MTH 787 - Lie Theory

1/17/18

Certain hard problems in mathematics are completely understood, and they have essentially the same solutions

1) Compact manifolds that are also groups

2) Affine varieties that are groups

3) Finite groups generated by reflections.

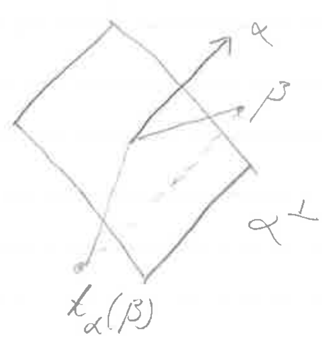
4) Root systems

Definition: Let V be a Euclidean space, i.e. $V = (\mathbb{R}^n, \langle , \rangle)$

A root system is a finite set of vectors $\Phi \subseteq V$ satisfying

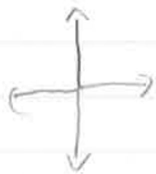
1) $\mathbb{R}\Phi \subseteq V$
2) $\forall \alpha \in \Phi$, $\Phi \cap \mathbb{R}\alpha = \{\pm \alpha\}$
3) $\forall \alpha \in V$, we define the reflection $t_\alpha : V \to V$ by
$$t_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$$

Then $\forall \alpha, \beta \in \Phi$, $t_\alpha(\beta) \in \Phi$

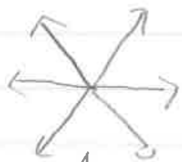


4) $\forall \alpha, \beta \in \Phi$, $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \mathbb{Z}$.

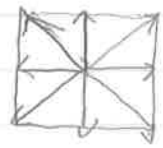
2D Classification:



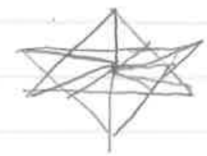
$A_1 \times A_1$



A_2



B_2 or C_2



G_2

Root Systems can be compressed into "Dynkin Diagrams":

A_n 

B_n 

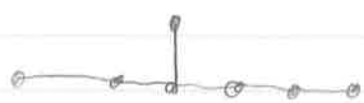
C_n 

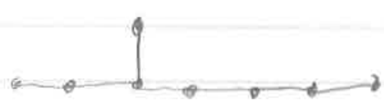
D_n 

 F_4

 G_2

 E_6

 E_7

 E_8

That's all!

Type	Simple Compact $\mathbb{R}$	Simple affine $\mathbb{C}$
A_n	$PSU(n+1)$	$PSL_{n+1}(\mathbb{C})$
B_n	$SO(2n+1)$	$SO_{n+1}(\mathbb{C})$
C_n	$PSp(n)$	$PSp_{2n}(\mathbb{C})$
D_n	$PSO(2n)$	$PSO_{2n}(\mathbb{C})$
G_2	Octonions	

Compact groups $\longleftrightarrow$ $\Lambda_R \subseteq ? \subseteq \Lambda_W$

weight lattice　　root lattice

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Classification:

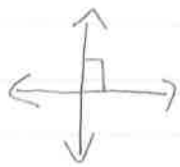
A_n　E_6, E_7, E_8
B_n　F_4
C_n　G_2
D_n
$\boxed{H_3, H_4, I_2[m]}$ $\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \notin \mathbb{Z}$

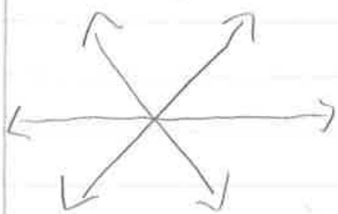
Recall: A root system $\Phi \subseteq V$ is a finite set s.t.

- $\mathbb{R}\Phi = V$
- $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$
- $\forall \alpha, \beta \in \Phi$, $t_\alpha(\beta) \in \Phi$, $t_\alpha(\beta) = \beta - \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$
- $\forall \alpha, \beta \in \Phi$, $\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Decomposition: $\Phi = \Phi_1 \sqcup \Phi_2$, s.t. $\forall\ \alpha \in \Phi_1,\ \beta \in \Phi_2,\ (\alpha,\beta)=0$

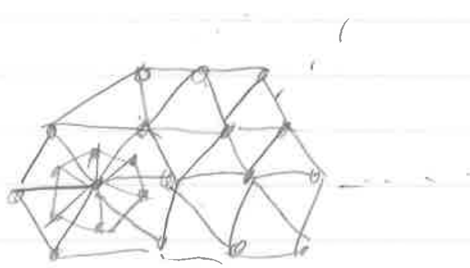
 $A_1, B_1 \to \overset{S^3}{SU(2)}, \overset{\mathbb{RP}^3}{SO(3)}, SL_2(\mathbb{C})$

 $A_1 \times A_1$

 $A_2 \to SL_3(\mathbb{C}), SU(3), PSU(3)$

Definition:

Let $\Phi \subset V$ be a root system.
The root lattice is $\Lambda_R = \mathbb{Z}\Phi \subseteq V$



The "weight lattice" $\Lambda_W = \{x \in V \mid (x,\alpha) \in \mathbb{Z},\ \forall \alpha \in \Phi\}$

Thm: Locally (group) isomorphic Compact Groups $\longleftrightarrow$ $\Lambda_R \subseteq$ Lattices $\subseteq \Lambda_W$

(abelian groups) π_1 (simply

Z(simply connected)

ex: A_1

$$\cdots \circ - \times - \circ - \times - \circ - \times - \circ - \times - \circ \cdots$$

$$\Lambda_W = \mathbb{Z}$$
$$\Lambda_R = 2\mathbb{Z}$$

Thm: $$\#\frac{\Lambda_W}{\Lambda_R} = \frac{\text{Covol}(\Lambda_W)}{\text{Covol}(\Lambda_R)}$$ ← Volume of fundamental domain

ex: If we look at A_2, $\frac{\text{Covol}(\Lambda_W)}{\text{Covol}(\Lambda_R)} = 3 \Rightarrow \frac{\Lambda_W}{\Lambda_R} \cong \mathbb{Z}_3$

ex: For A_1, $\frac{\Lambda_W}{\Lambda_R} \cong \mathbb{Z}_2$

$$Z(SU(2)) = \pm I$$
$$\pi_1(PSU(2) \cong SO(3)) = \mathbb{Z}_2$$

- $U(1) \simeq S^1$, $U(2) \simeq S^1 \times S^3$, $U(3) \simeq S^1 \times S^3 \times S^5$

- ~~Compact Simply connected E_8: $S^1 \times S^7 \times S^{11} \times S^{13} \times S^{\ldots}$~~

$A_1, A_2, \ldots$

abelian groups

Real compact groups $\longleftrightarrow$ affine complex groups

$SU(2)$ $\qquad\qquad$ $SL_2(\mathbb{C})$

How about projective?

Projective $\Rightarrow$ abelian

Let G be a group that is also a $\underbrace{\text{complex projective}}_{\Rightarrow \text{ compact}}$ variety.

Consider $\Phi: G \times G \longrightarrow G$

$(g,h) \longmapsto ghg^{-1}h^{-1}$

Φ is holomorphic on a compact set $\Rightarrow$ Φ is constant.

$\Rightarrow \Phi(g,h) = 1 \quad \forall g,h \in G.$

$\Rightarrow G = \mathbb{C}^n/\Lambda$ "elliptic curves"

Theorem: Compact real abelian $\Rightarrow$ $G = T^k \times \mathbb{R}^{n-k}$, where $T = \mathbb{R}/\mathbb{Z}$

aka $U(1) = e^{i\theta}$

Representations of abelian Lie groups. (Fourier Theory)

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$E(\lambda)^{\alpha} = \prod_i (1+x_i)^{\alpha}$

- Stephanie Frank Singer, *The Hydrogen Atom*
- Sternberg, *Group Theory and Physics*

1/24/17 *Type A_1:*

$$\Phi_{A_1} = \{\pm\alpha\},\ \alpha \neq 0$$

$$\Lambda_R \simeq \mathbb{Z}_{A_1} = \mathbb{Z}\alpha$$

$$\Lambda_W = \left\{ w \in \mathbb{R} \;\middle|\; \frac{2(w,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \right\}$$

$$\Updownarrow$$

$$\frac{2w}{\alpha} \in \mathbb{Z} \iff w \in \mathbb{Z}\frac{\alpha}{2}$$

"Fundamental Group" $= \Lambda_W / \Lambda_R = \mathbb{Z}_2$

Fundamental group of what?

Simple group is $SO(3) \cong PSU(2)$

$\pi_1(SO(3)) = \mathbb{Z}_2$, b/c $SO(3) \cong \mathbb{R}P^3$

Simply-connected group: $SU(2) \cong Spin(3) \xrightarrow{2:1} SO(3)$

$$Z(SU(2)) = \{\pm 1\} \cong \mathbb{Z}_2$$

Quantum Chemistry (orbitals)

Type A_2: (There's a picture here...)

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A_2 has roots $\alpha_1 = e_1 - e_2$
$\alpha_2 = e_2 - e_3$

where $\mathbb{R}^3 = \langle e_1, e_2, e_3 \rangle$

$\Phi_{A_2} = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$

$V = \mathbb{R}\Phi_{A_2} = (e_1+e_2+e_3)^{\perp} \subseteq \mathbb{R}^3$

$= \{c_1e_1 + c_2e_2 + c_3e_3 : c_1+c_2+c_3 = 0\}$

$\Lambda_R = \mathbb{Z}\Phi_{A_2} = \{c_1\alpha_1 + c_2\alpha_2 \mid c_1, c_2 \in \mathbb{Z}\} \cong \mathbb{Z}^2$

$\Lambda_W = \{w \in V \mid \frac{2(w,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}, \ i = 1,2\}$

$w = x\alpha_1 + y\alpha_2 = \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ -y \end{pmatrix} = \begin{pmatrix} x \\ y-x \\ -y \end{pmatrix}$

$(\alpha_i, \alpha_i) = 2$, so $\frac{2(w,\alpha_1)}{(\alpha_1,\alpha_1)} = (w, \alpha_1)$

Need $\begin{pmatrix} x \\ y-x \\ -y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 2x - y \in \mathbb{Z}$

$\Rightarrow$ $x + y \in \mathbb{Z}$

Similarly $\begin{pmatrix} x \\ y-x \\ -y \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 2y - x \in \mathbb{Z}$

$3x \in \mathbb{Z}$

$3y \in \mathbb{Z}.$

$$\begin{array}{c|cc} & \omega_1 & \omega_2 \\ \alpha_1 & \frac{2}{3} & \frac{1}{3} \\ \alpha_2 & \frac{1}{3} & \frac{2}{3} \end{array} \rightarrow \begin{array}{c|cc} & \omega_1+\omega_2 & \omega_2 \\ \alpha_1 & 1 & \frac{1}{3} \\ \alpha_2 & 1 & \frac{2}{3} \end{array}$$

$$\downarrow$$

$$\begin{array}{c|cc} & \omega_1+\omega_2 & \omega_2 \\ \alpha_1 & 1 & \frac{1}{3} \\ \alpha_2-\alpha_1 & 0 & \frac{1}{3} \end{array}$$

Λ_W has a Standard basis

$$\omega_1 = \frac{2\alpha_1+\alpha_2}{3} = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$$

$$\omega_2 = \frac{\alpha_1+2\alpha_2}{3} = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

$$\Lambda_W \cong \mathbb{Z}^2$$

$$\Lambda_W / \Lambda_R \cong \mathbb{Z}_3$$

Fact: $\#(\Lambda_W/\Lambda_R) = \det([\alpha_1]_W\ [\alpha_2]_W) = \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3$

"Cartan Matrix"

The Cartan Matrix:

- $C w_i = \alpha_i \quad \forall i$
- Duality: $(\omega_i, \alpha_j) = \delta_{ij}$

~~$\omega_i = C^{-1}\alpha_i \Rightarrow \alpha_j^T \omega_i = \alpha_j^T C^{-1}\alpha_i = \delta_{ij}$~~

~~Let $C^{-1} = (c_{ij})$~~

~~Then $w_i = c_{ij}\alpha_j$~~

~~$\alpha_k^T w_i = \alpha_k^T c_{ij}\alpha_j = \delta_{ik}$~~

$$C = (\alpha_i^T \alpha_j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$(\alpha_1, \alpha_1) = 2 = (\alpha_2, \alpha_2)$

$(\alpha_1, \alpha_2) = 2$

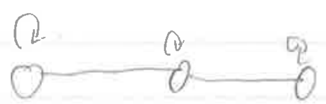
$$C = ([\alpha_1]_w, [\alpha_2]_v) = (\alpha_i^T \alpha_j)$$

Dynkin diagram:

Nodes are root basis

Edges record the "angle" (α_i, α_j)

A_2



$$A_3 \longleftrightarrow \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

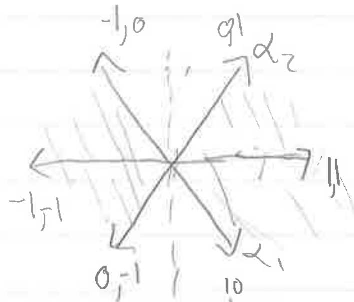
In physics, we have a basis for Λ_R

$L_i = e_i - \frac{1}{3}(e_1 + e_2 + e_3)$

1/26/18 The "Cartan Matrix"

Every root system Φ has a "simple basis":

- $\forall \alpha \in \Phi$, $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$, so a generic $\overset{\text{in}}{V}$ hyperplane cuts the root system into equal pieces



Π = simple basis

Every root β has a unique expression $\beta = \sum_{\alpha \in \Pi} n_{\beta\alpha}\alpha$, where $n_{\beta\alpha} \in \mathbb{Z}$ are either

- all nonnegative
- all nonpositive

The Cartan matrix records the angles b/w basis vectors:

$$C = (c_{ij}), \text{ where } c_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z}$$

Root system A_2

$$\Pi = \{\alpha_1, \alpha_2\}, \quad \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$C = A^t A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Lambda_R = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$$

Recall: $\Lambda_W = \{ v \in V \mid \langle w, \alpha \rangle \in \mathbb{Z} \ \forall \alpha \in \Pi \}$

<u>Dual basis</u>: $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$

A_2: Let ω_1, ω_2 be rows/cols of C^{-1}

<u>Fact</u>: $\ker(A^t A) = \ker(A)$

<u>Pf</u>: $Ax = 0 \Rightarrow A^t(Ax) = 0$ ✓

$$A^t A x = 0 \Rightarrow \|Ax\|^2 = (Ax)^t (Ax) = x^t (A^t A x) = 0 \Rightarrow Ax = 0.$$

□

<u>Now</u>: $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\omega_1 = \frac{1}{3}(2\alpha_1 + \alpha_2)$ (so ω_i is the i^{th} row of C^{-1})

$\omega_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$

<u>Thm</u>: $\Lambda_W = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \cdots$

<u>Pf</u>: Let $D = (d_{ij}) = C^{-1}$

Define $\omega_i = \sum_j d_{ij}\alpha_j$.

$$\langle \omega_i, \alpha_k \rangle = \langle \sum_j d_{ij}\alpha_j, \alpha_k \rangle = \sum_j d_{ij} \langle \alpha_j, \alpha_k \rangle$$

$$= \sum_j d_{ij} c_{jk} = (DC)_{ik} = \delta_{ik}$$

Since $\delta_{ik} \in \mathbb{Z}$, we have $\omega_i \in \Lambda_W$ $\forall i$, so $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \cdots \subseteq \Lambda_W$.

Conversely, note that $\mathbb{R}\omega_1 + \mathbb{R}\omega_2 + \cdots = \mathbb{R}\alpha_1 + \mathbb{R}\alpha_2 + \cdots = V$

Choose any $x_1\omega_1 + x_2\omega_2 + \cdots \in \mathbb{R}\omega_1 + \mathbb{R}\omega_2 + \cdots = V$

Then, if $\sum_i x_i \omega_i \in \Lambda_W$,

$\mathbb{Z} \ni \langle \sum_i x_i \omega_i , \alpha_k \rangle = \sum_i x_i \langle \omega_i , \alpha_k \rangle = x_k$,

so $x_k \in \mathbb{Z} \ \forall k$.

Hence $\Lambda_W \subseteq \mathbb{Z}\langle \omega_1, \dots \rangle$.

$\square$

Corollary: $|(\Lambda_W / \Lambda_R)| = \det C$

Idea: $|\Lambda_W / \Lambda_R| = \dfrac{\mathrm{vol}\ V/\Lambda_R}{\mathrm{vol}\ V/\Lambda_W}$

Note $C : \Lambda_W \to \Lambda_R$, so $\det C =$ (by Smith Normal form or something)

Systematic discussion of Lie groups:

Abelian Lie groups?

Thm: If G is abelian, $G = (\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-k}$

Pf: I.O.U.

A representation is a group homom $G \longrightarrow GL_n(\mathbb{C})$

Representations of $(\mathbb{R}, +)$?

$(\mathbb{R}/\mathbb{Z}, +)$?

John Baez Classical Mechanics

f.d. Countable all dimensions

Schur-Dixmier-Quillen Lemma: Let G be abelian and let U be an irreducible $\mathbb{C}G$-module $\mathbb{C}G \curvearrowright U$.

Then, every $g \in G$ acts like a scalar, i.e. $g(u) = \lambda_g u$ for all $g \in G$, $u \in U$.

Pf (for f.d. case): $g: U \to U$ is a matrix. It has an eigenvalue $\lambda \in \mathbb{C}$.

Thus, $g - \lambda I : U \to U$ is not invertible

However, since G is abelian, $g - \lambda I$ is a $\mathbb{C}G$-module map

$\ker(g - \lambda I) \subseteq U$ is a G-submodule

Since $\ker(g-\lambda I) \neq 0$, $\ker(g - \lambda I) = U$, so $g - \lambda I \equiv 0 \Rightarrow g \equiv \lambda I$. $\square$

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Structure Theory:

- What kind of Lie groups can there be?

General Fact (Cartan Decomposition / Iwasawa Decomposition)
Every Lie group G has a maximal compact subgroup $K \leq G$
s.t. any compact subgroup is conjugate to a subgroup of K.

Furthermore, $G \underset{\text{homeo}}{\cong} K \times (\mathbb{R}, +)^n$ "Polar decomposition" / "Gram-Schmidt Process"

"Polar decomposition" $\longrightarrow$ $z = r e^{i\theta}$ with $r \in \mathbb{R}$, $e^{i\theta} \in K$ Gram-Schmidt $\longrightarrow$ $g =$ unitary/orthogonal $\times$ pos. def

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If G is abelian, we have $G = K \times \mathbb{R}^n$ as groups.

Fact: Compact abelian groups are tori!

$K = (\mathbb{R}/\mathbb{Z})^n = \mathbb{T}^n$, where $\mathbb{T} = (\mathbb{R}/\mathbb{Z})$

Representation Theory = Group Theory (according to Physicists)

Q: Can we Realize Lie groups as matrices?
Not always, but mostly!

Schur-Dixmier-Quillen Lemma: Let G be abelian, $\varphi: G \to End_{\mathbb{C}}(U)$ be an irreducible $\mathbb{C}G$-module.

Then: $\forall g \in G$, $g(u) = \lambda_g u$ $\forall u \in U$ for some $\lambda_g \in \mathbb{C}$.

- Consequently, every vector subspace is preserved by G.
  $\Rightarrow U$ has no nontrivial subspaces
  $\Rightarrow \dim_{\mathbb{C}}(U) = 1$

Pf: Fix $g \in G$. Since $\mathbb{C}$ is alg. closed, the endomorphism $g: U \to U$ has an eigenvalue $\lambda_g \in \mathbb{C}$.

Consider $(g - \lambda_g I): U \to U$
This is not invertible, but it commutes with G-action:

$$(g - \lambda I)(hu) = (gh - \lambda_g h)(u) = (hg - \lambda_g h)(u)$$
$$= h(g - \lambda_g I)(u)$$

$\Rightarrow \ker(g - \lambda_g I) \subseteq U$ is a nonzero $\mathbb{C}G$-submodule
Since is U is irred., $\ker(g - \lambda_g I) = U$, so $g - \lambda_g I \equiv 0$
$\Rightarrow g \equiv \lambda_g I$.

