u(\bar{x}), v(\bar{x}))$$

Then $C = V(u,v) \subseteq \mathbb{R}^4$ is the 2D surface defined by equations

$$u(\bar{x}) = 0,$$
$$v(\bar{x}) = 0.$$

The tangent space at a point $\bar{p} \in \mathbb{R}^4$ is $\perp$ to the gradient vectors:

$$(\nabla u)_{\bar{p}} = (u_{x_1}, u_{y_1}, u_{x_2}, u_{y_2})_{\bar{p}},$$
$$(\nabla v)_{\bar{p}} = (v_{x_1}, v_{y_1}, v_{x_2}, v_{y_2})_{\bar{p}}.$$

Let's say that $\bar{a} = (a_1, b_1, a_2, b_2)$ is in the tangent plane, so that

$$\begin{cases} \nabla u \cdot \bar{a} = 0, \\ \nabla v \cdot \bar{a} = 0. \end{cases}$$

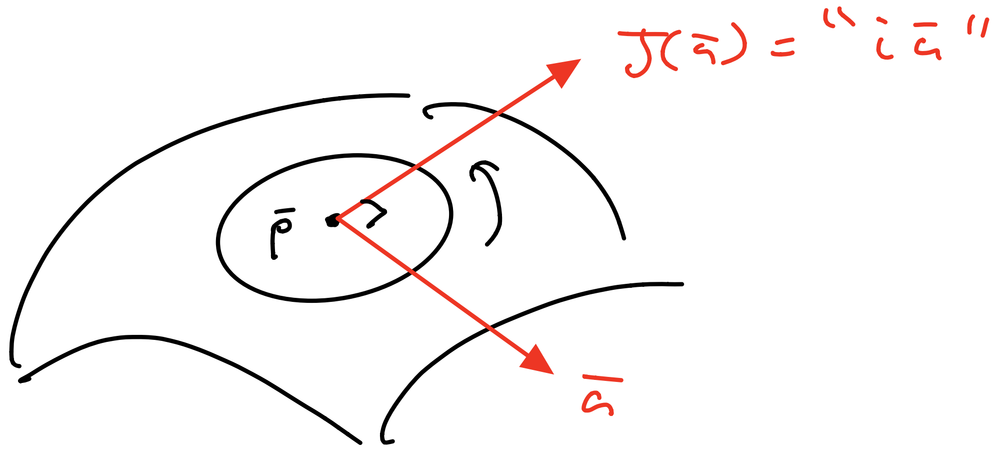
In coordinates:

$$(*) \begin{cases} u_{x_1} a_1 + u_{y_1} b_1 + u_{x_2} a_2 + u_{y_2} b_2 = 0, \\ v_{x_1} a_1 + v_{y_1} b_1 + v_{x_2} a_2 + v_{y_2} b_2 = 0. \end{cases}$$

Now we define "multiplication by i" $\mathbb{R}^4 \to \mathbb{R}^4$ as follows:

$$
\begin{aligned}
J(\bar{a}) &= \text{“} i\bar{a} \text{”} \\
&= \text{“} i(a_1, b_1, a_2, b_2) \text{”} \\
&= \text{“} i(a_1 + ib_1, a_2 + ib_2) \text{”} \\
&= \text{“} (-b_1 + ia_1, -b_2 + ia_2) \text{”} \\
&= (-b_1, a_1, -b_2, a_2).
\end{aligned}
$$

If $\bar{a} \in \mathbb{R}^4$ is tangent to the surface at $\bar{p} \in \mathbb{R}^4$ then I claim that $J(\bar{a}) \in \mathbb{R}^4$ is also tangent to C at the same point:



Indeed, since the polynomial $f: \mathbb{C}^2 \to \mathbb{C}$ is holomorphic in each coordinate, we have the Cauchy-Riemann equations:

$$u_{x_1} = v_{y_1} \qquad \& \qquad u_{x_2} = v_{y_2}$$
$$u_{y_1} = -v_{x_1} \qquad\quad\;\; u_{y_2} = -v_{x_2}.$$

[Exercise: Prove this (only for polynomial functions, so calculus is not necessary).]

Then substituting the C-R equations into $\circledast$ gives

$$\begin{cases} v_{y_1} a_1 - v_{x_1} b_1 + v_{y_2} a_2 - v_{x_2} b_2 = 0, \\ u_{y_1} a_1 - u_{x_1} b_1 + u_{y_2} a_2 - u_{x_2} b_2 = 0. \end{cases}$$

That is:

$$\begin{cases} \nabla u \cdot J(\bar{a}) = 0, \\ \nabla v \cdot J(\bar{a}) = 0. \end{cases}$$

