Algebraic Curves **Spring 2021**
Homework 2 **Drew Armstrong**

Turn in any one problem by Thurs Mar 25 on the Google classroom.. You may be able to find solutions in last semester's course notes, or elsewhere on my webpage.

Problem 1. Homogenization and Dehomogenization. Given $F(x,y,z) \in \mathbb{F}[x,y,z]$ we define $F_*(x,y) = F(x,y,1) \in \mathbb{F}[x,y]$, and given $f(x,y) \in \mathbb{F}[x,y]$ of degree d we define $f^*(x,y,z) = z^d f(x/z, y/z) \in \mathbb{F}[x,y,z]$. Prove the following:

(a) $(FG)_* = F_*G_*$ and $(fg)^* = f^*g^*$

(b) $(F+G)_* = F_* + G_*$ and $(f+g)^* = z^r f^* + z^s g^*$ where $r = \deg(g)$ and $s = \deg(f)$

(c) $(f^*)_* = f$ and $z^r(F_*)^* = F$ where $z^r|F$ and $z^{r+1} \nmid F$

(d) $(F_x)_* = (F_*)_x$ and $(F_y)_* = (F_*)_y$

Problem 2. Let $\mathbb{F}$ be an algebraically closed and let $f(x,y) \in \mathbb{F}[x,y]$ be nonzero.

(a) Prove that $\mathbb{F}$ has infinitely many elements.

(b) Prove that the curve $C = V(f)$ has infinitely many points.

Problem 3. Higher Cusps. Prove that the polynomial $x^m - y^n \in \mathbb{C}[x,y]$ is irreducible if and only if $\gcd(m,n) = 1$.[1]

Problem 4. Veronese Map. Prove that the map $\varphi_d : \mathbb{CP}^2 \to \mathbb{CP}^{d(d+3)/2}$ is injective, where

$$\varphi_d(x,y,z) := (x^d, x^{d-1}y, \ldots, z^d).$$

Problem 5. Hessian Identities. Let $F(x,y,z)$ be homogeneous of degree d.

(a) Prove Euler's identity: $xF_x + yF_y + zF_z = dF$.

(b) Use Euler's identity to prove the following:

$$\det\begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{xy} & F_{yy} & F_{yz} \\ F_{xz} & F_{yz} & F_{zz} \end{pmatrix} = \frac{d-1}{z} \det\begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{xy} & F_{yy} & F_{yz} \\ F_x & F_y & F_z \end{pmatrix} = \frac{(d-1)^2}{z^2} \det\begin{pmatrix} F_{xx} & F_{xy} & F_x \\ F_{xy} & F_{yy} & F_y \\ F_x & F_y & \frac{d}{d-1}F \end{pmatrix}$$

---

[1] https://math.stackexchange.com/questions/652392/xn-ym-is-irreducible-in-bbbcx-y-iff-gcdn-m-1