$\square$

Pf (of Countable-dimensional case): $g: U \to U$

Goal: Prove $\exists\, c \in \mathbb{C}$ where $(g - cI): U \to U$ is not invertible

Suppose $(g - cI): U \to U$ is invertible $\forall\, c \in \mathbb{C}$.
Then $p(g)$ $\forall$ polys $p \in \mathbb{C}[x]$, $p \neq 0$

$$P(g) = (g - c_1 I)(g - c_2 I)\cdots(g - c_n I)$$

Thus g gives a map
$$\mathbb{C}(x) \longrightarrow \mathrm{End}_{\mathbb{C}}(U)$$
$$\frac{p(x)}{q(x)} \longmapsto p(g)\,q(g)^{-1}$$

Furthermore, given a fixed nonzero vector $u \in U$, we have, for all $0 \neq r(x) \in \mathbb{C}(x)$ that $r(g)u \neq 0$

Hence, the map
$$\varphi_u: \mathbb{C}(x) \longrightarrow U$$
$$r(x) \longmapsto r(g)u$$

is injective and $\mathbb{C}$-linear
This is a contradiction, because $\left\{ \frac{1}{x-c} \mid c \in \mathbb{C} \right\}$ is linearly indep. $\square$

What are the irreps?

ex: $\varphi: \mathbb{R} \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^{\times}$
$\varphi: \mathbb{R} \longrightarrow \mathbb{C}^{\times}$
$\varphi(t_1 + t_2) = \varphi(t_1)\varphi(t_2)$
$\varphi(0) = 1$

$$\frac{d}{dt}\log(\varphi) = \varphi'(0)$$

Assume φ is continuous (it turns out that Cts $\Rightarrow$ Smooth in this case)

Actually, assume $\varphi'(0)$ exists.

Then, $\varphi'(t) = \lim_{\varepsilon \to 0} \frac{\varphi(t+\varepsilon) - \varphi(t)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\varphi(t)\varphi(\varepsilon) - \varphi(t)}{\varepsilon}$

$$= \varphi(t) \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon) - 1}{\varepsilon}$$

$$= \varphi(t) \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon} = \varphi(t)\varphi'(0)$$

$$\Rightarrow \varphi(t) = e^{\varphi'(0)t}$$

Thus, $\varphi(t) = e^{\alpha t}$ for some $\alpha \in \mathbb{C}$.

Bijection: irreps of $(\mathbb{R}, +) \longleftrightarrow \mathbb{C}$

irreps of $(\mathbb{R}/\mathbb{Z}, +) \longleftrightarrow$?

$\varphi(0) = \varphi(1)$

$1 = \varphi(1) = e^{\alpha}$

$\Rightarrow \alpha = 2\pi i k, \; k \in \mathbb{Z}$

So irreps of $(\mathbb{R}/\mathbb{Z}, +) \longleftrightarrow \mathbb{Z}$.

1/31/18: Why do we focus on compact groups?

Thm (Cartan – Iwasawa – Malcev)
Every Lie group G has a maximal compact $K \subseteq G$, and cosets:
$G/K \underset{\text{homeo}}{\cong} \mathbb{R}^n$

ex: $GL_1(\mathbb{C}) = \mathbb{R}^+ \times U(1)$
$\mathbb{C}^\times$

Polar Decomposition: Let $K \in \{\mathbb{R}, \mathbb{C}\}$. Every $g \in GL_n(K)$ has a unique decomp $g = pu$, where
- $p \in GL_n(\mathbb{R})$ is symmetric + pos. def.
- $u \in O(n)$, if $K = \mathbb{R}$
  $\in U(n)$ if $K = \mathbb{C}$.

Note: If P is the set of symmetric pos. def $n \times n$ $\mathbb{R}$-matrices, then
$P \underset{\text{homeo}}{\cong} \mathbb{R}^{n(n+1)/2}$

Corollary: $GL_n(\mathbb{C}) \underset{\text{homeo}}{\cong} U(n) \times \mathbb{R}^{n(n+1)/2}$

$GL_n(\mathbb{R}) \underset{\text{homeo}}{\cong} O(n) \times \mathbb{R}^{n(n+1)/2}$

$GL_n^+(\mathbb{R}) \underset{\text{homeo}}{\cong} SO(n) \times \mathbb{R}^{n(n+1)/2}$

Pf: For $g \in GL_n(K)$, eigenvalues of gg^* are real and positive.
Indeed, let $gg^* x = \lambda x$ for $\lambda \in \mathbb{C}$, $x \in \mathbb{C}^n \setminus \{0\}$. Then we have

$gg^* x = \lambda x$
$g^* g (g^* x) = \lambda (g^* x) \iff g^* g y = \lambda y \quad (y \neq 0)$

Then, $0 < \|gy\|^2 = (gy)^* (gy) = y^* (g^* g y) = y^* \lambda y = \lambda \|y\|^2 \Rightarrow \lambda > 0.$ ✓

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Want to write $g = pu$.
Want $p = $ "$+\sqrt{gg^*}$"

Spectral theorem $\Rightarrow$ gg^* is unitarily / orthogonally diagonalizable:

$$gg^* = h \Lambda h^*, \quad \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

So, define $p = +\sqrt{gg^*} := h\sqrt{\Lambda}h^*$, where $\sqrt{\Lambda} = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$,
and observe that, indeed,

$$p^2 = h\sqrt{\Lambda}h^* h\sqrt{\Lambda}h^* = h(\sqrt{\Lambda})^2 h^* = h\Lambda h^* = gg^* \quad \checkmark$$

Uniqueness? Define the unique degree n polynomial $f(x)$ s.t.
$f(\lambda_i) = \sqrt{\lambda_i}$.

Then $f(\Lambda) = \sqrt{\Lambda}$, and $f(h\Lambda h^*) = hf(\Lambda)h^* = h\sqrt{\Lambda}h^* = p$

$\Rightarrow f(gg^*) = p$

Now, suppose $p^2 = gg^* = q^2 \Rightarrow \dots ?$

Upshot, $pq = qp$, so p & q are simultaneously diagonalizable, so

$$\left.\begin{aligned} p &= z\sqrt{\Lambda}z^* \\ q &= z\sqrt{\Lambda}z^* \end{aligned}\right\} \Rightarrow p = q$$

$\checkmark$

Want $g = pu$. Is $u := p^{-1}g$ unitary?

WTS $u^*u = uu^* = I$.

Will show that $\forall y_1, y_2 \in \mathbb{C}^n$, $y_1^* u u^* y_2 = y_1^* y_2$

Then, plugging in $y_1 = e_i$, $y_2 = e_j$, we have

$$(uu^*)_{ij} = e_i^*(uu^*)e_j = e_i e_j = \delta_{ij}$$

Note that P is invertible, so we have $y_1 = Px_1$
$y_2 = Px_2$

$$y_1^*(uu^*)y_2 = (u^*y_1)^*(u^*y_2) \qquad (\text{since } u = p^{-1}g,\ u^* = g^*p^{-1})$$

$$= (g^*p^{-1}y_1)^*(g^*p^{-1}y_2)$$

$$= (g^*x_1)^*(g^*x_2) = x_1^*(gg^*)x_2$$

$$= x_1^* PP x_2$$

$$= (Px_1)^*(Px_2)$$

$$= y_1^* y_2$$

$\square$

If G is abelian, then $G = U(1)^k \times \mathbb{R}^n$

Last time: Schur's Lemma: All irreps $\varphi: G \longrightarrow GL_n(\mathbb{C})$ have $n=1$ if G is abelian.

Irreps $\varphi: \mathbb{R} \longrightarrow \mathbb{C}^\times$
cts $\quad \varphi(t) = e^{\alpha t}, \qquad \alpha \in \mathbb{C}.$

$\varphi: \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}^\times$
$1 = \varphi(0) = \varphi(1) = e^{\alpha} \Rightarrow \alpha = 2\pi i k,\ k \in \mathbb{Z}$

"quantization"

$\binom{2n-2}{\cdots}$

Multiplicative: $\mathbb{R}/\mathbb{Z} = U(1)$
$t \longleftrightarrow e^{2\pi i t}$

$\varphi: U(1) \longrightarrow U(1)$
$e^{2\pi i t} \longmapsto e^{2\pi i k t} \quad k \in \mathbb{Z}$
$\varphi_k: z \longmapsto z^k$

$\varphi: U(1)^n \longmapsto U(1)$
$(z_1, \ldots, z_n) \longmapsto z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$

2/5/18 Maximal Compact Subgroups
Characters of tori (Fourier analysis):

Thm (Cartan - Iwasawa - Malcev): Existence of max. cpt subgroup $K \trianglelefteq G$.

Last time: Cartan Decomp: $G = PU$ "Polar"

Iwasawa Decomposition: $G = KAN$

Classical types (A, B, C, D):

Let $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and let $G = GL_n(F)$

Note: $q = (x_1 + ix_2 + jx_3 + kx_4)$, $q^* = (x_1 - ix_2 - jx_3 - kx_4)$, $|q|^2 = qq^*$,

$$\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \sum q_i u_i^*$$

Let $K = O_n(F) = \{ g \in GL_n(F) \mid g^* g = I \}$, $O_n(\mathbb{R}) = O(n)$, $O_n(\mathbb{C}) = U(n)$, $O_n(\mathbb{H}) = Sp(n)$

This is a (maximal) compact subgroup of $GL_n(F)$.

$\dim(GL_n(\mathbb{R})) = n^2$
$\dim(GL_n(\mathbb{C})) = 2n^2$
$\dim(GL_n(\mathbb{H})) = 4n^2$

Let $A = \{ \operatorname{diag}(r_1, \dots, r_n) \mid r_i \in \mathbb{R},\ r_i > 0 \}$

$$A = \mathbb{R}_+^n$$

$$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \;\middle|\; * \in F \right\}$$

$$N = \begin{cases} \mathbb{R}^{n(n-1)/2}, & F = \mathbb{R} \\ \mathbb{R}^{n(n-1)}, & F = \mathbb{C} \\ \mathbb{R}^{2n(n-1)}, & F = \mathbb{H} \end{cases}$$

<u>Theorem</u>: The multiplication map

$$K \times A \times N \longrightarrow G$$
$$(k, a, n) \longmapsto kan$$

is a diffeomorphism.

<u>Corollaries</u>: $GL_n(\mathbb{C}) = U(n) \times \mathbb{R}_+^n \times \mathbb{R}^{n(n-1)}$ $\qquad A$

$GL_n(\mathbb{R}) = O(n) \times \mathbb{R}_+^n \times \mathbb{R}^{n(n-1)/2}$ $\qquad B + D$
$GL_n^+(\mathbb{R}) = SO(n) \times \mathbb{R}_+^n \times \mathbb{R}^{n(n-1)/2}$ $\qquad$ n odd $\quad$ n even

$GL_n(\mathbb{H}) = Sp(n) \times \mathbb{R}_+^n \times \mathbb{R}^{2n(n-1)}$ $\qquad C$

Note: As fiber bundles, we have

$$0 \longrightarrow \underset{\substack{\uparrow \\ \text{contractible}}}{AN} \longrightarrow \underset{kan \longmapsto k}{G} \longrightarrow K \longrightarrow 0$$

Pf of Iwasawa: $K \times A \times N \longrightarrow G$ is smooth by definition.

Invertible?

Injective: $A \cap N = \{I\} \implies A \times N \longrightarrow AN$ is injective.

Indeed, $a_1 n_1 = a_2 n_2 \implies a_2^{-1} a_1 = n_2 n_1^{-1} = I$ ✓

$K \cap AN = \{I\}$?

$$g \in AN \implies g = \begin{pmatrix} r_1 & & g_{ij} \\ & \ddots & \\ 0 & & r_n \end{pmatrix}, \quad r_i > 0, \quad g_{ij} \in F$$

If $g \in K$, $(g_{-,1})^* \cdot (g_{-,2}) = 0 \iff r_1^* g_{12} = 0 \implies g_{12} = 0$

Proceeding by induction, we see that $g_{1j} = 0 \ \forall j$, and then continuing the induction we see $g_{ij} = 0 \ \forall i,j$.

Similarly, $(g_{\cdot i})^* g_{\cdot i} = 1 \iff r_i^* r_i = 1 \iff r_i^2 = 1 \implies r_i = 1$, since $r_i > 0$.

By the same argument as above, we see that this implies that

$$K \times (A \times N) \longrightarrow G \text{ is injective, so}$$

$K \times A \times N \longrightarrow G$ is injective ✓

Surjective: Given $g \in G$, can we write $g = kan$?

Gram-Schmidt: Let $g_i \in F^n$ be the i^{th} column of g.
To get an orthonormal basis:

$h_1 = g_1 / \|g_1\|$

$h_2 = (g_2 - \mathrm{Proj}_{h_1}(g_2)) / \|\cdot\|$

$= \dfrac{g_2 - (g_2, h_1)h_1}{\|g_2 - (g_2, h_1)h_1\|}$

$\vdots$

$h_k = \dfrac{1 \, g_k - \sum_{i=0}^{k-1} (g_k, h_i) h_i}{\|\cdot\|}$ $\in \mathbb{R}^+$

Now, note that $(h_k, h_k) = 1$ by def.

(proceed by induction on k)

If $j < k$, $(h_k, h_j) = \frac{1}{\|\cdot\|}\left((g_k, h_j) - \sum_i (g_k, h_i) \underbrace{(h_i, h_j)}_{\delta_{ij}} \right)$

$= \frac{1}{\|\cdot\|}((g_k, h_j) - (g_k, h_j)) = 0.$

Thus, we have $g \underset{(\text{col})}{an} = h \in$, $h \in O_n(F)$

(column operations)

$\Updownarrow$

$g = h a^{-1} (a n^{-1} a^{-1})$

$h \in K$, $a^{-1} \in A$, $(a n^{-1} a^{-1}) \in N$

Gram-Schmidt is smooth, so we have a smooth inverse!

$\square$

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[dr, dash] & & B \\
 & K \cap B = T &
\end{tikzcd}$$

all upper triangular matrices

$K \cap B = T \leftarrow$ a torus!

$\Rightarrow \frac{K}{T} \cong \boxed{G/B}$ Flag manifold

compact!

Line bundles on $G/B = \Lambda_W$

2/7/18 Last time:

Iwasawa: Let $G = GL_n(F)$, $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$

Let $K \leq G$ be the subgroup $K = \{g \in GL_n(F) \mid g^* g = I\}$

$= O(n), U(n), Sp(n)$

$A = \{\operatorname{diag}(a_1, \ldots, a_n) \mid 0 < a_1, \ldots, a_n \in \mathbb{R}\}$

$N = \left\{\begin{pmatrix} 1 & & n_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \;\middle|\; n_{ij} \in F\right\}$

Theorem: $K \times A \times N \longrightarrow G$

$(k, a, n) \longmapsto kan$

is a diffeomorphism.

Proof sketch: Clearly it's a smooth map.

Injective: $A \cap N = \{I\}$

$K \cap AN = \{I\}$

Surjective: (Gram-Schmidt) $g \in G$, $g = \begin{pmatrix} | & | & & | \\ g_1 & g_2 & \cdots & g_n \\ | & | & & | \end{pmatrix}$, $g_i \in F^n$

$$gna = \begin{pmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{pmatrix} \begin{pmatrix} 1 & & n_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = \begin{pmatrix} | & & | \\ k_1 & \cdots & k_n \\ | & & | \end{pmatrix} = k$$

$a_i = \frac{1}{\|k_i\|} \in \mathbb{R}^+$, $n_{ij} = -(k_i, g_j) \in F$ $\qquad$ $k^* k = I$

$g = k a^{-1} n^{-1}$

$a^{-1} = \operatorname{diag}\left(\frac{1}{a_1}, \dots, \frac{1}{a_n}\right) \in A$ ✓

$n^{-1} = \begin{pmatrix} 1 & & ? \in F \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in N$ ✓

$g \longmapsto (k, a^{-1}, n^{-1})$ is smooth.

$\square$

Corollary (Flag manifold/Variety)

$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$ $\qquad$ $B \supseteq AN = \left\{ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \; 0 < a_1, \dots, a_n \in \mathbb{R} \right\}$

$\uparrow$

"Borel Subgroup" (Conjugates of B in G)

$T = K \cap B$ is a "maximal torus".

In the case of $GL_n(\mathbb{C})$, we have $T = \left\{ t = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \; t^* t = I \right\}$

$$= \left\{ t = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \; t^* t = I \right\}$$

$$= \left\{ \begin{pmatrix} e^{it_1} & & 0 \\ & \ddots & \\ 0 & & e^{it_n} \end{pmatrix} \right\} = U(1)^n$$

$$\begin{tikzcd}[column sep=small] & G \; \boxed{= KB} \arrow[dl, dash] \arrow[dr, dash] & \\ K \arrow[dr, dash] & & B \arrow[dl, dash] \\ & K \cap B = T & \end{tikzcd}$$

"2nd Isomorphism Theorem" $K/T = G/B$ as manifolds.

Pf: $K/T \longrightarrow G/B$

$kT \longmapsto kB$

Well-defined? $k_1 T = k_2 T \Rightarrow k_2^{-1} k_1 \in T$

$\Rightarrow k_2 k_1^{-1} \in B \Rightarrow k_1 B = k_2 B$

Invertible? $G/B \longrightarrow K/T$

$gB = kanB \longmapsto kT$

$= kB$

$k_1 B = k_2 B \iff k_2^{-1} k_1 \in B \cap K = T$

$\iff k_1 T = k_2 T$

$\square$

Example: Let $Fl(\mathbb{R}^n) = \{ (V_0, V_1, \dots, V_n) \mid 0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{R}^n \}$
(note: $\dim V_i = i$)

$GL_n(\mathbb{R}) \curvearrowright Fl(\mathbb{R}^n)$ transitive.
$g(V_i) = gV_i$

$\mathrm{Stab}(\underset{}{V_0} \subsetneq \underset{e_1}{V_1} \subsetneq \underset{e_1, e_2}{\cdots} \subsetneq \underset{e_1, \dots, e_n}{V_n}) = B$

So Orbit-Stabilizer $\Rightarrow$ $G/B \longleftrightarrow$ Flags

Abelian Lie Groups / Classical Fourier Analysis

Recall: All abelian Lie groups are products of $(\mathbb{R},+)$ and $(T,+)$ where $T = \mathbb{R}/\mathbb{Z}$.

Representation theory studies "abstract representations"

$$\varphi : G \longrightarrow GL_n(\mathbb{C}) \quad \leftarrow \text{alg. closed, char. } 0$$

Alternatively we look at ring homomorphisms

$$\underset{\text{group ring}}{\mathbb{C}G} \longrightarrow \underset{\mathbb{C}\text{-vector space}}{End(V)}$$

Thm (Density Theorem): Action $G \curvearrowright V$ is irreducible $\iff$ $\mathbb{C}G \longrightarrow End(V)$ is surjective.

Representations of $(\mathbb{R},+)$:
$$\varphi : \mathbb{R} \longrightarrow GL_n(\mathbb{C})$$

Theorem (see T. Tao "Hilbert's Fifth Problem"): Under mild conditions:
- $\varphi(s+t) = \varphi(s)\varphi(t)$
- φ cts

Then $\varphi(t) = e^{At}$ for unique $A \in Mat_n(\mathbb{C})$

n-dim'l $\mathbb{C}$-reps of $\mathbb{R} \longleftrightarrow Mat_n(\mathbb{C})$

Such homs
$$\varphi_A : \mathbb{R} \longrightarrow GL_n(\mathbb{C})$$
$$t \longmapsto e^{At}$$

are called "One-parameter subgroups" of $GL_n(\mathbb{C})$.

Jargon: $Mat_n(\mathbb{C})$ is the Lie algebra of the Lie group $GL_n(\mathbb{C})$.

Idea: $(n=1)$ $\varphi: \mathbb{R} \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$

$\varphi(0) = 1$

$\varphi(s+t) = \varphi(s)\varphi(t)$

φ cts.

For t near 0, i.e., $\varphi(t)$ near 1, we can define

$f(t) = \log(\varphi(t))$

We get a cts function $f: \mathbb{R} \longrightarrow \mathbb{C}$

$f(0) = 0$

$f(s+t) = f(s) + f(t)$

Claim: $f(t) = \alpha t$ for some scalar $\alpha \in \mathbb{C}$.