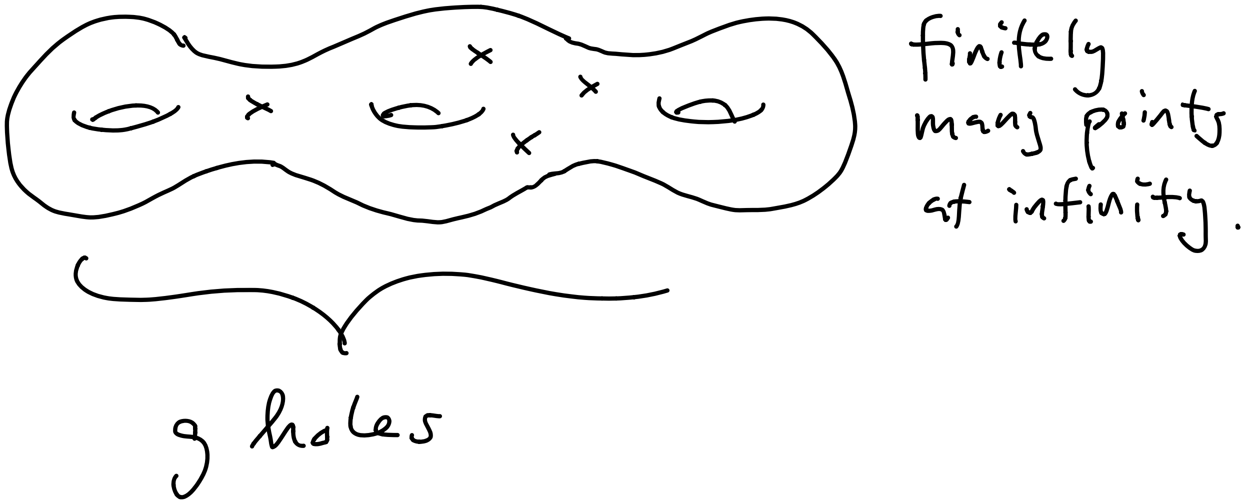
Since $J : \mathbb{R}^4 \to \mathbb{R}^4$ is continuous this gives a contistent notion of "counterclockwise rotation" on the whole surface. //

So the surface $C \subseteq \mathbb{R}^4$ is connected, orientable & smooth.

As we know, by adding finitely many "points at infinity" we obtain a compact set $\overline{C} \subseteq \mathbb{CP}^2 \subseteq \mathbb{RP}^4$

Assuming these points are non-singular then $\overline{C}$ is connected, orientable, smooth & compact. Hence it has only one topological invariant:

the genus g.



Given an irreducible homogeneous polynomial $F(x,y,z) \in \mathbb{C}[x,y,z]$ with dehomogenization $f(x,y) = F(x,y,1)$, we have smooth affine & projective curves:

$$\begin{array}{ccc}
\mathbb{C}^2 & \hookrightarrow & \mathbb{CP}^2 \\
\cup\!| & & \cup\!| \\
V(f) = C & \hookrightarrow & \overline{C} = V(F)
\end{array}$$

Last time we considered the projection

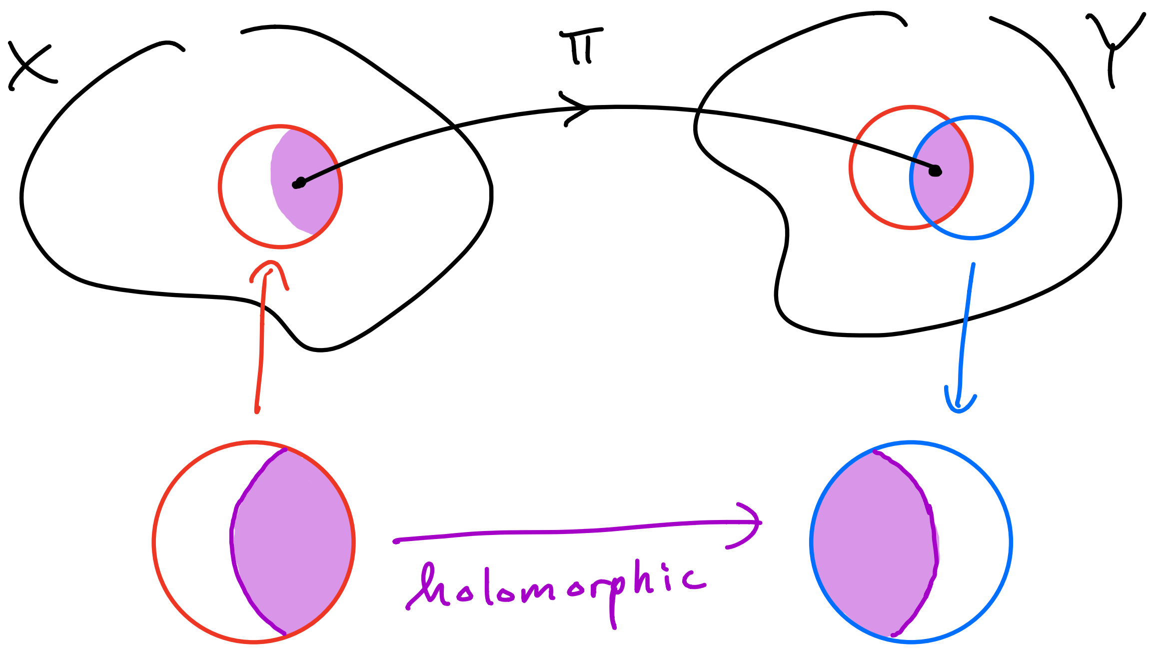
$$\begin{aligned}
\pi : C &\twoheadrightarrow \mathbb{C} \\
(x,y) &\mapsto x
\end{aligned}$$

But it is probably more natural to consider a general projection

$$\pi : \overline{C} \twoheadrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$$

Such a projection is an example of a "meromorphic function".

Definition: Let X, Y be smooth 1D complex manifolds. A function $\pi: X \to Y$ is called a holomorphic map if its restrictions to charts are holomorphic:



[Note that this notion is independent of coordinates.]

A holomorphic map $\pi: X \to \mathbb{C}$ is called a <u>holomorphic function</u> on X

A holomorphic map $\pi : X \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is called a <u>meromorphic function</u> on X.

The sets $\pi^{-1}(0), \pi^{-1}(\infty) \subseteq X$ are the "zeros" and "poles" of the meromorphic function π. If X is compact then these sets are <u>finite</u>. [Discrete subset of compact set is finite.]

A zero $p \in X$ of $\pi : X \to \mathbb{CP}^1$ has <u>order d</u> if for any chart $\gamma : \mathbb{C} \to X$, $\gamma(0) = p$, we have a Taylor series of the form

$$\pi(\gamma(t)) = t^d + \text{higher terms}$$

$$[\pi(p) = \pi(\gamma(0)) = 0 \quad \checkmark]$$

Next we observe that the set of meromorphic functions on X has the structure of a field:

$$\mathbb{C}(X) = \{\pi : X \to \mathbb{CP}^1\}$$

Multiplication & addition are pointwise, assuming $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$, etc.

Observe that

$$\{\text{poles of } \pi\} = \{\text{zeros of } \tfrac{1}{\pi}\}$$

We say that a pole $p \in X$ of π has order d if the corresponding zero of $\frac{1}{\pi}$ has order d.

Convention: A pole of order d is also called a zero of order $-d$.

That is, for any point $p \in X$ and meromorphic function $\pi : X \to \mathbb{CP}^1$ we have well-defined integer

$$\mathrm{ord}_p(\pi) \in \mathbb{Z}$$

called "order of π at p".

There are three cases:

$\mathrm{ord}_p(\pi) > 0$ zero

$\mathrm{ord}_p(\pi) < 0$ pole

$\mathrm{ord}_p(\pi) = 0$ neither.

This function $\mathrm{ord}_p : \mathbb{C}(X) \to \mathbb{Z}$ satisfies the following properties:

- $\mathrm{ord}_p(\text{nonzero constant}) = 0$
- $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$
- Hence $\mathrm{ord}_p\left(\frac{1}{g}\right) = -\mathrm{ord}_p(g)$
  and $\mathrm{ord}_p(f/g) = \mathrm{ord}_p(f) - \mathrm{ord}_p(g)$
- $\mathrm{ord}_p(f+g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}$.

Such a function $\mathbb{C}(X) \to \mathbb{Z}$ is called a "discrete valuation" of the field $\mathbb{C}(X)$.

Example: Let $f(x) \in \mathbb{C}[x]$ have degree d:

$$f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_d.$$

By defining $f(\infty) = \infty$ we obtain a meromorphic function

$$f : \mathbb{CP}^1 \to \mathbb{CP}^1$$

Indeed, to show that f is meromorphic at infinity, we examine the function $1/f$ in the chart $x \mapsto 1/x$ (switching $0 \leftrightarrow \infty$).

$$f\left(\frac{1}{x}\right) = \frac{a_0}{x^d} + \frac{a_1}{x^{d-1}} + \cdots + \frac{a_{d-1}}{x} + a_d$$

$$= \frac{1}{x^d}\left(a_0 + a_1 x + \cdots + a_d x^d\right).$$

$$\frac{1}{f(1/x)} = \frac{x^d}{a_0 + a_1 x + \cdots + a_d x^d}$$

This is holomorphic in a neighborhood of $x = 0$ with Taylor series

$$\frac{1}{f(1/x)} = \frac{1}{a_0} x^d + \text{higher terms}.$$

In other words, $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ has a pole of order d at ∞.

The fundamental theorem of algebra then tells us that

$$\sum_{p \in \mathbb{CP}^1} \mathrm{ord}_p(f) = 0.$$

$$\left[\ \underbrace{d}_{\substack{\text{sum of}\\ \text{orders of}\\ \text{the roots}\\ \text{of } f(x)}} - \underbrace{d}_{\substack{\text{order of}\\ \text{the single}\\ \text{pole at}\\ \infty.}} = 0\ \right]$$