$\Rightarrow \varphi(t) = e^{\alpha t}$

Indeed, $2f(t/2) = f(t/2) + f(t/2) = f(t)$

$\Rightarrow \frac{m}{n} f(t) = f(\frac{m}{n} t) \quad \forall \frac{m}{n} \in \mathbb{Q} \Rightarrow f$ is linear

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2/9/18

Type A:

$G = GL_n(\mathbb{C})$, $K = \{g \in G \mid g^* g = I\} = U(n)$

$B = \{\text{upper-triangular}\}$

$T = K \cap B = \text{Unitary + upper-triangular} = \text{Unitary + diagonal}$

$= \{\text{diag}(a_1, \dots, a_n) \mid a_i^* a_i = |a_i|^2 = 1\}$

$= \{\text{diag}(e^{it_1}, \dots, e^{it_n}) \mid t_i \in \mathbb{R}\} = U(1)^n$.

Claim: If $H \leq G$ is any compact abelian subgroup, then $\exists g \in G$ s.t. $gHg^{-1} \leq T$

Pf: Consider the inclusion $H \hookrightarrow GL_n(\mathbb{C})$
$\nearrow$ cpt abelian

We first find a matrix $g \in G$ s.t. $gHg^{-1} \subseteq U(n)$

① Since H is compact, $\exists !$ translation-invariant measure, called the Haar measure

Consider the standard Hermitian form $(\,,\,)$ on $\mathbb{C}^n$.

If $H \not\subseteq U(n)$, then we have $(hx, hy) \neq (x,y)$, $h \in H$, $x,y \in \mathbb{C}^n$.

Let's fix this! $\forall x,y \in \mathbb{C}^n$, define

$$(x,y)' := \int_{h \in H} (hx, hy)\, dh$$

Since the Haar measure is translation-invariant, we have $\forall g \in H$, $f: H \to \mathbb{C}$,

$$\int_H f(h)\, dh = \int_H f(gh)\, dh = \int_H f(hg)\, dh$$

Then $\forall h' \in H$, $\forall x,y \in \mathbb{C}^n$, we have

$$(h'x, h'y)' = \int_H (h'hx, h'hy)\, dh = \int_H (hx, hy)\, dh = (x,y)'.$$

Choose a basis for $\mathbb{C}^n$ that is orthonormal wrt $(\,,\,)'$.

Now change bases:

$gHg^{-1} \subseteq U(n)$

Remark: We just proved that $U(n)$ is maximal compact!

(2) Every element of $U(n)$ is diagonalizable (in fact, unitarily diagonalizable)

Lemma: Consider a unitary matrix $g \in U(n)$ + $V \subseteq \mathbb{C}^n$. Then
g stabilizes $V \iff g^*$ stabilizes $V^\perp \iff g^{-1}$ stabilizes $V^\perp \iff g$ stabilizes $V^\perp$.

Pf: For all $x \in V$, $y \in V^\perp$, we have
$(gx, y) = (x, g^* y)$

If g stabilizes V, then $gx \in V$, so $0 = (gx, y) = (x, g^* y)$ $\forall x \in V$
$\Rightarrow g^* y \in V^\perp \Rightarrow g^*$ stabilizes $V^\perp$. Also, $g^* = g^{-1}$.

Furthermore, for any subspace $U \subseteq \mathbb{C}^n$, we claim that
g stabilizes U iff g^{-1} stabilizes U.

Indeed, consider the subspace $g(U) \subseteq U$. $\ker(g) = 0$ by invertibility.
By finite-dimensionality, $g(U) = U$.

Let $y \in U$, so $y = gx$ for some $x \in U$. Then $g^{-1}y = x \in U$, so g^{-1} stabilizes U. Converse follows by the same argument. $\square$

Given $g \in U(n)$, $\exists$ eigenvector $gv = \lambda v$. Extend v to an orthonormal basis to get

$$PgP^{-1} = \left(\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & g' & \\ 0 & & & \end{array}\right)$$

Since pgp^{-1} is unitary, (Since p, g, p^{-1} are), $\lambda \in U(1)$ and $g' \in U(n-1)$. Proceed by induction.

Next, use abelian

(3) Elements of H are diagonalizable and commute, so they are simultaneously diagonalizable.

Pf: Let $h_1, h_2 \in H \subseteq U(n)$

Suppose $h_1 v = \lambda v$ for some $0 \neq v \in \mathbb{C}^n$.

Then
$$h_1(h_2 v) = h_2(h_1 v) = h_2(\lambda v) = \lambda h_2(v).$$

Thus, h_2 preserves the eigenspaces of h_1.

Diagonalize h_1:

$$uh_1u^{-1} = \begin{pmatrix} \boxed{\lambda_1 I} & & & \\ & \boxed{\lambda_2 I} & & \\ & & \ddots & \\ & & & \boxed{\lambda_d I} \end{pmatrix}, \qquad uh_2u^{-1} = \begin{pmatrix} \boxed{S_1} & & & \\ & \boxed{S_2} & & \\ & & \ddots & \\ & & & \boxed{S_d} \end{pmatrix}$$

Since $uh_2u^{-1} \in U(n) \Rightarrow$ each S_i is unitary, so we have

$$\underbrace{w_i}_{\text{unitary!}} S_i w_i^{-1} = \mu_i I.$$

Define
$$W = \begin{pmatrix} \boxed{w_1} & & \\ & \ddots & \\ & & \boxed{w_d} \end{pmatrix}$$

Then $wuh_1u^{-1}w^{-1} = \begin{pmatrix} \boxed{\lambda_1 I} & & \\ & \ddots & \\ & & \boxed{\lambda_d I} \end{pmatrix}$ a_{n_1}

$wuh_2u^{-1}w^{-1} = \begin{pmatrix} \boxed{\mu_1 I} & & \\ & \ddots & \\ & & \boxed{\mu_d I} \end{pmatrix}$

2/12/18

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, dash] \arrow[dr, dash, "\text{Solvable}"] & & \\ \text{Compact } K \arrow[dr, dash] & & B \arrow[dl, dash, "\text{torus}"'] \arrow[dr, dash, "\text{nilpotent}"] & \\ & T \arrow[dr, dash] & & AN \arrow[dl, dash] \\ & & I & \end{tikzcd}$$

Want to fill in this picture

$G = GL_n(F)$, $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

$K = \{g \in G \mid g^* g = I\}$, Closed (Yes, by continuity)

Define the absolute value of a matrix

$$|g|^2 = \sum_{i,j} |g_{ij}|^2 \qquad (\text{Viewing } G \subseteq \mathbb{R}^{n^2 \dim(F)})$$

$$= \sum_j \left(\sum_i |g_{ij}|^2 \right) = \sum_j (g_{\bullet j}, g_{\bullet j}) = n$$

$\Rightarrow |K| = \sqrt{n} \Rightarrow$ Closed + bdd $\Rightarrow$ Cpt.

Let $H \subseteq G$ be any cpt subgroup.

Claim: $\exists g \in G$, $gHg^{-1} \subseteq K$

Pf (Haar measure): Let F^n, $(,)$ be the standard Hermitian fm,

$\forall x, y \in F^n$, we define
$$(x,y)' = \int_{h \in H} (hx, hy)\, dh$$

Translation invariance $\Rightarrow$ $\forall h \in H$, $x, y \in F^n$, $(hx, hy)' = (x,y)'$

Let g be the change of basis from standard to some orthonormal basis for $(,)'$.

$\Rightarrow gHg^{-1} \subseteq K$.

$\square$

Corollary: Let $H \leq G$ be max'l cpt. Then $\exists g \in G$ s.t.
$gHg^{-1} \leq K \Rightarrow H \subseteq g^{-1}Kg$ cpt $\Rightarrow H = g^{-1}Kg$.

Top-down approach

Let G be a connected Lie group, $K \leq G$ maximal cpt

Claim: Any compact subgroup H of G can be conjugate into K;

Proof?: $H \curvearrowright \boxed{G/K}$ contractible (by generalization of Iwasawa)

Want $HgK = gK$.

183 To be continued

Iwasawa:

$G = K \times \underbrace{A \times N}_{}$ contractible

Next: Maximal tori

$T \subseteq K \subseteq G$

"Simultaneous diagonalization"

"Principal Axis Theorem"

Last time we showed that, for Type A,

Any cpt abelian subgroup of G can be conjugated into

$$T = \{ \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) : \theta_i \in \mathbb{R} \}$$

Top down: Let $T \subseteq K \leq G$ be a maximal torus. (connected) ("cpt, connected, abelian")

Claim: Every $k \in K$ can be conjugated into T. "diagonalizable"

pf: Given any $h \in K$, we have a cts map

$$\mu_h : K/T \to K/T$$
$$kT \mapsto hkT$$

Want a fixed point, i.e. $hkT = kT \iff k^{-1}hkT = T$
$\Rightarrow k^{-1}hk \in T$

Lefschetz fixed-point theorem: Cts $\varphi: X \longrightarrow X$ for X cpt with $\chi(X) \neq 0$ and $\varphi \simeq id_X$

Then $\exists$ fixed point for φ.

Claim: $\chi(K/T) =$ size of Weyl group (Proof TBA) > 0

Define $\varphi: [0,1] \longrightarrow K$

$0 \longmapsto I$

$1 \longmapsto h$

$\mu_{\varphi(t)}: [0,1] \times K/T \longrightarrow K/T$

$(t, kT) \longmapsto \varphi(t) kT$

$\mu_{\varphi(0)} = id_{K/T}$, $\mu_{\varphi(1)} = \mu_h$

$\square$

Conjugates of T fill up K.

Example: Hopf fibration: S^3 filled up by circles.

Group characters $K \longrightarrow \mathbb{C}$ constant on conjugacy classes

So it suffices to consider a char $T \longrightarrow \mathbb{C}$.

2/14/18 IOU: Using the idea of the Lie Algebra one can show that abelian varieties are just $\mathbb{C}^n$/sublattice (these are still subtle)

Abelian Lie groups, however, are just $\mathbb{R}^n$/sublattice (easy)

Choose a basis for the lattice to get $\mathbb{R}^n/\text{lattice} \cong \mathbb{R}^k \times \underbrace{(\mathbb{R}/\mathbb{Z})^{n-k}}_{\text{noncpt} \quad \text{cpt}}$

Compact + Connected abelian Lie groups are called *tori*.

$$\mathbb{T} = \mathbb{R}/\mathbb{Z} = S^1 = U(1) \simeq SO(2)$$

Euler $\quad e^{i\theta} \overset{\simeq}{\longleftrightarrow} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

Fact about tori: Every torus has a "topological generator", i.e. $\exists\, \alpha \in \mathbb{R}^n$ s.t. $\mathbb{Z}\alpha/\mathbb{Z}^n \underset{\text{dense}}{\subseteq} \mathbb{R}^n/\mathbb{Z}^n$

We are interested in maximal tori $T \subseteq G$ for two reasons:

1) Tori are easy to understand
2) Lots of information can be lifted from $T \longrightarrow G$.

In type A:

$$GL_n(\mathbb{C}) = U(n) \times \mathbb{R}^?$$

Define $T \subseteq U(n)$ to be $T = \{ \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \mid \theta_1, \dots, \theta_n \in \mathbb{R}\}$.
Claim that T is a maximal torus.

Pf: Let $H \subseteq G$ be any maximal torus (cpt, connected, abelian).
By compactness, + connectedness $\exists\, g \in G$ s.t. $gHg^{-1} \subseteq U(n)$.

By abelian-ness, the elts of gHg^{-1} can be simultaneously diagonalized: $\exists\, u \in U(n)$ w/
$$ugHg^{-1}u^{-1} \subseteq T \iff H \subseteq \underbrace{(ug)^{-1}T(ug)}_{\text{Torus}}$$

By maximality of H, we have $H = (ug)^{-1}T(ug)$.

$\square$

General theorem: Every connected Lie group G has a max'l torus $T \leq G$. Furthermore, all max'l tori are conjugate.

Idea: "Simultaneous diagonalization"

Existence: Finite-dimensionality (the trivial subgroup is always a torus)

Conjugation??

Sketch: Any torus $H \leq G$, by cptness + connectedness, can be conjugated into a choice of max'l cpt subgroup $K \leq G$ (Haar measure argument).

Let $T \leq K \leq G$ be a max'l torus.

gHg^{-1} has a topological generator k (so we write $gHg^{-1} = \overline{\langle k \rangle}$)

Claim: $\exists\, u \in K$ s.t. $u^{-1}ku \in T$

pf: $\mu_k : K/T \to K/T$, $uT \mapsto kuT$

If this has a fixed point, then $\exists u$ with $uT = kuT \Rightarrow u^{-1}ku \in T$

$\mu_K \simeq id_{K/T}$ + $\chi(K/T) \neq 0$, so fixed point exists by Lefschetz fixed-pt thm.

$\mu_k \simeq id_{K/T}$ is easy by path-connectivity of K.

$\chi(K/T) = |W|$, where $W = N_G(T)/T$

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Finally, we have that $u^{-1}Ku \in T$, so $\overline{\langle u^{-1}Ku \rangle} \leq T$

$$\begin{aligned} &\overline{\langle u^{-1}Ku\rangle} \leq T \\ &\quad \| \\ &u^{-1}\overline{\langle K\rangle}u \leq T \\ &\quad \| \\ &u^{-1}gHg^{-1}u \leq T. \end{aligned}$$

$\square$

Corollary: K is a union of tori, i.e.,

$$K = \bigsqcup_{\text{some } g} gTg^{-1} \qquad \underline{\text{Fibration}} \quad T \longrightarrow K \longrightarrow K/T$$

$$G \neq \bigsqcup gTg^{-1}$$

Small examples: • $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{C})$ is not diagonalizable

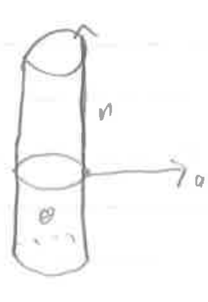
• $GL_2(\mathbb{R}) = O(2) \times \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
$\qquad a,b > 0 \qquad n \in \mathbb{R}$

noncompact

$$SL_2(\mathbb{R}) = SO(2) \times \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

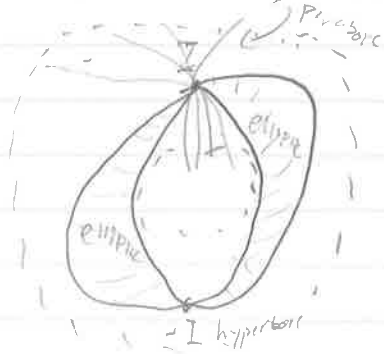
$$= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \times \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$= S^1 \times \mathbb{R}^+ \times \mathbb{R}$$



$$SL_2(\mathbb{R}) = SU(1,1)$$

Inside of Solid torus



$$SO(3) \supseteq T = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = SO(2)$$

Euler's rotation theorem

2/16/18

Examples: $SL_2(\mathbb{R}) = \underbrace{\boxed{\begin{pmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos(\theta) \end{pmatrix}}}_{S^1} \underbrace{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}}_{\mathbb{R}_+} \underbrace{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}}_{\mathbb{R}}$ (rank 1)

Max'l torus

$GL_2(\mathbb{C}) = U(2) \times \mathbb{R}^2$

$T \subseteq U(2)$, $T = \{ \mathrm{diag}(e^{i\theta}, e^{i\theta_2}) \mid \theta_i \in \mathbb{R} \}$, $T \cong U(1)^2$ (rank 2)

$$SL_2(\mathbb{C}) = SU(2) \times \mathbb{R}^?$$
$$\quad\quad\quad\quad \cup\!| \quad T$$

$T = \{ \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}) \mid \theta_i \in \mathbb{R} + \theta_1 + \theta_2 \equiv 0 \pmod{2\pi} \}$

$T \cong \{ \mathrm{diag}(e^{i\theta}, e^{-i\theta}) \mid \theta_i \in \mathbb{R} \} \cong U(1)$ (rank 1).

$T \subseteq SU(3)$, $T = \{ \mathrm{diag}(e^{2\pi i x_1}, e^{2\pi i x_2}, e^{2\pi i x_3}) \mid x_i \in \mathbb{R},\ x_1 + x_2 + x_3 = 0 \}$

Let $V = \mathbb{R}^3 / (1,1,1)$

$$\underset{\text{Lie alg.}}{(V,+)} \xrightarrow[\exp]{} \underset{\text{Lie group}}{T \subseteq SU(3)}$$
$$(x_1, x_2, x_3) \longmapsto \mathrm{diag}(e^{2\pi i x_1}, e^{2\pi i x_2}, e^{2\pi i x_3})$$

Kernel = root lattice $\Lambda \subseteq V$, $\Lambda = \mathbb{Z}\underset{\alpha_1}{(1,-1,0)} + \mathbb{Z}\underset{\alpha_2}{(0,1,-1)}$

The center of $SU(3)$ is

$$Z(SU(3)) = \{ \omega^k I \mid \omega = e^{2\pi i/3},\ k \in \mathbb{Z} \}.$$

1st iso. thm: $V/\Lambda_R \cong T$
$$\cup\!| \qquad\qquad \cup\!|$$
$$\Lambda_W/\Lambda_R \cong Z(T)$$

In general, consider $T \subseteq K \subseteq G$.

<u>Claim</u>: $Z(K) = \bigcap_{k \in K} kTk^{-1}$

Pf: Let $c \in Z(K) \subseteq K$. $\exists\ k \in K$ s.t. $kck^{-1} \in T \Rightarrow c \in T$. ✓ 190

Conversely, if $c \in \bigcap_{k \in K} kTk^{-1}$, then $\forall\, h \in K$, h is in some torus with c.

Since tori are abelian, we're done.

$\square$

Later, we can use the tori to show '$\forall_{n}$, $PSU(n)$ is simple'

Idea: Any normal subgroup is finite; any finite normal subgroup is in the center, hence trivial.

Theorem (Maximal torus in $SO(n)$): We can choose a maximal torus $T \subseteq SO(n)$

n = odd: $\quad T = \begin{pmatrix} 1 & & & \\ & R_{\theta_1} & & \\ & & \ddots & \\ & & & R_{\theta_{\frac{n-1}{2}}} \end{pmatrix}$ $\quad$ rank $= \frac{n+1}{2}$

B

n = even: $\quad T = \begin{pmatrix} R_{\theta_1} & & & \\ & R_{\theta_2} & & \\ & & \ddots & \\ & & & R_{\theta_{\frac{n}{2}}} \end{pmatrix}$ $\quad$ rank $= \frac{n}{2}$

D

Pf: Induction

$SO(n) \curvearrowright \mathbb{R}^n$, preserving dot product.

If n is odd, then 1 is an eigenvalue.

Let $K \in SO(n)$. Want $\det(K - I) = 0$.

$$\det(K-I) = \det(K - KK^T) = \det(K)\det(I-K^T)$$
$$= \det(I-K)$$
$$= (-1)^n \det(K-I) = -\det(K-I)$$

$\Rightarrow \det(K-I) = 0.$ ✓

Then we can conjugate any $u \in SO(n)$ into the form

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & K' & \\ 0 & & & \end{array}\right),$$ where $K' \in SO(n-1)$.

n even: Let $K \in SO(n)$. Since $\mathbb{C}$ is alg. closed, $\exists$ eigenvector $Ku = \lambda u$, $\lambda \in \mathbb{C}$, $u \in \mathbb{C}^n$.

Claim: K stabilizes the plane spanned by the real & imaginary parts of u.

$$K \circlearrowright \quad \frac{u+\bar{u}}{2} \in \mathbb{R}^n, \quad \frac{u-\bar{u}}{2i} \in \mathbb{R}^n$$

Indeed, $Ku = \lambda u \Rightarrow K\bar{u} = \bar{\lambda}\bar{u}$. Say $\lambda = (r+is) \in \mathbb{R}$.
Then $\forall \ \alpha, \beta \in \mathbb{R}$, we have

$$K\left[\alpha\left(\frac{u+\bar{u}}{2}\right) + \beta\left(\frac{u-\bar{u}}{2i}\right)\right] = \underbrace{(r\alpha + s\beta)}_{\in \mathbb{R}}\underbrace{\left(\frac{u+\bar{u}}{2}\right)}_{\in \mathbb{R}} + \underbrace{(r\beta - s\alpha)}_{\in \mathbb{R}}\underbrace{\left(\frac{u-\bar{u}}{2i}\right)}_{\in \mathbb{R}}$$

$\Rightarrow K$ can be written $\left(\begin{array}{c|c} SO(2) & \\ \hline & K' \end{array}\right)$, completing the induction. $\square$

2/19/18 The exponential map:

$$\mathbb{R}^n \xrightarrow{\exp} \mathbb{R}^n/\mathbb{Z}^n = T \subseteq U(n) \subseteq SL_n(\mathbb{C}) \subseteq GL_n(\mathbb{C})$$

$$(x_1, \dots, x_n) \longmapsto \operatorname{diag}(e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$$

This map is surjective onto T

Kernel of exp is called the "root lattice"

$\Lambda = \operatorname{Ker}(\exp) \subseteq (\mathbb{R}^n, +)$
Discrete additive subgroup.

$$\mathbb{R}^n_0 \twoheadrightarrow \mathbb{R}^n_0/\mathbb{Z}^n = T_0 \subseteq SU(n)$$

$$(x_1, \dots, x_n) \longmapsto \operatorname{diag}(e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$$
$$\det = e^{2\pi i (x_1 + \dots + x_n)} = 1$$
$$\Rightarrow x_1 + \dots + x_n \in \mathbb{Z}.$$

Root lattice of type A:

$\mathbb{R}^n_0 \subseteq \mathbb{R}^n$
(coords sum to zero)

Choose a standard basis:
$$\left.\begin{aligned} \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= e_2 - e_3 \\ &\vdots \\ \alpha_{n-1} &= e_{n-1} - e_n \end{aligned}\right\} \text{"simple roots"}$$

Root lattice: $\Lambda_R := \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \dots + \mathbb{Z}\alpha_{n-1} = \mathbb{R}^n_0 \cap \mathbb{Z}^n$

Where is the weight lattice?

Claim: We get the weight lattice by projecting $\mathbb{Z}^n$ orthogonally onto the hyperplane $\mathbb{R}^n_0$.

To project from $\mathbb{R}^n$ to $\mathbb{R}^n_0$, let

$$A = \underbrace{\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & & 1 \\ & & & -1 \end{pmatrix}}_{n-1} \Big\} n$$

Projection matrix: $P = A(A^TA)^{-1}A$

$$P_{\text{hyp } \alpha^\perp} = I - P_{\text{line } \alpha}$$

$$\mathbb{R}^n_0 = (e_1 + e_2 + \cdots + e_n)^\perp = \mathbb{1}^\perp$$

$$P_{\text{line}(\alpha)} = \alpha(\alpha^T\alpha)^{-1}\alpha^T = \frac{\alpha\alpha^T}{\alpha^T\alpha}$$

$$P_{\text{line } \mathbb{1}} = \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}} = \frac{1}{n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & \vdots \\ \vdots & & \ddots & \\ 1 & \cdots & & 1 \end{pmatrix}$$

Thus, $P_{\mathbb{R}^n_0} = I - \frac{1}{n}\mathbb{1}\mathbb{1}^T$

To project $\mathbb{Z}^n$, project the basis 'elts'

$$L_1 := Pe_1 = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{1}{n}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1 - \frac{1}{n}(e_1 + \cdots + e_n)$$

$L_i := Pe_i = e_i - \frac{1}{n}(e_1 + \cdots + e_n)$

Define $L = \operatorname{im}(\operatorname{Proj} \mathbb{Z}^n \text{ onto } \mathbb{R}^n_0) \subseteq \mathbb{R}^n_0$.

Claim: $L =$ weight lattice

$$= \left\{ \beta \in \mathbb{R}^n_0 \;\middle|\; \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \quad \forall \alpha \in \Phi \right\}$$

$$= \Lambda_\Omega$$

Proof: Clearly, L is spanned by the images $L_1, L_2, \ldots, L_n$ of the basis:

$$P(a_1 e_1 + \cdots + a_n e_n) = a_1 L_1 + \cdots + a_n L_n$$

In fact, the subset $L_1, L_2, \ldots, L_{n-1}$ is a spanning set, b/c

$$P(e_1 + \cdots + e_n) = 0$$

$$L_1 + \cdots + L_n = 0 \iff L_n = -L_1 - L_2 - \cdots - L_{n-1}$$

Two steps: Step 1: Switch to a new basis:

$\omega_1 = L_1$ "Unimodular change of basis"

$\omega_2 = L_1 + L_2$

$\vdots$

$\omega_{n-1} = L_1 + \cdots + L_{n-1}$

The proof will be done if we can show that

$$\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$$

$= (\omega_i, \alpha_j)$ in this case.

First, we observe that

$$(L_i, L_j) = (e_i - \tfrac{1}{n}\mathbb{1}, e_j - \tfrac{1}{n}\mathbb{1})$$
$$= (e_i, e_j) - \tfrac{1}{n}(e_i, \mathbb{1}) - \tfrac{1}{n}(\mathbb{1}, e_j) + \tfrac{1}{n^2}(\mathbb{1}, \mathbb{1})$$
$$= \delta_{ij} - \tfrac{1}{n} - \tfrac{1}{n} + \tfrac{1}{n} = \delta_{ij} - \tfrac{1}{n}.$$

Also, observe that $L_i - L_j = (e_i - \tfrac{1}{n}\mathbb{1}) - (e_j - \tfrac{1}{n}\mathbb{1}) = e_i - e_j$.

Thus, we have

$$(w_i, \alpha_j) = (L_1 + \cdots + L_i, e_j - e_{j+1}) = (L_1 + \cdots + L_i, L_j - L_{j+1})$$
$$= (e_1 + e_2 + \cdots + e_i, e_j - e_{j+1})$$
$$= \begin{cases} 0, & j < i \\ 1, & j = i \\ 0, & j > i \end{cases} = \delta_{ij}.$$

$\square$

Types B & D: $SO(n)$

I claim that the following subgroups $T \subseteq SO(n)$ are max'l tori:

$\underline{n \text{ odd}}$: $T = \begin{pmatrix} 1 & & & \\ & R_{\theta_1} & & \\ & & \ddots & \\ & & & R_{\theta_{\frac{n-1}{2}}} \end{pmatrix}$ (Type B)

$\underline{n \text{ even}}$: $T = \begin{pmatrix} R_{\theta_1} & & & \\ & R_{\theta_2} & & \\ & & \ddots & \\ & & & R_{\theta_{\frac{n}{2}}} \end{pmatrix}$ (Type D)

Pf: $\underline{\text{Step 1}}$: Given any $h \in SO(n)$, show that $\exists\, g \in SO(n)$ s.t. $ghg^{-1} \in T$.

$\underline{n \text{ odd}}$: $ghg^{-1} = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \underset{SO(n-1)}{\overset{\uparrow}{h'}} \\ 0 & \end{array}\right)$

$\underline{n \text{ even}}$: $ghg^{-1} = \left(\begin{array}{c|c} \overset{R_\theta}{SO(2)} & 0 \\ \hline 0 & SO(n-2) \end{array}\right)$

Induction!

✓

$\underline{\text{Step 2}}$: Let $H \subseteq SO(n)$ be $\underline{\text{any}}$ torus. Then H has a topological generator: $H = \overline{\langle h \rangle}$

By Step 1, $\exists\, g \in SO(n)$ s.t. $ghg^{-1} \in T$.

$$\Rightarrow \overline{\langle ghg^{-1} \rangle} \subseteq T$$
$$\overline{\langle ghg^{-1} \rangle} = g\overline{\langle h \rangle}g^{-1} \subseteq T$$
$$\Rightarrow H \subseteq T.$$

$\square$

2/21/18 $SO(n)$ is <u>not</u> simply-connected

$\pi_1(SO(n)) = \mathbb{Z}/2$

$$\Rightarrow \exists \text{ double-cover } \underset{\text{"Dirac's Spinor"}}{\overset{\tilde{T} \\ \cap}{Spin(n)}} \xrightarrow{2:1} \overset{T \\ \cap}{SO(n)}$$

<u>Spin:</u> How do we construct $Spin(n)$?

Turns out $Spin(n) \not\hookrightarrow GL(n)$

<u>Model:</u> <u>Quaternions</u>

$Spin(3) = Sp(1) =$ Unit Quaternions
$Spin(4) = Sp(1) \times Sp(1)$

Quaternions $\rightsquigarrow$ Clifford Algebras

<u>Recall:</u> $\mathbb{H} = \mathbb{R}^4/(\text{relations}) = \dfrac{\{1a + ib + jc + kd \mid a,b,c,d \in \mathbb{R}\}}{(i^2 = j^2 = k^2 = ijk = -1)}$

Let's write $a + ib + jc + kd = a + \underset{(b,c,d)}{\vec{x}}$

$$\Rightarrow \mathbb{H} = \underset{\text{real}}{\mathbb{R}} \oplus \underset{\text{imaginary}}{\mathbb{R}^3}$$

Define Conjugation: $\overline{a + \vec{x}} = a - \vec{x}$

Define a norm by $\|a + \vec{x}\|^2 = (a + \vec{x})(a - \vec{x})$

Turns out, for $\vec{x}, \vec{y} \in \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$,

$$\underset{}{\vec{x}\vec{y}} = \underbrace{-\vec{x}\cdot\vec{y}}_{\text{real}} + \underbrace{\vec{x}\times\vec{y}}_{\text{imaginary}}$$

Thus, the norm satisfies $\|a+\vec{x}\|^2 = (a+\vec{x})(a-\vec{x})$

$$= a^2 - \vec{x}^2 = a^2 + \vec{x}\cdot\vec{x} - \cancel{\vec{x}\times\vec{x}}$$

$$= a^2 + \|\vec{x}\|^2$$

real, and agrees w/ std. norm on $\mathbb{R}^4 \supseteq \mathbb{R}^3$

Furthermore, the norm is multiplicative:

For $\vec{x}, \vec{y} \in \mathbb{R}^3$, $\|\vec{x}\vec{y}\|^2 = \|\underset{\mathbb{R}}{-\vec{x}\cdot\vec{y}} + \underset{\mathbb{R}^3}{\vec{x}\times\vec{y}}\|^2$

$$= (\vec{x}\cdot\vec{y})^2 + \|\vec{x}\times\vec{y}\|^2$$
$$= (\|\vec{x}\|\|\vec{y}\|\cos\theta)^2 + (\|\vec{x}\|\|\vec{y}\|\sin\theta)^2$$

$$= \|\vec{x}\|^2\|y\|^2.$$

✓

Euler: $U(1) \cong SO(2)$

$$a+ib \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\mathbb{H} \hookrightarrow \mathrm{Mat}_2(\mathbb{C})$$

Cayley: $Sp(1) \xrightarrow{\sim} SU(2)$

$$a+ib+jc+kd \longmapsto \begin{pmatrix} a+id & -b-ic \\ b-ic & a-id \end{pmatrix}$$

norm $\rightsquigarrow$ determinant

Inverses: $(a+\vec{x})^{-1} = \frac{a-\vec{x}}{\|a+\vec{x}\|} = \frac{a-\vec{y}}{a^2+\|\vec{x}\|^2}$

Claim: $\mathbb{H}^\times \curvearrowright \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$

$q \curvearrowright \vec{x}$

$\vec{x} \longmapsto q\vec{x}q^{-1}$

Proof: Use anti-multiplicative conjugation!
idea

$\square$

Furthermore: $\mathbb{H}^\times \curvearrowright \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ by isometries!

$$\|q(\vec{x}-\vec{y})q^{-1}\| = \|q\| \|\vec{x}-\vec{y}\| \|q\|^{-1}$$

Group hom: $\mathbb{H}^\times \longrightarrow O(3)$

$$Sp(1) \xrightarrow{2:1} SO(3)$$

Pf: $q\vec{x}q^{-1} = \left(\frac{q}{\|q\|}\right)\vec{x}\left(\frac{q}{\|q\|}\right)^{-1}$

Consider $u \in Sp(1)$ $\quad 1 = \|u\| = u_{re}^2 + \|\vec{u}_{im}\|^2$

$$= \cos\theta + \frac{\vec{u}_{im}}{\|\vec{u}_{im}\|}\sin(\theta),$$

Claim: $\pm(\cos\theta + \vec{u}\sin\theta) \curvearrowright \mathbb{R}^3$ is rotation by θ around axis $\vec{u}$.

Since $Sp(1) \underset{\text{homeo}}{\cong} S^3$, we see $SO(3) \underset{\text{homeo}}{\equiv} \mathbb{RP}^3$.

$\square$

Conclusion: $\mathrm{Spin}(3) = Sp(1) = SU(2)$

But we get more: $Sp(1)\times Sp(1) \curvearrowright \mathbb{H}=\mathbb{R}^4$

$u, v \qquad q \longmapsto u^{-1} q v$

by isometries

Get a group hom $Sp(1)\times Sp(1) \xrightarrow{2:1} SO(4)$

$\Rightarrow Sp(1)\times Sp(1) = Spin(4)$

$$Cl_n = \frac{\mathbb{R}\langle 1, e_1, \dots, e_n\rangle}{\begin{pmatrix} e_i^2 = -1 \ \forall i \\ e_i e_j = -e_j e_i \ \forall i \neq j \end{pmatrix}}$$

$Cl_0 = \mathbb{R}, \quad Cl_1 = \mathbb{R}\langle 1, e_1\rangle / (e_1^2 = -1) = \mathbb{C}$

$Cl_2 = \mathbb{R}\langle 1, e_1, e_2\rangle / (e_1^2 = e_2^2 = -1, \ e_1 e_2 = -e_2 e_1) = \mathbb{H}$

$\dim(Cl_n) = 2^n$, basis $e_{i_1} e_{i_2} \cdots e_{i_k}$, $i_1 < i_2 < \cdots < i_k$

2/26/18 Filling in some gaps:

G connected Lie group

$K \subseteq G$ max'l cpt subgroup

Then K is necessarily connected

Corollary of Iwasawa

Homeomorphism $G \cong K \times \mathbb{R}^3$

G connected $\Rightarrow K$ connected

Type A: $GL_n(\mathbb{C}) \supseteq U(n) \supseteq T$ ← diagonal

max'l cpt, max'l torus

$$\begin{array}{ccccc} \cup\!\mid & & & & \\ SL_n(\mathbb{C}) & \supseteq & SU(n) & \supseteq & T\cap SL_n(\mathbb{C}) \\ & & \| & & ? \\ & & U(n)\cap SL_n(\mathbb{C}) & & \\ & & ? & & \end{array}$$

Let $H \leq SL_n(\mathbb{C})$ be compact, Haar measure $\Rightarrow \exists\, g \in GL_n(\mathbb{C})$ s.t. $gHg^{-1} \leq U(n)$

Observe: Conjugation preserves determinant!

Thus, $gHg^{-1} \leq SL_n(\mathbb{C}) \Rightarrow gHg^{-1} \leq U(n)\cap SL_n(\mathbb{C}) = SU(n)$

Now, suppose $SU(n) \underset{\text{cpt}}{\subsetneq} H \leq SL_n(\mathbb{C})$. $\exists\, g \in GL_n(\mathbb{C})$, $gHg^{-1} \leq SU(n) \subsetneq H$

$gHg^{-1} \subsetneq H$, a contradiction

Furthermore, we can take $g \in SL_n(\mathbb{C})$ by replacing g with $\frac{g}{\sqrt[n]{\det(g)}}$.

Let $A \leq SL_n(\mathbb{C})$ be a torus.
Then $\exists\, g \in GL_n(\mathbb{C})$ s.t. $gAg^{-1} \leq T$, but since conjugation preserves determinants
$gAg^{-1} \leq SL_n(\mathbb{C})$

$\Longrightarrow T \cap SL_n(\mathbb{C})$ is a max'l torus

$T = \{\operatorname{diag}(e^{2\pi i x_1}, e^{2\pi i x_2}, \dots, e^{2\pi i x_n}) \mid x_i \in \mathbb{R}\}$

Need $\exp(2\pi i(\sum x_j)) = 1 \Longleftrightarrow x_1 + \dots + x_n \in \mathbb{Z}$
Claim: Only need $x_1 + \dots + x_n = 0$.

Indeed: Define $T' = T \cap SL_n(\mathbb{C}) = \{ \operatorname{diag}(e^{2\pi i x_1}, \dots, e^{2\pi i x_n}) \mid x_i \in \mathbb{R}, x_1 + \dots + x_n = 0 \}$

$T_0 := \{ \operatorname{diag}(e^{2\pi i x_1}, \dots, e^{2\pi i x_n}) \mid x_i \in \mathbb{R}, \ x_1 + \dots + x_n = 0 \}$

Claim that $T' = T_0$.

Pf: $T_0 \subseteq T'$ ✓

$T' \subseteq T_0$? If $\operatorname{diag}(e^{2\pi i x_1}, \dots, e^{2\pi i x_n}) \in T'$ with $x_1 + \dots + x_n = s$,

Then $\underset{\substack{\uparrow \\ T_0}}{\operatorname{diag}(e^{2\pi i (x_1 - s)}, e^{2\pi i x_2}, \dots, e^{2\pi i x_n})} = \operatorname{diag}(e^{2\pi i (x_1)}, \dots, e^{2\pi i x_n})$

$\square$

$$\mathbb{R}^n \xrightarrow{\exp} \underset{\substack{\| \\ U(1)^n}}{T} \subseteq GL_n(\mathbb{C}) \quad \text{rank } n-1$$

$$\mathbb{R}^n_{\mathbb{Z}} \xrightarrow{\exp} T' \subseteq SL_n(\mathbb{C}) \quad \text{rank } n-1$$

$$\Lambda \hookrightarrow \mathbb{R}^n_0 \xrightarrow{\exp} T_0 = U(1)^{n-1}$$

Root lattice

Define a group

$U(1)_j = \{ \operatorname{diag}(1, 1, \dots, 1, \underset{j}{e^{2\pi i \theta}}, \underset{j+1}{e^{-2\pi i \theta}}, 1, \dots, 1) \mid \theta \in \mathbb{R} \} \cong \{ e^{2\pi i \theta} \mid \theta \in \mathbb{R} \}$

Claim $U(1)_1 \times U(1)_2 \times \dots \times U(1)_{n-1} \xrightarrow{\cong} T_0$

Surjective b/c vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}$ span $\mathbb{R}^n_0$, and

203 injective b/c $U(1)_j \cap U(1)_k = I \quad \forall j \neq k$. $\square$

- $SU(n)$ & $SL_n(\mathbb{C})$ are simply-connected
- $PSU(n)$ & $PSL_n(\mathbb{C})$ are simple

Relationship: $\pi_1(PSU(n)) = \pi_1(PSL_n(\mathbb{C})) = \mathbb{Z}/n\mathbb{Z}$ " Ω/Λ (weight lattice, root lattice)

$Z(SU(n)) = Z(SL_n(\mathbb{C})) = \mathbb{Z}/n\mathbb{Z}$ "

$Z(GL_n(\mathbb{C})) = \{\lambda I \mid \lambda \in \mathbb{C}^\times\}$

Pf: Let $g \in Z(GL_n(\mathbb{C}))$. Then $g(I+e_{ij}) = (I+e_{ij})g \quad \forall i \neq j$
(note $(I+e_{ij})^{-1} = I - e_{ij}$)

$\Rightarrow g + ge_{ij} = g + e_{ij}g \Rightarrow \boxed{ge_{ij} = e_{ij}g} \Rightarrow g$ scalar

$\square$

$Z(SL_n(\mathbb{C})) = \{\omega I \mid \omega^n = 1\}$.

Pf: $\det(I+e_{ij}) = 1$ ✓

$\Rightarrow g \in Z(SL_n(\mathbb{C}))$ is scalar!

Since $\det(g) = 1 \Rightarrow g = \omega I$, where $\omega^n = 1$.

$\square$

$Z(U(n)) = \{\lambda I \mid |\lambda| = 1\} = U(1)$.

Pf: $g \in Z(U(n))$. $\exists\ u \in U(n)$ s.t. $ugu^{-1} \in T$. But $g = ugu^{-1}$,
So $g \in T$.

Permutations are unitary! Let $\pi_{ij} = \mathrm{Mat}((i,j))$. (transposition)

Since g is central, $\pi_{ij}\, g\, \pi_{ij}^{-1} = g \quad \forall i \neq j$

$\Rightarrow g_{ii} = g_{jj} \quad \forall\ i \neq j.$

Since the cols. of g are orthonormal, we have $|g_{ii}| = 1 \quad \forall i.$

$\blacksquare$

$Z(SU(n)) = \{ \omega I \mid \omega^n = 1 \}$

Pf: $g \in Z(SU(n))$

$\exists\ u \in U(n)$ s.t. $ugu^{-1} \in T$

Two cases: If $u \in SU(n)$, then done. ✓

Factor: $U(n) = T \_\ SU(n)$
?

2/28/16

Type A: $GL_n(\mathbb{C}),\ SL_n(\mathbb{C})$
$U(n) \qquad SU(n)$

and quotients of these by discrete normal subgroups.

Fact: A discrete normal subgroup must be contained in the center.
Proof later.

Thm: $Z(GL_n(\mathbb{C})) = \{ \lambda I \mid \lambda \neq 0 \} \cong \mathbb{C}^\times$
$Z(SL_n(\mathbb{C})) = \{ \lambda I \mid \lambda^n = 1 \} \cong \mathbb{Z}/n\mathbb{Z}$
$Z(U(n)) = \{ \lambda I \mid |\lambda| = 1 \} \cong U(1)$
$Z(SU(n)) = \{ \lambda I \mid \lambda^n = 1 \} = \mathbb{Z}/n\mathbb{Z}$

Pf: 1^{st} two done last time. Recapping 3^{rd}.

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Let $g \in Z(U(n))$. From max. torus thm (Spectral Thm) $\exists$ $u \in U(n)$ s.t. $ugu^{-1} \in T \Rightarrow g \in T$

Trick: Permutations are unitary!

$$\begin{array}{cc} & \begin{array}{cc} \quad\quad i & j \end{array} \\ \Pi_{ij} = \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 1 & & & & \\ & 0 & \cdots & 1 & \\ & \vdots & 1 & \vdots & \\ & 1 & \cdots & 0 & \\ & & & & 1 \end{pmatrix} \end{array} \qquad \Pi_{ij}^{-1} = \Pi_{ij}$$

Since g commutes with Π_{ij}, we have

$$g = \Pi_{ij} \, g \, \Pi_{ij}^{-1},$$

So by taking ii^{th} entry on both sides, we have

$$g_{ii} = g_{jj} \quad \forall\, i \neq j$$

$\Rightarrow g = \lambda I$ for some $\lambda \in \mathbb{C}$

Finally, each row + col of g has length 1 $\Rightarrow |\lambda| = 1$. ✓

Next, let $g \in Z(SU(n))$.

We know that $\exists$ $v \in U(n)$ with vgv^{-1} diagonal.

<u>Note</u>: $1 = \det(I) = \det(v^* v) = \det(v)^* \det(v) = |\det(v)|^2$

Let λ be any fixed n^{th} root of $\det(v)$ $\Rightarrow \lambda^* \lambda = |\lambda|^2 = 1$

Define $w = \dfrac{v}{\lambda}$ and note that $\det(w) = \det\left(\dfrac{v}{\lambda}\right) = \dfrac{1}{\lambda^n}\det(v) = \dfrac{\det(v)}{\det(v)} = 1$

$$w^* w = \left(\frac{v}{\lambda}\right)^* \frac{v}{\lambda} = \frac{v^* v}{\lambda^* \lambda} = v^* v = I$$

Since $g \in Z(SU(n))$ + $w \in SU(n)$, we have

$g = wgw^{-1} = \left(\frac{v}{\lambda}\right) g \left(\frac{v}{\lambda}\right)^{-1} = vgv^{-1} \in T. \Rightarrow g$ is diagonal.

Consider the $\underline{3\text{-cycles}}$ $\Pi_{ijk} = \Pi_{ij}\Pi_{jk}$

$\det(\Pi_{ijk}) = \det(\Pi_{ij})\det(\Pi_{jk}) = (-1)^2 = 1$

Hence $g = \Pi_{ijk}\, g\, \Pi_{ijk}^{-1} \Rightarrow g_{ii} = g_{jj} = g_{kk}$ $(i<j<k)$

$\Rightarrow g = \lambda I$ for some $\lambda \in \mathbb{C}$

$1 = \det(g) = \det(\lambda I) = \lambda^n.$

$\square$

Root & Weight Lattice in General:

Let K be a simply-connected compact Lie group with maximal torus $T \subseteq K$.

We have a surjective group hom

$$\exp : (\underset{\text{vector space}}{\mathfrak{t}}, +) \twoheadrightarrow T \subseteq K$$

The kernel is called the root lattice $\Lambda \subseteq \mathfrak{t}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\ker(\exp)}$

Λ is discrete.

Recall (Lefschetz fixed-point thm) $Z(K) = \bigcap_{k \in K} kTk^{-1}$

$\Rightarrow Z(K) \subseteq T.$

$\exp^{-1}(Z(K))$ is called the weight lattice Ω.

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$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 2 - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

From 1^{st} isom. thm, we have $\exp : \Omega \twoheadrightarrow Z(K)$

$$\Rightarrow \frac{\Omega}{\mathrm{Ker}(\exp)} = \frac{\Omega}{\Lambda} \cong Z(K)$$

Subgroups of Ω/Λ = discrete normal subgroups of K.

Type A:

$SU(n)$ is simply-connected (proof for π_1 contd.)

$$\exp : \mathbb{R}^n_0 \longrightarrow T_0$$
$$(x_1, \dots, x_n) \longmapsto \mathrm{diag}(e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$$

$$\Lambda = \mathrm{Ker}(\exp) = \mathbb{Z}^n \cap \mathbb{R}^n_0$$

$$\left.\begin{aligned} \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= e_2 - e_3 \\ &\vdots \\ \alpha_{n-1} &= e_{n-1} - e_n \end{aligned}\right\} \text{Standard root basis}$$

$$\Omega = \exp^{-1}(Z)$$

Claim: $\Omega = \mathrm{Proj}_{\mathbb{R}^n_0}(\mathbb{Z}^n)$

Basis of fundamental weights:
$$\begin{aligned} w_1 &= L_1 \\ w_2 &= L_1 + L_2 \\ &\vdots \\ w_{n-1} &= L_1 + L_2 + \dots + L_{n-1} \end{aligned}$$

where $L_i = e_i - \frac{1}{n}\mathbb{1} = e_i - \frac{1}{n}(e_1 + e_2 + \dots + e_n)$

$$\left|\frac{\Omega}{\Lambda}\right| = \det \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} = n, \text{ but } \frac{\Omega}{\Lambda} \cong Z(SU(n)) \cong \mathbb{Z}/n\mathbb{Z}.$$

What's missing?

$\pi_1(SU(n)) = $ trivial

$\pi_1(PSU(n)) = \pi_1(PSL_n(\mathbb{C})) = \mathbb{Z}/n\mathbb{Z}$

$PSU(n)$, $PSL_n(\mathbb{C})$ are simple.

3/2/18

Topic: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$

$Mat_n(\mathbb{R})$ is an associative $\mathbb{R}$-algebra with an anti-automorphism
$$* : Mat_n(\mathbb{R}) \longrightarrow Mat_n(\mathbb{R})$$
and a monoid homom
$$\det : Mat_n(\mathbb{R}) \longrightarrow (\mathbb{R}, \times)$$

$\mathbb{C} = \mathbb{R}\langle \underline{1}, \underline{i} \rangle / (\underline{i}^2 = -\underline{1}, \underline{1}\,\underline{i} = \underline{i}\,\underline{1})$

$\mathbb{C} = \mathbb{R}\underline{1} + \mathbb{R}\underline{i}$ as a vector space

$\mathbb{R}$-algebra homomorphism
$$\mathbb{C} \overset{\rho}{\hookrightarrow} Mat_2(\mathbb{R})$$
$$\underline{1} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\underline{i} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$a\underline{1} + b\underline{i} \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Observe: $\rho(\underline{1})^* = \rho(\underline{1})$, $\rho(\underline{i})^* = \rho(-\underline{i})$

Pull back to get an (anti-) automorphism
$$* : \mathbb{C} \longrightarrow \mathbb{C}$$
$$a\underline{1} + b\underline{i} \longmapsto a\underline{1} - b\underline{i}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, leftrightarrow] \arrow[rrr, bend right, "|\cdot|^2"'] & \mathbb{C} \arrow[r, hook, "\rho"] & Mat_2(\mathbb{R}) \arrow[r, "\det"] & \mathbb{R} \end{tikzcd}$$

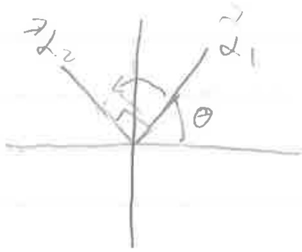
Note: $\alpha^* \alpha = \det(\alpha) I$

It follows that
$$\alpha^+ \alpha = |\alpha|^2$$
$$\alpha^{-1} = \frac{\alpha^*}{|\alpha|^2}$$

and that unit-length cpx #'s form a group

$$\mathbb{C} \hookrightarrow Mat_2(\mathbb{R})$$
$$U(1) \hookrightarrow SO(2) \leftarrow \text{Surjective?}$$

Consider $A \in SO(2)$. Cols of A are an orthonormal basis for $\mathbb{R}^2$



$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \rho(1\cos(\theta) + i\sin(\theta)) \checkmark$$

$$U(1) \cong SO(2)$$

Since $R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2}$, we have a surjective group homom

$$(\mathbb{R}, +) \xrightarrow{\exp} U(1) \xrightarrow[\rho]{\cong} SO(2)$$
$$\theta \longmapsto e^{i\theta} \longleftrightarrow \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = e^{\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}}$$

Whole story generalizes to

$$Mat_n(\mathbb{C}) \hookrightarrow Mat_{2n}(\mathbb{R}) \qquad \boxed{\alpha^+ \alpha = \det(\alpha) I}$$

$$U(n) \hookrightarrow SO(2n)$$

Picture: $e^{i\theta}$

Quaternions: $\mathbb{H} = \dfrac{\mathbb{R}\langle \underline{1}, \underline{i}, \underline{j} \rangle}{\left(\underline{i}^2 = \underline{j}^2 = -1,\ \underline{1}\,\underline{i} = \underline{i}\,\underline{1},\ \underline{1}\,\underline{j} = \underline{j}\,\underline{1},\ \underline{i}\,\underline{j} = -\underline{j}\,\underline{i} \right)}$

$\mathbb{H} = \mathbb{R}\underline{1} + \mathbb{R}\underline{i} + \mathbb{R}\underline{j} + \mathbb{R}(\underline{i}\,\underline{j})$

$$\mathbb{H} \overset{\chi}{\hookrightarrow} Mat_2(\mathbb{C}) \overset{\rho}{\hookrightarrow} Mat_4(\mathbb{R})$$

$$\underline{1} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{i} \longmapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\underline{j} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\underline{k} \longmapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right)$$

$$a\underline{1}+b\underline{i}+c\underline{j}+d\underline{k} \longmapsto \begin{pmatrix} a\underline{1}+b\underline{i} & c\underline{1}+d\underline{i} \\ -c\underline{1}+d\underline{i} & a\underline{1}-b\underline{i} \end{pmatrix}$$

$$\Big\downarrow$$

$$\left(\begin{array}{cc|cc} a & -b & c & -d \\ -b & a & d & c \\ \hline -c & -d & a & b \\ d & -c & -b & a \end{array}\right)$$

Note: $\chi(\underline{1})^* = \chi(\underline{1})$

$\chi(\underline{i})^* = \chi(-\underline{i})$

$\chi(\underline{j})^* = \chi(-\underline{j})$

$\chi(\underline{k})^* = (\chi(\underline{i})\chi(\underline{j}))^* = \chi(\underline{j})^*\chi(\underline{i})^* = \chi(-\underline{j})\chi(-\underline{i})$

$= \chi(\underline{ji}) = \chi(-\underline{k}).$

Miracle: $\chi(\alpha^*\alpha) = \chi(\alpha)^*\chi(\alpha) = \det(\chi(\alpha))\cdot I$

$$\begin{tikzcd} \mathbb{R}^4 = \mathbb{H} \arrow[r, hook] \arrow[rrr, bend right, "|\cdot|^2"'] & Mat_2(\mathbb{C}) \arrow[r, hook] & Mat_4(\mathbb{R}) \arrow[r, "\det"] & \mathbb{R} \end{tikzcd}$$

$\Rightarrow$ length is a multiplicative function!

Restricting to unit length:

$$Sp(1) \xhookrightarrow{\cong} SU(2) \hookrightarrow SO(4)$$

Furthermore, we can lift everything to $Mat_n(\mathbb{H}) \xhookrightarrow{\chi} Mat_{2n}(\mathbb{C}) \xhookrightarrow{\rho} Mat_{4n}(\mathbb{R})$

$$Sp(n) \hookrightarrow SU(2n) \hookrightarrow SO(4n)$$

3/5/18 Basic miracle: $\mathbb{R}, \mathbb{C}, \mathbb{H}$

Embeddings of associative $\mathbb{R}$-algebras:

$$\underset{*}{\mathrm{Mat}_n(\mathbb{H})} \xhookrightarrow{\kappa} \underset{*}{\mathrm{Mat}_{2n}(\mathbb{C})} \xhookrightarrow{\rho} \underset{T}{\mathrm{Mat}_{4n}(\mathbb{R})}$$

$$\underset{\|\cdot\|^2}{a+ib+jc+kd} \longmapsto \underset{\det}{\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}} \longmapsto \underset{\det}{\left(\begin{array}{cc|cc} a & -b & c & -d \\ b & a & d & c \\ \hline -c & -d & a & b \\ d & -c & -b & a \end{array}\right)}$$

(When $n=1$, all determinants are equal)

Know: $GL_n(\mathbb{H}) \hookrightarrow GL_{2n}(\mathbb{C}) \hookrightarrow GL_{4n}(\mathbb{R})$

Furthermore $Sp(n) \hookrightarrow U(2n) \hookrightarrow O(4n)$, because

$$\alpha^*\alpha = I \longmapsto \alpha^*\alpha = I \longmapsto \alpha^T\alpha = I$$

When $n=1$:

$$\begin{tikzcd} Sp(1) \arrow[r, hook] & SU(2) \arrow[r, hook] & SO(4) \\ Spin(3) \arrow[u] & Spin(4) \arrow[u] & \end{tikzcd}$$

$\mathbb{R}^4 \longrightarrow \mathbb{R}^4$

For all $q \in \mathbb{H}^\times$, define $[q]: \mathbb{H} \longrightarrow \mathbb{H}$

$$x \longmapsto q^{-1}xq$$

This preserves length, b/c $\|q^{-1}xq\|^2 = \det(q^{-1}xq) = \det(x) = \|x\|^2$

We may assume that $\|q\|=1$, so we obtain a group homom.

$$Sp(1) \longrightarrow O(4)$$

Note: This action fixes the "real subspace"

$$Re(\mathbb{H}) = \mathbb{R}^1 \subseteq \mathbb{H}$$

$x \in \mathbb{R}e(\mathbb{H})$ is central, so $x \longmapsto q^{-1}xq = x$

Hence, each $q \in \mathbb{H}$ stabilizes the "imaginary part"

$Im(\mathbb{H}) = Re(\mathbb{H})^{\perp}$

Hence, we have a map $Sp(1) \longrightarrow O(3)$

In fact, this is a map $Sp(1) \longrightarrow SO(3)$

Polar form:

$q \in \mathbb{H} \Rightarrow q = r(\cos(\theta) + u\sin(\theta))$

where $r \in \mathbb{R}$, $u \in$ Unit imaginary, i.e.,

$u = xi + yj + zk$, $x^2+y^2+z^2=1$

Then $[q]: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the rotation by 2θ around $\vec{u}$

$v \longmapsto \bar{q}vq$

Pf: Let $q = \cos(\theta) + i\sin(\theta)$, $v = xi + yj + zk \in \mathbb{R}^3$

Then $\bar{q}vq = \bar{q}(xi)q + \bar{q}(yj+zk)q$ (note: $jq = \bar{q}j$, $kq = \bar{q}k$)

$= \bar{q}q(xi) + \bar{q}^2(yj+zk)$

$= xi + ((\cos(2\theta) + i\sin(2\theta))(yj+zk)$

Rotation around i-axis by 2θ

✓

Now have to check that kernel of $Sp(1) \longrightarrow SO(3)$ is ± 1,

So $Sp(1) \xrightarrow{2:1} SO(3)$.

Q: Is every elt of $SO(3)$ a rotation?

A: Yes.

Thm (Cartan–Dieudonné 1948): Let (V,Q) ($\dim(V)=n$) be a quadratic vector space (i.e., Q is a non-degenerate quadratic form). Then every "orthogonal transformation" in the group $\mathrm{Isom}(V,Q)$ can be expressed as a composition of at most n reflections.

reflection: $x \longmapsto y - \frac{2B(x,y)}{B(y,y)}x$

Def: Let Q = dot product. A simple rotation is a product of two reflections.

F_2F_1 is a simple rotation by 2θ about the intersection of hyperplanes

Corollary: Every elt. of $SO(n)$ is a product of at most $\lfloor \frac{n}{2} \rfloor$ simple rotations

$\lfloor \frac{3}{2} \rfloor = 1 \Rightarrow SO(3)$ consists of simple rotations

$Spin(3) \xrightarrow{2:1} SO(3)$

$= Sp(1) \xrightarrow{2:1}$

$Sp(1) \times Sp(1) \circlearrowright \mathbb{R}^4 = \mathbb{H}$

Lemma: $q \in Sp(1)$, $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$, $x \longmapsto -q\bar{x}q$ is a reflection across $q^\perp$.

Pf: For $u \in \mathbb{H}$, $|u|=1$, $uq \longmapsto -q\bar{u}\bar{q}q = -q\bar{q}\bar{u}q = -\bar{u}q$

If $u=1$, $q \longmapsto -q$

$u=i$, $iq \longmapsto -\bar{i}q = iq$

$u=j$, $jq \longmapsto -\bar{j}q = jq$

$u=k$, $kq \longmapsto -\bar{k}q = kq$ $\square$

3/7/18 Theorem (Cartan-Dieudonné, 1937, 1948): Every $g \in O(n)$ can be expressed as a product of reflections across hyperplanes. In fact, the minimum # of reflections needed is $n - \dim(\ker(g-I))$

Pf: Suppose g can be expressed as a product of r reflections. Then $\bigcap \{r \text{ hyperplanes}\} \subseteq \ker(g-I)$

$$n-r \leq \dim(\cap \text{ hyperplanes}) \leq \dim(\ker(g-I))$$

$$\Rightarrow \underbrace{n - \dim(\ker(g-I))}_{\text{minimum possible \# of reflections.}} \leq r$$

First, suppose $\dim(\ker(g-I)) = k \geq 1$

Since g is orthogonal and fixes the subspace $\ker(g-I)$, then it also stabilizes the subspace $\ker(g-I)^{\perp}$. Hence, $\exists\ h \in O(n)$ with

$$hgh^{-1} = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & g' \end{array}\right)$$

Note: $g' \in O(n-k)$ and $\dim \ker(g'-I) = 0$.

By induction, $\exists$ reflections $f_i \in O(n-k)$ s.t. $g' = f_1 \cdots f_{n-k}$

Hence, $g = f_1' f_2' \cdots f_{n-k}'$, where $f_i' = h^{-1}\left(\begin{array}{c|c} I_k & \\ \hline & f_i \end{array}\right)h$ ← reflections in $O(n)$. ✓

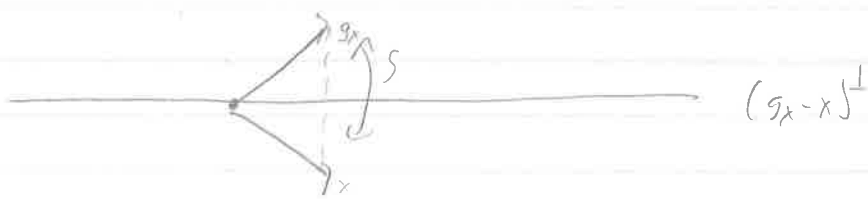
(Base case: $O(1) = \{\pm 1\}$ ✓)

Notation: reflection: $g \in O(n)$ conj. to $\left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & I \end{array}\right)$

rotation: $g \in O(n)$ conj. to $\left(\begin{array}{cc|c} \cos & -\sin & 0 \\ \sin & \cos & \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right)$

Finally, Suppose $\dim(\ker(g-I)) = 0$. WTS that g is a product of n reflections.

By definition, $\exists\, 0 \neq x \in \mathbb{R}^n$ with $gx \neq x$. Let $s \in O(n)$ be the reflection across hyperplane $(gx - x)^{\perp}$



We see $(sg)x = s(g(x)) = x$, so $sg \in O(n)$ fixes the line $\mathbb{R}x$.

By induction, $sg = f_1 \cdots f_\ell$ $(\ell \leq n-1)$

Finally, $g = s f_1 f_2 \cdots f_\ell$ $(\Rightarrow \ell = n-1)$
So g is a product of n reflections.

$\square$

Corollary: Every $g \in SO(n)$ is a product of simple rotations.
Minimum necessary: $\dfrac{n - \dim(\ker(g-I))}{2}$

Pf: $g \in SO(n)$. By Cartan-Dieudonné, $g = f_1 f_2 \cdots f_k$, where k is even and $k = n - \dim(\ker(g-I))$

Then, $g = (f_1 f_2)(f_3 f_4) \cdots (f_{k-1} f_k)$, where each $(f_i f_{i+1})$ is a simple rotation.

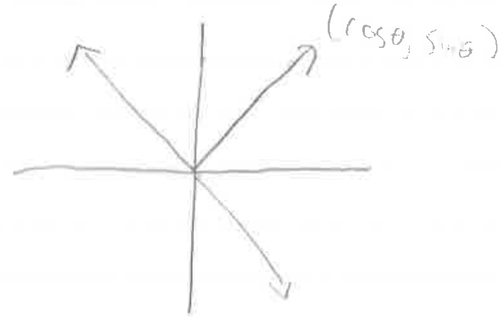
$\square$

$$\cos(\alpha+\beta) = Re(e^{i(\alpha+\beta)}) = Re((\cos(\alpha)+i\sin(\alpha))(\cos(\beta)+i\sin(\beta)))$$
$$= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Ex: $O(2)$: Columns of $O(2)$ are Orthonormal

$g \in O(2)$, Suppose $g = \begin{pmatrix} \cos\theta & ? \\ \sin\theta & ? \end{pmatrix}$

Then $g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} R_\theta$ rotation or



$g = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} F_\theta$

reflection about line of angle $\theta/2$

Relations: • $R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2}$

• $F_\alpha F_\beta = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ \sin(\beta) & -\cos(\beta) \end{pmatrix}$

$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) & \\ \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) & \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{pmatrix} = R_{\alpha-\beta}$$

• $R_\alpha F_\beta = F_{\alpha+\beta}$

3/9/18 Theorem (Orthogonal "Diagonalization" of Orthogonal Matrices (types B + D))

[Remark: A real matrix is orthog. diagonalizable iff it's symmetric]

Given $g \in O(n)$, $\exists\ h \in SO(n)$ s.t. hgh^{-1} is block diagonal with blocks of the form

$$1,\ -1,\ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Proof: If $gv = \lambda v$ for $\lambda \in \mathbb{C}$, $0 \neq v \in \mathbb{C}^n$, then

$$0 \neq (v, v) = (gv, gv) = (\lambda v, \lambda v) = |\lambda|^2 (v, v)$$

$$\Rightarrow |\lambda| = 1 \Rightarrow \lambda = \pm 1 \ \text{or} \ \lambda = \cos\theta \pm i\sin\theta$$

If $gv = \lambda v$ with g, λ real, v complex,

$$g\bar{v} = \lambda\bar{v} \Rightarrow g(v + \bar{v}) = \lambda(v + \bar{v}),$$ and $v + \bar{v}$ is real

If g has eigenvalue ± 1, then $\exists$ real eigenvector $v \in \mathbb{R}^n$, $gv = \pm v$

$\Rightarrow g$ stabilizes the line $\mathbb{R}v$, so it also stabilizes the hyperplane $(\mathbb{R}v)^\perp$

By choosing an orthonormal basis, $\exists\ h \in O(n)$ with

$$hgh^{-1} = \left(\begin{array}{c|c} \pm 1 & 0 \\ \hline 0 & g' \end{array}\right)$$

Since $g' \in O(n-1)$, we're done by induction ✓

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Otherwise, we have $gv = \lambda v$ where λ, v complex

Let $\lambda = \alpha + i\beta$, $v = r + is$. Then

$g(r+is) = (\alpha + i\beta)(r+is)$

$gr + igs = (\alpha r - \beta s) + i(\alpha s + \beta r)$

Taking real & imaginary parts,

$\begin{cases} gr = \alpha r - \beta s \\ gs = \beta r + \alpha s \end{cases} \Rightarrow g$ stabilizes the subspace $\mathbb{R}r + \mathbb{R}s$

Claim $r \perp s$, and we can choose them to be length 1

Since $|\lambda| = 1$, can write $\alpha = \cos\theta$, $\beta = \sin\theta$, so

$\begin{cases} gr = \cos\theta r - \sin\theta s \\ gs = \sin\theta r + \cos\theta s \end{cases}$

$(r,r) = (gr, gr) = (\cos\theta r - \sin\theta s, \cos\theta r - \sin\theta s)$
$= \cos^2\theta (r,r) - 2\cos\theta\sin\theta (r,s) + \sin^2\theta (s,s)$

$(s,s) = \sin^2\theta (r,r) + 2\cos\theta\sin\theta (r,s) + \cos^2\theta (s,s)$

$(r,s) = \cos\theta\sin\theta (r,r) + (\cos^2\theta - \sin^2\theta)(r,s) - \cos\theta\sin\theta (s,s)$

$\Rightarrow$ Trig stuff $\Rightarrow$ $(r,s) = 0$

$\Rightarrow (s,s) = (r,r)$.

$\Rightarrow \exists h \in O(n)$ with $hgh^{-1} = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & \\ \hline 0 & & g' \end{array}\right)$ done by induction. $\square$

3/19/18 The Fundamental Theorem of $O(n)$:

Let $g \in O(n)$. Then $\exists\, h \in SO(n)$ s.t. hgh^{-1} is block diagonal, with blocks of the form

$$\pm 1 \quad \text{or} \quad \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \theta \neq k\pi$$

Proof: $Mat_{m,n}(\mathbb{R}) \subseteq Mat_{m,n}(\mathbb{C})$

$\bar{g}$ = Conjugate matrix
g^T = transpose matrix
g^* = Conjugate transpose

Consider $g \in O(n) \subseteq U(n)$.

$\exists\, gu = \lambda u$ with $\lambda \in \mathbb{C}$, $0 \neq u \in \mathbb{C}^n$

$\|u\|^2 = u^* u = u^* g^T g u = u^* g^* g u = (gu)^*(gu) = (\lambda u)^*(\lambda u) = \bar{\lambda}\lambda\, u^* u = |\lambda|^2 u^* u$
$= |\lambda|^2 \|u\|^2$

Since $u \neq 0$, we have $\|u\| \neq 0$, so $|\lambda|^2 = 1 \Rightarrow |\lambda| = 1$.

Case 1: If $\lambda = \pm 1$, we can assume u is real.
Indeed, if $u \in i\mathbb{R}^n$, then $gu = \pm u \Rightarrow g(iu) = \pm iu$, $iu \in \mathbb{R}^n$.
Otherwise, if $u \notin i\mathbb{R}^n$, then

$gu = \pm u \Rightarrow \overline{gu} = \pm \bar{u}$
$g\bar{u} = \pm \bar{u} \Rightarrow g(u+\bar{u}) = \pm(u+\bar{u})$ where $u+\bar{u}$ is real + nonzero. ✓

Now g stabilizes the line $\mathbb{R}u \subseteq \mathbb{R}^n$, and hence also stabilizes $(\mathbb{R}u)^\perp \subseteq \mathbb{R}^n$.

By choosing an orthonormal basis, $\exists\, h \in O(n)$ with

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$$hgh^{-1} = \left(\begin{array}{c|c} \pm 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & g' \in O(n-1) \\ 0 & \end{array}\right) \in O(n)$$

By induction, $\exists\, w \in O(n-1)$ with $wg'w^{-1}$ of the correct form.

Then $w' = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & w \end{array}\right) \in O(n)$ and

$$\begin{array}{c} w'(hgh^{-1})w'^{-1} \\ \shortparallel \\ (w'h)g(w'h)^{-1} \end{array} = \left(\begin{array}{c|c} \pm 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & wg'w^{-1} \\ 0 & \end{array}\right)$$

✓

Case 2: $g \in O(n)$ has no real eigenvalues

Let $gu = \lambda u$, $\quad g(v+iw) = (\alpha + i\beta)(v+iw)$

$$(gv) + i(gw) = (\alpha v - \beta w) + i(\beta v + \alpha w)$$

Since g is real, $\begin{cases} gv = \alpha v - \beta w \\ gw = \beta v + \alpha w \end{cases}$

Claim: $\|v\| = \|w\|$, and $v^T w = 0$

To see this, we expand $u^T u$ in two ways:

- $u^Tu = u^Tg^Tgu = (gu)^T(gu) = (\lambda u)^T(\lambda u) = \lambda^2 u^Tu$

Since $\lambda \neq \pm 1$, we have $\lambda^2 \neq 1 \Rightarrow u^Tu = 0 + i0$

- On the other hand,

$$0 = u^Tu = (v+iw)^T(v+iw) = (v^T+iw^T)(v+iw)$$
$$= v^Tv - w^Tw + 2i\,v^Tw$$
$$= (\|v\|^2 - \|w\|^2) + 2i(v^Tw)$$

$\Rightarrow \|v\| = \|w\|$ and $v^Tw = 0$. ✓

Know $|\lambda| = \sqrt{\alpha^2+\beta^2} = 1$

$\alpha = \cos\theta$

$\beta = \sin\theta$

Multiply v & w by $\frac{1}{\|v\|} = \frac{1}{\|w\|}$ to make them orthonormal

Thus, g acts on $(\mathbb{R}v + \mathbb{R}w)$ by rotation and stabilizes $(\mathbb{R}v+\mathbb{R}w)^\perp \subseteq \mathbb{R}^n$.

Extend to an orthonormal basis to get $h \in O(n)$ s.t.

$$hgh^{-1} = \left(\begin{array}{cc|ccc} \cos\theta & -\sin\theta & 0 & \cdots & 0 \\ \sin\theta & \cos\theta & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & g' \in O(n-2) & \\ 0 & 0 & & & \end{array}\right) \Rightarrow \text{Done by induction.}$$

$\square$

Finally we WTS that we can choose $h \in SO(n)$:

If n is <u>odd</u>, $\det(-h) = (-1)^n \det(h) = -\det(h)$

$$hgh^{-1} = (-1)g(-h)^{-1} \quad ✓$$

$$\underset{✓}{hgh^{-1}} = \begin{array}{c} \quad p \qquad q \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix} \end{array} \quad h \notin SO(n)$$

Let n be even. If g has only real eigenvalues and $h \notin SO(n)$ then replace h by

$$\underbrace{\begin{array}{c} p \text{ on } p\text{th} \\ \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix} \end{array} h}_{\in SO(n)} \quad d$$

$$d(hgh^{-1})d^{-1} = (dh)g(dh)^{-1} \quad ✓$$

Finally, suppose hgh^{-1} has a 2×2 rotation block in bottom right.
Let

$$\Pi = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix},$$

If $h \in O(n) \setminus SO(n)$, then $\Pi h \in SO(n)$

$(\Pi h)g(\Pi h)^{-1} = \Pi(hgh^{-1})\Pi^{-1}$ ⟵ Final block is <u>transposed</u>

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(\theta) \end{pmatrix}.$$

$\blacksquare$

Corollary (Max'l Tori): Define $T \subseteq SO(n)$ as the group of matrices:

- n odd: $r = \frac{n-1}{2}$, $R_{1, \theta_1, \dots, \theta_r} = \begin{pmatrix} 1 & & & \\ & \boxed{R_{\theta_1}} & & \\ & & \ddots & \\ & & & \boxed{R_{\theta_r}} \end{pmatrix}$, where R_{θ_i} is a rotation.

- n even: $r = \frac{n}{2}$, $R_{\theta_1, \dots, \theta_r} = \begin{pmatrix} R_{\theta_1} & & \\ & \ddots & \\ & & R_{\theta_r} \end{pmatrix}$

Then T is a max'l torus.

Pf.: T is a torus because of the exponential homom

$$\exp: (\mathbb{R}, +) \longrightarrow SO(2)$$
$$\theta \longmapsto \exp\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\exp: (\mathbb{R}^r, +) \longrightarrow T$$

$$\ker(\exp) = (2\pi\mathbb{Z})^r \Rightarrow T \cong (\mathbb{R}/2\pi\mathbb{Z})^r \cong (\mathbb{R}/\mathbb{Z})^r \quad \checkmark$$

Let $H \subseteq O(n)$ be any torus

Trick: $H = \overline{\langle h \rangle}$. $\exists\, k \in SO(n)$ so $khk^{-1} \in T$

Then $\overline{\langle khk^{-1} \rangle} \subseteq T \Rightarrow kHk^{-1} \subseteq T$

$$\overline{\langle khk^{-1} \rangle} = k\overline{\langle h \rangle}k^{-1} \subseteq T$$

$\square$

3/21/18 Theorems:

- Let $\mathbb{D}$ be f.d. associative division $\mathbb{R}$-algebra. Then
  $\mathbb{D} \cong \underbrace{\mathbb{R} \text{ or } \mathbb{C}}_{\text{commutative}}$ or $\mathbb{H}$

Proof (commutative case): Let $\mathbb{D}$ be a f.d. commutative division $\mathbb{R}$-algebra.
Then, $\forall \alpha \in \mathbb{D}$, we have a minimal polynomial

$$t_0 + t_1\alpha + \cdots + t_m\alpha^m = 0, \quad t_i \in \mathbb{R}$$
$$\varphi(\alpha) = 0$$

Subalgebra $\mathbb{R}[x]/(\varphi(x)) \cong \mathbb{R}(\alpha) \subseteq \mathbb{D}$ of dimension m

Let $\gamma \in \mathbb{C}$ be any complex root of $\varphi(x) \in \mathbb{R}[x]$. Get an injective map of $\mathbb{R}$-algebras

$$\mathbb{R}(\alpha) \hookrightarrow \mathbb{C}$$
$$\alpha \longmapsto \gamma$$

$\Rightarrow m = 1$ or 2. If $\dim_{\mathbb{R}}(\mathbb{R}(\alpha)) = 1 \ \forall \alpha$ then $\alpha \in \mathbb{R}$

$\Rightarrow \mathbb{D} = \mathbb{R}$ ✓

Otherwise, $\exists \beta \in \mathbb{D}$ s.t. $\dim \mathbb{R}(\beta) = 2$, and hence $\mathbb{R}(\beta) \cong \mathbb{C}$.

$\Rightarrow \mathbb{C} \cong \mathbb{R}(\beta) \subseteq \mathbb{D}$.

$\therefore$ $\mathbb{D}$ is a f.d. associative commutative division $\mathbb{C}$-algebra

$\forall \delta \in \mathbb{D}$, the min. poly over $\mathbb{C}$ is irreducible$/\mathbb{C}$ $\Rightarrow$ min poly has degree 1 $\Rightarrow \delta \in \mathbb{C}$.

$\square$

Two Constructions:

Cayley-Dickson "Doubling"

$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, Sedenions, ...

Clifford algebras

$Cl_0 \quad Cl_1 \quad Cl_2$

$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H}\times\mathbb{H}, M_2(\mathbb{H}), M_4(\mathbb{C}), M_8(\mathbb{R}), M_8(\mathbb{R})\times M_8(\mathbb{R}), M_{16}(\mathbb{R}), \ldots$

Bott Periodicity: $Cl_{n+8} \cong Mat_{16}(Cl_n)$

Def: $$Cl(n) = \frac{\mathbb{R}\langle 1, e_1, e_2, \ldots, e_n\rangle}{(e_i^2=-1,\ e_ie_j=-e_je_i \ \forall i\neq j)}$$

$$\dim_{\mathbb{R}}(Cl(n)) = 2^n$$

basis: monomials $e_{i_1}e_{i_2}\cdots e_{i_r}$ $(0\leq r\leq n) \longleftrightarrow$ Subsets of $\{1,\ldots,n\}$

Universal Property: $$\begin{aligned} \mathbb{R}^n &\xhookrightarrow{\iota} Cl_n \\ e_i &\longmapsto e_i \end{aligned}$$

Note: $\iota(\vec{x})^2 = \left(\sum_i x_ie_i\right)^2 = -\left(\sum_i x_i^2\right) = -|\vec{x}|^2\cdot 1$

Given any $\mathbb{R}$-algebra A and a linear map $f:\mathbb{R}^n \longrightarrow A$ satisfying $f(\vec{x})^2 = -|\vec{x}|^2\cdot 1$,

we have

$$\begin{tikzcd} Cl_n \arrow[r, dashed, "\exists!"] & A \\ \mathbb{R}^n \arrow[u, "\iota"] \arrow[ur, "f"'] \arrow[u, phantom, shift left=4, "\circlearrowleft"] & \end{tikzcd}$$

Application: Define a linear map

$$\alpha : Cl_n \longrightarrow Cl_n$$
$$e_i \longmapsto -e_i$$

$$\alpha(\vec{x})^2 = \left(-\sum_i x_i e_i\right)^2 = \left(\sum_i x_i e_i\right)^2 = -|\vec{x}|^2$$

We get a unique $\mathbb{R}$-algebra automorphism $\alpha : Cl_n \longrightarrow Cl_n$
$$e_{i_1} \cdots e_{i_r} \longmapsto (-1)^r e_{i_1} \cdots e_{i_r}$$

Gives a $\mathbb{Z}_2$-grading on Cl_n.

Example: $\alpha : \mathbb{H} \longrightarrow \mathbb{H}$
$$i \longmapsto -i$$
$$j \longmapsto -j$$
$$k \longmapsto k$$
$$ij$$

$\Rightarrow \alpha$ is not conjugation

Conjugation is define

$$Cl_n \longrightarrow Cl_n \qquad \text{(anti-automorphism)}$$
$$e_{i_1} \cdots e_{i_r} \longmapsto (-1)^r e_{i_r} \cdots e_{i_1}$$

Def: $Spin(n) = \{ u \in Cl_n^\times \mid \forall \vec{x} \in \mathbb{R}^n \ \alpha(u)\vec{x}u^{-1} \in \mathbb{R}^n, \ u\bar{u} > 0 \}$

3/26/18 Define $Cl_n^0 = \{ u \in Cl_n \mid \alpha(u) = u \}$

$$Cl_n^1 = \{ u \in Cl_n \mid \alpha(u) = -u \}$$

$$Cl_n = Cl_n^0 \oplus Cl_n^1$$
$$u \qquad \frac{u+\alpha(u)}{2} \qquad \frac{u-\alpha(u)}{2}$$

The adjoint representation:

$$Cl_n^\times \times Cl_n \xrightarrow{\rho} Cl_n$$
$$(u, v) \longmapsto uvu^{-1} \quad \text{(Chevalley)}$$
$$\boxed{\alpha(u)vu^{-1}} \quad \text{(Atiyah-Bott-Shapiro)}$$

Define a subgroup $\Gamma_n \leq Cl_n^\times$ as the stabilizer of $\mathbb{R}^n \subseteq Cl_n$

$$\Gamma_n := \{ u \in Cl_n^\times \mid \underset{\alpha(u)xu^{-1}}{\rho_u(x)} \in \mathbb{R}^n \quad \forall x \in \mathbb{R}^n \}$$

Get a group homomorphism

$$\rho: \underset{\text{Lie group}}{\boxed{\Gamma_n}} \longrightarrow GL_n(\mathbb{R})$$

Lemma: $Ker(\rho) = \mathbb{R}^\times \cdot 1$

Pf: Let $u \in ker(\rho)$, $\rho_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$
$\rho_u = id$

$$\rho(x) = x \quad \forall x \in \mathbb{R}^n \subseteq Cl_n^1$$
$$\alpha(u)xu^{-1} = x \iff \alpha(u)x = xu$$

Let $u = u^0 + u^1$, $\alpha(u) = u^0 - u^1$

$$\therefore \alpha(u)x = xu \iff (u^0 - u^1)x = x(u^0 + u^1)$$
$$\underset{\text{odd}}{u^0x} - \underset{\text{even}}{u^1x} = \underset{\text{odd}}{xu^0} + \underset{\text{even}}{xu^1}$$

$u^0 x = x u^0$

Write $\underset{\text{even}}{u^0} = a^0 + e_1 b^1$

no e_1 no e_1

Observe that $a^0 e_1 = e_1 a^0$, because $a^0 e_1 = (-1)^{\text{even}} e_1 a^0$

Similarly, $b^1 e_1 = -e_1 b^1$

Let $x = e_1$. Then

$u^0 e_1 = e_1 u^0$
$(a^0 + e_1 b^1) e_1 = e_1 (a^0 + e_1 b^1)$

$e_1 a^0 - e_1^2 b^1 = e_1 a^0 + e_1^2 b^1$

$\Rightarrow b^1 = -b^1 \Rightarrow b^1 = 0.$

We conclude that u^0 contains no e_1.

Similarly, u^0 contains no $e_2, e_3, \ldots, e_n$
$\Rightarrow u^0 \in \mathbb{R} \cdot 1$. ✓

Similar argument shows that $u^1 \in \mathbb{R} \cdot 1 \cap Cl^0 \Rightarrow u^1 = 0$.

$\Rightarrow u = u^0 \in \mathbb{R} \cdot 1 \cap Cl_n^\times$

$\Rightarrow u \in \mathbb{R}^\times \cdot 1$.

$\square$

$\rho : \Gamma_n \longrightarrow GL_n(\mathbb{R})$, $\ker(\rho) = \mathbb{R}^{\times} \cdot 1$

Claim: $\Gamma_n / \mathbb{R}^{\times} \cdot 1 \cong O(n)$

Theorem: We have another group homom

$$N : \Gamma_n \longrightarrow \mathbb{R}^{\times} \cdot 1$$
$$u \longmapsto u\bar{u}$$

Pf: We show $u\bar{u} \in \ker(\rho)$

For any $x \in \mathbb{R}^n$, we have $\rho_{u\bar{u}}(x) = \alpha(u\bar{u})x(u\bar{u})^{-1}$

$$= \alpha(u)\alpha(\bar{u})\, x\, \bar{u}^{-1}u^{-1}$$
$$= \alpha(u)\alpha(\bar{u}\alpha(x)\alpha(\bar{u}^{-1}))u^{-1}$$
$$= \alpha(u)\alpha(\bar{u}\bar{x}\,\overline{\alpha(u^{-1})})u^{-1}$$
$$= \alpha(u)\alpha(\underbrace{\overline{\alpha(u^{-1})x u}}_{\in \mathbb{R}^n})u^{-1}$$
$$= \alpha(u)\alpha(u^{-1})x u u^{-1}$$
$$= \alpha(uu^{-1})x uu^{-1}$$
$$= 1 \cdot x \cdot 1 = x.$$

✓

This also implies that $\bar{u}u \in \mathbb{R}^{\times} \cdot 1$

WTS: $N(uv) = N(u)N(v)$

- $uv\overline{uv} = uv\bar{v}\bar{u} = u\bar{u}v\bar{v} = N(u)N(v)$.

□

Def: $Pin(n) = Ker(N)$

$Spin(n) = Ker N \cap Cl_n^0$

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Def: $Cl(V, Q) = T(V)/(V \otimes V + Q(v)\cdot 1)$

$Cl_n(\mathbb{R}^n, \cdot) = T(\mathbb{R}^n)/(v \otimes v + \|v\|^2 \cdot 1)$

$= \mathbb{R}(1, e_1, \dots, e_n)/(e_i^2 = -1,\ e_i e_j = -e_j e_i)$

Canonical automorphism

$\alpha : Cl_n \longrightarrow Cl_n$ $\qquad \alpha^2 = id$

$e_i \longmapsto -e_i$

Canonical anti-automorphism

$t : Cl_n \longrightarrow Cl_n^{op} = Cl_n$ $\qquad t^2 = id$

Conjugation: $\alpha \circ t = t \circ \alpha$

Grading: $Cl_n = Cl_n^0 \oplus Cl_n^1$, $\quad Cl_n^i = \{u \in Cl_n \mid \alpha(u) = (-1)^i u\}$

Canonical injection: $i : \mathbb{R}^n \hookrightarrow Cl_n^1 \subseteq Cl_n$

$e_1, \dots, e_n$

$Cl_n^\times \curvearrowright Cl_n$ by "conjugation"

$u(v) := \alpha(u) v u^{-1} = \rho_u(v)$

Defn: $\Gamma(n) = Stab_{Cl_n^\times}(\mathbb{R}^n)$

$$u \longmapsto \rho_u$$

We get a representation $\rho : \Gamma(n) \longrightarrow GL_n(\mathbb{R})$
"adjoint representation"

Why is ρ_u invertible?

$\rho_u(x) = 0 \Rightarrow \alpha(u)xu^{-1} = 0 \Rightarrow x = \alpha(u^{-1}).0.u^{-1} = 0$

$\mathrm{Ker}(\rho_u) = 0 \Rightarrow \rho_u^{-1}$ exists.

In fact, $\rho_u^{-1} = \rho_{u^{-1}}$

Recall: $\mathrm{Ker}(\rho) = \mathbb{R}^{\times}\cdot 1$

Corollary: $\forall u \in \Gamma(n)$, we can check that $u\bar{u} \in \mathrm{Ker}(\rho) \Rightarrow u\bar{u} \in \mathbb{R}^{\times}\cdot 1$

Thus, $N : \Gamma(n) \longrightarrow \mathbb{R}^{\times}\cdot 1$ is a well-defined group homom
$u \longmapsto u\bar{u}$

We have four Lie groups:

$S\Gamma(n) = \Gamma(n) \cap Cl_n^0$
$Pin(n) = \mathrm{Ker}(N)$
$Spin(n) = \mathrm{Ker}(N) \cap Cl_n^0$

$$\begin{tikzcd}
 & \Gamma(n) \arrow[dl, dash] \arrow[dr, dash] \arrow[drr, bend left] & & \\
Pin(n) \arrow[rrr, bend left, "\text{adjoint rep}"] & & S\Gamma(n) & GL_n(\mathbb{R}) \\
 & Spin(n) \arrow[ul, dash] \arrow[ur, dash] \arrow[urr, bend right, "\text{adjoint rep}"'] & &
\end{tikzcd}$$

How does this picture relate to $O(n)$ and $SO(n)$?

Lemma: Given a nonzero $x \in \mathbb{R}^n$,

$$\rho_x : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is the reflection across the hyperplane $x^\perp$.

Proof: $y \longmapsto y - \dfrac{2(y,x)}{(x,x)} \cdot x$ is the reflection across $x^\perp$.

Facts:

- $0 \neq x \in \mathbb{R}^n$, $x^2 = -\|x\|^2 \cdot 1 \implies x^{-1} = \dfrac{-x}{\|x\|^2}$
- $\forall x, y \in \mathbb{R}^n \subseteq Cl_n$, $xy + yx = -2(x,y) \cdot 1$

Indeed, $\left(\sum x_i e_i\right)\left(\sum y_j e_j\right) + \left(\sum y_i e_i\right)\left(\sum x_j e_j\right)$

$$= \sum_{i,j} x_i y_j e_i e_j + \sum_{i,j} y_i x_j e_i e_j = -2 \sum x_i y_i + \sum_{i,j} x_i y_j \underbrace{(e_i e_j - e_i e_j)}_{0}$$

$$= -2(x,y) \quad \checkmark$$

Now, $y - \dfrac{2(y,x)}{\|x\|^2} = y + 2(y,x)x^{-1}$

$$= y - (xy + yx)x^{-1}$$

$$= y - xyx^{-1} - yxx^{-1}$$

$$= -xyx^{-1}$$

$$= \alpha(x)yx^{-1}$$

$\square$

Theorem: Consider the adjoint rep $\rho: \Gamma(n) \longrightarrow GL_n(\mathbb{R})$

We already know that $\ker(\rho) = \mathbb{R}^\times \cdot 1$

Claim further that $\mathrm{im}(\rho) = O(n)$

Pf:

Fact: $\forall\, u \in \Gamma(n)$, $N(\alpha(u)) = N(u)$

Indeed, $N(\alpha(u)) = \alpha(u)\overline{\alpha(u)} = \alpha(u)\alpha(\bar{u}) = \alpha(u\bar{u})$

$$= \alpha(N(u))$$
$$= N(u)$$

$\square$

Thus, $\forall\, u \in \Gamma(n)$, $x \in \mathbb{R}^n$, we have

$$N(\rho_u(x)) = N(\alpha(u)xu^{-1}) = N(\alpha(u)N(x)N(u^{-1})) = N(\alpha(u))N(x)N(u^{-1})$$
$$= N(u)N(x)N(u)^{-1}$$
$$= N(x)$$

Finally, $N(x) = x\bar{x} = x(-x) = -x^2 = \|x\|^2 \cdot 1$. ✓

This shows that $\mathrm{im}(\rho) \subseteq O(n)$

Conversely, recall that any $g \in O(n)$ is a product of reflections

$$g = \rho_{x_1}\rho_{x_2}\cdots\rho_{x_n} = \rho_{x_1 \cdots x_n} \in \mathrm{im}(\rho)$$

Cartan-Dieudonné

$\square$

Since $\rho_x = \rho_{\frac{x}{\|x\|}}$, we find that $\rho|_{\mathrm{Pin}(n)} : \mathrm{Pin}(n) \twoheadrightarrow O(n)$

Claim: $\ker(\rho|_{\mathrm{Pin}(n)}) = \{\pm 1\}$

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Indeed, $\ker(\rho|_{Pin(n)}) = \ker(\rho) \cap Pin(n) = \mathbb{R}^\times \cdot 1 \cap Pin(n)$
$= \{\pm 1\}$

$\square$

Thus, $\rho : Pin(n) \xrightarrow{2:1} O(n)$

$\pm$ unit normals to hyperplanes

Finally, Claim that $\rho|_{Spin(n)} : Spin(n) \xrightarrow{2:1} SO(n)$

$\ker(\rho) = \{\pm 1\} \subseteq Spin(n)$ ✓

Note that even monomials $\twoheadrightarrow SO(n)$ by Cartan–Dieudonné

$\Rightarrow im(\rho|_{Spin(n)}) \supseteq SO(n)$ ✓

Conversely, Suppose that $\exists\, u \in Spin(n)$ with $\rho_u \in O(n) \setminus SO(n)$

Then we can write $\rho_u = \rho_{x_1} \cdots \rho_{x_{2k+1}}$

$\Rightarrow \rho_{u^{-1}x_1 \cdots x_{2k+1}} = I \Rightarrow u^{-1}x_1 \cdots x_{2k+1} \in \ker(\rho) = \mathbb{R}^\times$

$\Rightarrow u^{-1}x_1 \cdots x_{2k+1} = \lambda \cdot 1 \Rightarrow \underbrace{x_1 \cdots x_{2k+1}}_{\text{odd}} = \underbrace{\lambda u}_{\text{even}}.$

This is a Contradiction.

$\square$

finite products

<u>Summary</u>: $Pin(n) = \langle S^{n-1} \rangle$

$Spin(n) =$ even products $\langle S^{n-1} \rangle$

<u>Next</u>: • $\pi_1(Spin(n)) = 1$

• Maximal torus in $Spin(n)$
• Root system

3/30/18 $\Gamma(n) = \{ u \in Cl_n^\times \mid \alpha(u) x u^{-1} \in \mathbb{R}^n \ \forall x \in \mathbb{R}^n \}$

$\alpha, \overline{(\ )} : \Gamma(n) \longrightarrow \Gamma(n)$
$N : \Gamma(n) \longrightarrow \mathbb{R}^\times$

$\Gamma(n) = S\Gamma(n) \sqcup \Gamma(n)^1$
$Pin(n) = Ker(N)$ "length 1"
$= \langle S^{n-1} \subseteq \mathbb{R}^n \rangle$

$Pin(n) \xrightarrow{2:1} O(n)$
$\pm u \longmapsto$ reflection across $u^\perp$

$Pin(n) = Spin(n) \sqcup$ Odd Stuff

Products of even #'s of PB on S^{n-1}

$$\begin{array}{ccc} Spin(n) & \xrightarrow{2:1} & SO(n) \\ \cup\!| & & \cup\!| \\ \tilde{T} & \xrightarrow{2:1} & T \end{array}$$

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Observe: $(e_ie_j)^2 = e_ie_je_ie_j = -1$ (as long as $i \neq j$)

$$N(e_ie_j) = e_ie_j\overline{e_ie_j} = e_ie_je_je_i = 1$$

$\Rightarrow e_ie_j \in Spin(n)$

We get a 1-parameter subgroup

$$(\mathbb{R},+) \longrightarrow Spin(n)$$
$$\theta \longmapsto \cos(\theta) + e_ie_j\sin(\theta)$$

Theorem: Let $n = 2r$ or $2r+1$. Define a subgroup $\tilde{T} \leq Spin(n)$ by
$$\tilde{T} = \prod_{i=1}^{r}(\cos(\theta_i) + e_{2i-1}e_{2i}\sin(\theta_i)).$$

This is a torus, and in fact is maximal.

Pf: For $i \neq j$, $(e_{2i-1}e_{2i})(e_{2j-1}e_{2j}) = (-1)^4(e_{2j-1}e_{2j})(e_{2i-1}e_{2i})$

$\Rightarrow \tilde{T}$ is a direct product of circles, and hence it's a torus. ✓

Sketch of proof of maximality

Show that adjoint rep restricts to

$$\rho : \tilde{T} \xrightarrow{2:1} T$$

Consider $X^{\pm} = \pm e_i\cos\theta + e_j\sin\theta \in Pin(n)$

ρ_X is reflection across $(0,\dots,\pm\cos\theta,0,\dots,0,\sin\theta,\dots,0)^{\perp}$

$\rho_{x^\pm} = \rho_{x^+}\rho_{x^-}$ is rotation around $(\mathbb{R}e_i + \mathbb{R}e_j)^\perp$

Rotation by what angle? $x^+ \cdot x^- = \sin^2\theta - \cos^2\theta = -\cos(2\theta) = \cos(\pi \pm 2\theta)$

$$(e_i \cos\theta + e_j \sin\theta) = (\cos^2\theta - \sin^2\theta) + 2\cos\theta\sin\theta\, e_i e_j$$

$$= \cos(2\theta) + e_i e_j \sin(2\theta) \longrightarrow SO(n)$$

This gives a 2:1 surjection onto $T \subseteq SO(n)$ $\square$

Center of the spin groups:

$$Z(\mathrm{Spin}(n)) = \begin{cases} \{\pm 1\}, & n \text{ odd} \\ \{\pm 1, \pm e_1 e_2 \cdots e_n\} & n \text{ even} \end{cases}$$

$$\cong \begin{cases} \mathbb{Z}_2, & n \text{ odd} \\ \mathbb{Z}_4, & n \equiv 0 \bmod 4 \\ \mathbb{Z}_2 \times \mathbb{Z}_2, & n \equiv 2 \bmod 4 \end{cases}$$

Pf: $\rho : Z(\mathrm{Spin}(n)) \xrightarrow{2:1} Z(SO(n)) = \begin{cases} I, & n \text{ odd} \\ \pm I, & n \text{ even} \end{cases}$

2 elts n odd

4 elts n even

Check: $(\pm e_1 \cdots e_n)^2 = e_1 \cdots e_n e_1 \cdots e_n = (-1)^{\binom{n}{2}} (-1)^n = (-1)^{\binom{n+1}{2}} = \begin{cases} 1, & n \equiv 0 \bmod 4 \\ -1, & n \equiv 2 \bmod 4 \end{cases}$

$\square$

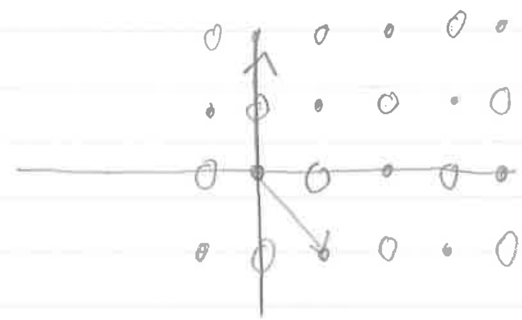
Spin(5): Type B_2

$(\mathbb{R}^2, +) \longrightarrow \tilde{T} \leq Spin(5)$

$(\theta_1, \theta_2) \longmapsto (\cos\theta_1 + e_1 e_2 \sin\theta_1)(\cos\theta_2 + e_3 e_4 \sin\theta_2)$

$$= \cos(2\pi\theta_1)\cos(2\pi\theta_2) + \cos 2\pi\theta_1 \sin 2\pi\theta_2\, e_3 e_4 + \cos 2\pi\theta_2 \sin 2\pi\theta_1\, e_1 e_2 + \sin 2\pi\theta_1 \sin 2\pi\theta_2\, e_1 e_2 e_3 e_4 = 1$$

$\text{Kernel} = \Lambda^\vee = \{ \mathbb{Z}^2 \mid \theta_1 + \theta_2 \in 2\mathbb{Z} \}$



$\exp^{-1}(\text{center}) = \Omega^\vee = \{ \mathbb{Z}^2 \}$

$\Omega^\vee / \Lambda^\vee \cong Z(Spin(5)) \cong \mathbb{Z}_2$

Spin(4): Type $D_2 = A_1 \times A_1$

$\Lambda^\vee$ = Same as before

$\Omega^\vee$ = Same as before $\sqcup \{(\theta_1, \theta_2) \in \frac{\mathbb{Z}}{2}\}$

Accidents: $B_1 = A_1 \Rightarrow Spin(3) \cong SU(2)$

$D_2 \cong A_1 \times A_1 \Rightarrow Spin(4) \cong SU(2) \times SU(2)$

$D_3 = A_3 \Rightarrow Spin(6) \cong SU(4)$

4/2/18 **Maximal Tori:**

$$\begin{tikzcd} \tilde{T} \arrow[r, hook] \arrow[d, "2:1"] & Spin(n) \arrow[d, "2:1"] & u \in \mathbb{R}^n \arrow[d] \\ T \arrow[r, hook] & SO(n) & \rho_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n \end{tikzcd}$$

Let $T = \begin{pmatrix} R_{\theta_1} & & & & \\ & R_{\theta_2} & & & \\ & & \ddots & & \\ & & & R_{\theta_r} & \\ & & & & I \end{pmatrix}$

Theorem: $\tilde{T} = \prod_{i=1}^{r} \left(\cos\frac{\theta_i}{2} + e_{2i-1} e_{2i} \sin\frac{\theta_i}{2} \right)$

Pf: $\rho\left(\cos\frac{\theta_i}{2} + e_{2i-1} e_{2i} \sin\frac{\theta_i}{2} \right) = \begin{pmatrix} I & & & & & & \\ & I & & & & & \\ & & I & & & & \\ & & & R_{\theta_i} & & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{pmatrix}$

$$\left(\cos\left(\frac{\theta_i}{2}\right) - e_{2i-1} e_{2i} \sin\left(\frac{\theta_i}{2}\right) \right) = \underbrace{\left(e_{2i} \cos\left(\frac{\theta_i}{4}\right) + e_{2i-1} \sin\frac{\theta_i}{4} \right)}_{Pin(n)} \underbrace{\left(-e_{2i} \cos\left(\frac{\theta_i}{4}\right) + e_{2i-1} \sin\left(\frac{\theta_i}{4}\right) \right)}_{Pin(n)}$$

$$= \left(\cos^2\frac{\theta_i}{4} - \sin^2\frac{\theta_i}{4} \right) - 2 e_{2i-1} e_{2i} \sin\frac{\theta_i}{4} \cos\frac{\theta_i}{4}$$

Rotation by θ_i in the e_{2i-1}, e_{2i} plane $=$ Product of two reflections with angle $\pi \pm \frac{\theta_i}{2}$

$\square$

Recall: $Z(SO(n)) = \begin{cases} I, & n \text{ odd} \\ \pm I, & n \text{ even} \end{cases}$

$$Z(Spin(n)) = \begin{cases} \pm 1, & n \text{ odd} \\ \pm 1, \pm e_1 e_2 \cdots e_n, & n \text{ even} \end{cases}$$

$$\cong \begin{cases} \mathbb{Z}_2, & n \text{ odd} \\ \mathbb{Z}_4, & n \equiv 2 \mod 4 \\ \mathbb{Z}_2 \times \mathbb{Z}_2, & n \equiv 0 \mod 4 \end{cases}$$

Fibration: $Spin(n-1) \longrightarrow Spin(n) \longrightarrow S^{n-1}$

$$\Rightarrow \underbrace{\pi_1(\underset{S^3}{Spin(3)})}_{\text{trivial}} \cong \pi_1(Spin(n)) \quad \forall n \geq 3$$

by homotopy LES: $\cdots \to \cancel{\pi_2(S^{n-1})}^{0} \longrightarrow \pi_1(Spin(n-1)) \longrightarrow \pi_1(Spin(n)) \to \cancel{\pi_1(S^{n-1})}^{0}$

Fibration: $\{\pm 1\} \longrightarrow Spin(n) \longrightarrow SO(n)$

$$\underset{1}{\pi_1(Spin(n))} \longrightarrow \underset{\mathbb{Z}_2}{\pi_1(SO(n))} \longrightarrow \underset{2}{\pi_0(\{\pm 1\})} \longrightarrow \underset{1}{\pi_0(SO(n))}$$

$O(n)$ has $\pi_0 = \mathbb{Z}_2$
kill π_0 to get $SO(n)$
$SO(n)$ has $\pi_1 = \mathbb{Z}_2$
kill π_1 to get $Spin(n)$

$Spin(n)$ has $\pi_2 = 1$, $\pi_3 = \mathbb{Z}$
kill π_3 to get $String(n)$

$String$ has $\pi_3 = \pi_4 = \pi_5 = \pi_6 = 1$

$\mathrm{Pin}(1) = \mathbb{Z}_4$ $\qquad\qquad\qquad$ $\mathrm{Pin}(3) \cong ?$

$\mathrm{Spin}(1) = \mathbb{Z}_2$ $\qquad\qquad\qquad$ $\mathrm{Spin}(3) \cong SU(2)$

$\mathrm{Pin}(2) = U(1) \times U(1)$

$\mathrm{Spin}(2) = U(1)$

$$\begin{array}{ccc} \mathrm{Spin}(4) & + & \mathrm{Spin}(5) \\ & \downarrow & \\ \mathrm{Spin}(2n) & + & \mathrm{Spin}(2n+1) \\ D_n & & B_n/C_n \end{array}$$

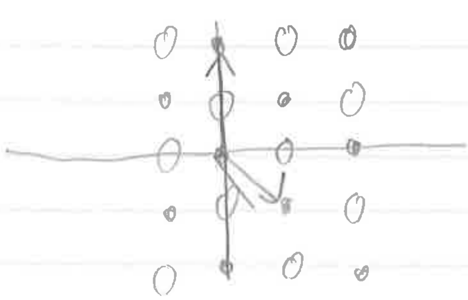
Torus in $\mathrm{Spin}(4)$ + $\mathrm{Spin}(5)$:

$\mathrm{Spin}(5)$:
$$\big(\cos(\pi s) + e_1 e_2 \sin(\pi s)\big)\big(\cos(\pi t) + e_3 e_4 \sin(\pi t)\big)$$

$$\begin{aligned} = \cos(\pi s)\cos(\pi t) &+ e_1 e_2 \cos(\pi t)\sin(\pi s) \\ &+ e_3 e_4 \cos(\pi s)\sin(2\pi t) \\ &+ e_1 e_2 e_3 e_4 \sin(\pi s)\sin(2\pi t) \end{aligned}$$

$$\left.\begin{array}{l} \mathbb{R}^2 \xrightarrow{\exp} \tilde{T} \\ (s,t) \longmapsto 1 \end{array}\right\} \quad \begin{aligned} \Lambda^\vee &= \mathrm{Ker}(\exp) \\ &= \{(s,t) \in \mathbb{Z}^2 \mid s+t \text{ even}\} \end{aligned}$$

$$\Omega^\vee = \exp^{-1}(Z) = \{(s,t) \in \mathbb{Z}^2\}$$



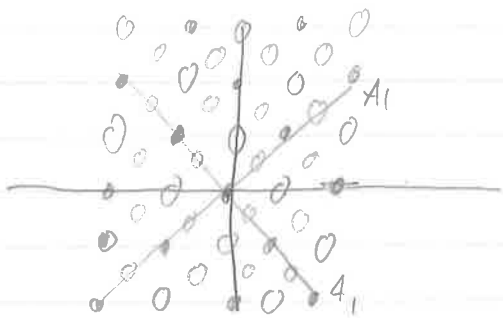
root basis: $x_1 - x_2,\ 2x_2$

weight basis: $x_1,\ x_1 + x_2$

Spin(4) : $e_1e_2e_3e_4 \in Z$

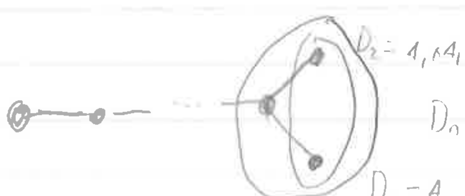
$\Lambda^\vee = \{ (s,t) \in \mathbb{Z}^2 \mid$ both even or both odd $\}$

$\Omega^\vee = \Lambda^\vee \cup \{ (s,t) \in \mathbb{R}^2 \mid s,t \in \frac{1}{2}\mathbb{Z} \}$



Accident: $Spin(4) \cong Spin(3) \times Spin(3)$

$D_2 \cong A_1 \times A_1$



$D_3 = A_3 \hookrightarrow Spin(6) \cong SU(4)$

In general: $\tilde{T} \subseteq Spin(2n)$
$Spin(2n+1)$

$$\tilde{T} = \underbrace{\prod_{i=1}^{r} \cos(\pi\theta_i)}_{0,\pm 1} + (\text{middle terms}) + e_1 \cdots e_n \underbrace{\prod_{i=1}^{r} \sin(\pi\theta_i)}_{0,\pm 1}$$

$Spin(2r+1)$: $\Lambda^\vee = \{(\theta_1,\dots,\theta_r) \in \mathbb{Z}^r \mid \theta_1+\dots+\theta_r \in 2\mathbb{Z}\}$

$$\Omega^\vee = \mathbb{Z}^r$$

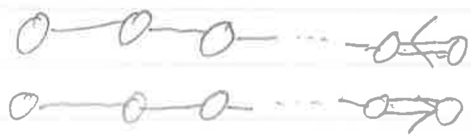
Root basis: $\alpha_1, \alpha_2, \dots, \alpha_{r-1}, \alpha_r$ $\longleftrightarrow$ 
x_1-x_2 $x_2-x_3, \dots, x_{r-1}-x_r,$ $2x_r$ (or x_r)

Spin representation

Weight basis: $\omega_1, \omega_2, \dots, \omega_r$
x_1 $x_1+x_2,$ $x_1+\dots+x_r$ or $\frac{1}{2}(x_1+\dots+x_r)$

Observe: $\langle \omega_i, \alpha_j \rangle = \dfrac{2(\omega_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$

4/4/18

Type $C(r) \to Sp(r)$
$B(r) \to Spin(2r+1)$ $\Big\} \longrightarrow$  $C(r)$ $B(r)$

dimensions $r(2r+1)$

The Weyl Group:

Let G be a topological Lie group, G_1 be the connected component of $1 \in G$. Then $G_1 \trianglelefteq G$, hence

$$\pi_0(G) = G/G_1$$

In fact, $\forall\ g \in G$, $gG = g \cdot G_1 = G_1 g$ is the path connected component of g.

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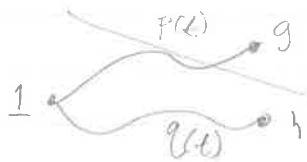
In particular, if G is compact, then the components of G are an open cover, hence $\pi_0 G$ is a <u>finite</u> group.

<u>Example</u>: Given $T \subseteq K$, the finite group $\pi_0(N_K(T))$ is called the Weyl group.

<u>Proof</u>: First, $G_1 \subseteq G$ is a subgroup. ~~Let $g,h \in G_1$~~



Then $p(t)q(t)$ is a path

$\forall\, g \in G_1$, the set gG_1^{-1} is connected and contains 1, hence $gG_1^{-1} \subseteq G_1 \Rightarrow gh^{-1} \in G_1 \;\forall\, g,h \in G_1$.

<u>Second</u>: G_1 is normal

$\forall\, g \in G$, gG_1g^{-1} is a connected set ctg 1, hence $gG_1g^{-1} \subseteq G_1$.

Conversely, every $h \in G_1$ can be written as $g(g^{-1}hg)g^{-1}$, where $g^{-1}hg \in G_1$.

∎

<u>Definition</u>: Let $T \subseteq K$, Define the <u>Weyl group</u> of K (relative to T) as

$$N_K(T)/T = W_K(T)$$

<u>Claim</u>: For maximal tori T, T', $W_K(T) \cong W_K(T')$

Pf: We know $T' = kTk^{-1}$ for some $k \in K$. Hence

$$N_K(T') = k N_K(T) k^{-1}$$

$$N_K(T)/T \longrightarrow N_{K'}(T')/T'$$

$$gT \longmapsto k(gT)k^{-1}$$

$$= kgk^{-1}kTk^{-1} = (kgk^{-1})T'$$

□

Claim: $Z_K(T) = T$

$$= \{k \in K \mid ktk^{-1} = t \ \forall t \in T\}$$

Pf: Let $g \in Z_K(T)$ and consider the abelian group
$T \subseteq \langle T, g \rangle$

WTS $\langle T, g \rangle \subseteq T'$ for some torus T'
Maximality of $T \Rightarrow g \in T$.

Let $A = \overline{\langle T, g \rangle}$

By compactness of A, $A/A_1 = \pi_0 A$ is finite.

Note that A_1 is a torus and $A_1 \supseteq T$.

Every elt $a \in A$ is of the form $g^d t$ for $d \in \mathbb{Z}$, $t \in T$.

Hence every coset looks like $aA_1 = g^d t A_1 = g^d A_1$

Get a surjective group hom

$$(\mathbb{Z}, +) \twoheadrightarrow A/A_1$$
$$d \longmapsto g^d A_1$$

Claim that A/A_1 is cyclic

Since A/A_1 is cyclic, we will find a torus $A \subseteq T' \subseteq K$.

Let $A/A_1 = \langle gA_1 \rangle$ of order d, so $g^d \in A_1$. Let $A_1 = \overline{\langle a \rangle}$ be a top. generator. Choose $b \in A_1$ s.t. $ag^{-d} = b^d$

$$\Rightarrow a = (gb)^d$$

$$\Rightarrow (gb)^{qd} \text{ dense in } A_1$$

$$(gb)^{qd+r} \text{ dense in } g^r A_1$$

Hence $\overline{\langle gb \rangle} \supseteq A$, but gb is contained in some maximal torus T',

So $A \subseteq \overline{\langle gb \rangle} \subseteq T'$

$\blacksquare$

4/9/18

Let $T \subseteq K$ be a max'l torus in a cpt Lie group.
Recall that

$$K = \bigcup_k kTk^{-1}, \quad Z(K) = \bigcap_k kTk^{-1}$$

"Weyl Reduction" $\quad T \overset{?}{\rightsquigarrow} K$
Weyl Group

$$K \underset{\text{conjugation}}{\overset{\text{transitive}}{\curvearrowright}} \{\text{max. tori}\}$$

Fix a max'l torus T, $\mathrm{Stab}_K(T) = N_K(T)$

Orbit-Stabilizer Thm: $\{\text{max'l tori}\} \longleftrightarrow K/N_K(T)$

"Space of max'l tori"

$$\pi_1(K/N_K(T)) \cong \underbrace{N_K(T)/T}_{\text{Weyl group}}$$

Define the Weyl group $W_K(T) = N_K(T)/T$

We've seen that $W_K(T) \cong W_K(T')$ $\forall$ max. tori T, T'.

We've also seen that $Z_K(T) = T$

$$N_K(T) \longrightarrow \mathrm{Aut}(T)$$
$$g \longmapsto (t \mapsto gtg^{-1})$$

Kernel is exactly $Z_K(T) = T$

We conclude that $W_K = N_K(T)/T$ (cpt) is isomorphic to a subgroup of $\mathrm{Aut}(T)$. (discrete)

W_K = automorphisms of T that come from the action of K.

Claim: that $\mathrm{Aut}(T)$ is discrete. Assuming this, we have

$$N_K(T)_1 = T$$

Hence, $W_K = \pi_0(N_K(T)) = N_K(T)/T$

Pf: $T \subseteq N_K(T)_1$.

Conversely, let $Q \subseteq N_K(T)$ be any connected subgroup

$Q \curvearrowright T$ by conjugation

$\underset{\substack{\uparrow \\ \text{connected}}}{\frac{Q}{Q \cap T}} \hookrightarrow \mathrm{Aut}(T) \implies \frac{Q}{Q \cap T}$ is trivial $\implies Q \subseteq T$

(since $\mathrm{Aut}(T)$ is discrete)

In particular, $Q = N_K(T)_1 \subseteq T$.

$\square$

Now, to show that $\mathrm{Aut}(T)$ is discrete:

Idea: All tori (cpt connected abelian group) are isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$

$\mathbb{T} = \mathbb{R}/\mathbb{Z}$

Automorphisms $\varphi: \mathbb{T} \longrightarrow \mathbb{T}$ (group hom, invertible, differentiable)

$0+\mathbb{Z} \qquad 0+\mathbb{Z}$

$\exists!$ lift $\tilde{\varphi}: \mathbb{R} \longrightarrow \mathbb{T}$

$0 \longmapsto 0+\mathbb{Z}$

$\mathbb{Z} \longmapsto 0+\mathbb{Z}$

Claim that $\tilde{\varphi}$ is a group homom

Earlier, we showed that $\tilde{\varphi}(t) = \alpha t$ for some $\alpha \in \mathbb{Z}$ $\exists \cdot$ $\mathrm{End}(\mathbb{T}) \cong \mathbb{Z}$

$Aut(\mathbb{T}) \cong \mathbb{Z}^\times \cong \{\pm 1\} \cong \mathbb{Z}/2$

More generally, we claim that $Aut(\mathbb{T}^n) \cong GL_n(\mathbb{Z})$

If K is a rank r Lie group (i.e. $T \subseteq K$)

$$\begin{array}{c} T \\ \text{\rotatebox{90}{$\cong$}} \\ \mathbb{T}^r \end{array}$$

Then $W_K \hookrightarrow GL_r(\mathbb{Z}) = Aut(\mathbb{T}^r)$

$W_K \hookrightarrow GL(\Lambda)$

root lattice

Recall: W_K is generated by the reflections in the simple roots

Type A:

$SU(n)$: $T = \{diag(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}) \mid \theta_1 + \dots + \theta_n = 0\}$

$N_{SU(n)}(T) \overset{\text{?}}{\cong}$ permute eigenspaces $= \mathfrak{S}_n$

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Example: Weyl group of type A_1.

$SU(2) \xrightarrow{2:1} SO(3)$

$\pi_1 = 0 \qquad Z = 0$

$\tilde{T} \qquad T$

251 $\tilde{T} = \{diag(e^{2\pi i\theta}, e^{-2\pi i\theta}) \mid \theta \in \mathbb{R}\}$ $\quad T = \left\{ \left(\begin{array}{cc|c} \cos 2\pi\theta & -\sin 2\pi\theta & 0 \\ \sin 2\pi\theta & \cos 2\pi\theta & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \mid \theta \in \mathbb{R} \right\}$

$$W_{SU(2)} = N_{SU(2)}(\tilde{T})/\tilde{T} \hookrightarrow Aut(\tilde{T}) = \mathbb{Z}_2$$

To find $N_{SU(2)}(\tilde{T})$: Let $t_\theta = \mathrm{diag}(e^{2\pi i\theta}, e^{-2\pi i\theta})$

Let $g \in N_{SU(2)}(\tilde{T})$. Then $g t_\alpha g^{-1} = t_\beta$

$$\Updownarrow$$

$$g t_\alpha = t_\beta g$$

i.e., $g[t_\alpha \begin{pmatrix}1\\0\end{pmatrix}] = t_\beta[g\begin{pmatrix}1\\0\end{pmatrix}]$

$$g\begin{pmatrix}e^{2\pi i\alpha}\\0\end{pmatrix} = t_\beta[g\begin{pmatrix}1\\0\end{pmatrix}]$$

$$e^{2\pi i\alpha}[g\begin{pmatrix}1\\0\end{pmatrix}] = t_\beta[g\begin{pmatrix}1\\0\end{pmatrix}]$$

t_β has eigenvalues $\pm 2\pi i\beta$. Hence, $e^{2\pi i\alpha} = e^{\pm 2\pi i\beta}$ Or any unit-length complex #

$g\begin{pmatrix}1\\0\end{pmatrix}$ is a unit-length eigenvector of t_β, hence $g\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\pm1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\\pm1\end{pmatrix}$

Since $g\begin{pmatrix}0\\1\end{pmatrix} \perp g\begin{pmatrix}1\\0\end{pmatrix}$, we have (over $\mathbb{R}$),

$g = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}$ (over the reals)

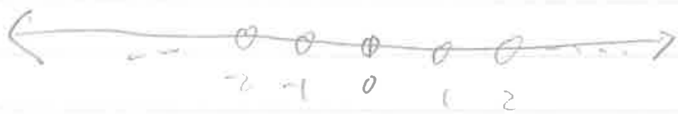
$$\boxed{\begin{pmatrix}e^{i\theta}&0\\0&e^{-i\theta}\end{pmatrix}, \begin{pmatrix}-e^{i\theta}&0\\0&-e^{-i\theta}\end{pmatrix}} \boxed{\begin{pmatrix}0&-e^{i\theta}\\e^{-i\theta}&0\end{pmatrix}, \begin{pmatrix}0&e^{i\theta}\\-e^{-i\theta}&0\end{pmatrix}}$$ (in general)

$\leftarrow \tilde{T}$

We conclude that $N_{SU(2)}(\tilde{T})/\tilde{T} = \{\begin{pmatrix}1&0\\0&1\end{pmatrix}\tilde{T} \sqcup \begin{pmatrix}0&-1\\1&0\end{pmatrix}\tilde{T}\}$

$$N_{SU(n)}(\tilde{T})/\tilde{T} = \left\{ \begin{matrix} P\tilde{T} & \det(P)=1 \\ \begin{pmatrix} -1 & 0 \\ 0 & \ddots & 1 \end{pmatrix} P\tilde{T} & \det(P)=-1 \end{matrix} \right\}$$

How does $W_{SU(2)}$ act on the root system?



$$\exp : \mathbb{R} \xrightarrow{t} \tilde{T} \subseteq SU(2)$$
$$\theta \longmapsto \operatorname{diag}(e^{2\pi i\theta}, e^{-2\pi i\theta})$$

Kernel is $\Lambda \subseteq \mathbb{R}$
$\quad\;\, \overset{\shortparallel}{\mathbb{Z}}$

W acts on $\tilde{T}$ by conjugation

$$w^{\tilde{T}} \curvearrowright \tilde{T}$$
$$g \longmapsto w g w^{-1}$$

$$w \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} w^{-1} \underset{\text{DEF}}{=:} \begin{pmatrix} e^{2\pi i(w\theta)} & 0 \\ 0 & e^{-2\pi i(w\theta)} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \curvearrowright \tilde{T} \quad \rightsquigarrow \quad \mathbb{R}, \quad \theta \longmapsto \theta$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \curvearrowright \tilde{T} \qquad \theta \longmapsto -\theta$$

$SU(n)$ $\qquad\qquad$ $\mathfrak{su}(n)$

$[W] \curvearrowright T$ $\qquad\quad$ $[W] \curvearrowright \mathbb{R}^n$

Weyl integral formula:

$T \subseteq K$

$f: K \longrightarrow \mathbb{R}$ Constant on Conj. Classes

$$\int_{SU(2)} f(k)\,dk = \frac{1}{2}\int_T f\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \left| e^{i\theta} - e^{-i\theta} \right|^2 \frac{d\theta}{2\pi}$$

$$= 2\int_0^{2\pi} f(\theta)\sin^2(\theta)\frac{d\theta}{2\pi}$$

$f: K \longrightarrow \mathbb{R}$

$$\int_K f(k)\,dk = \frac{1}{|W|}\int_T f(t)g(t)\,dt,$$ where $g(t) =$ determinant coming from root system

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4/13/18

Representations of a <u>Compact</u> Lie group K.

A representation is a group homom

$$\varphi: K \longrightarrow GL_n(\mathbb{C})$$

$$K \underset{\varphi}{\curvearrowright} \mathbb{C}^n$$

$$\varphi: \mathbb{C}K \longrightarrow Mat_n(\mathbb{C})$$

$\boxed{\mathbb{C}K\text{-modules}}$ Tensor Category

Let V be a $\mathbb{C}K$-module, V is <u>irreducible</u> if it has no nontrivial $\mathbb{C}K$-submodule, and is <u>indecomposable</u> if it's not a direct sum of $\mathbb{C}K$-submodules.

Schur's Lemma (over $\mathbb{C}$): Let U, V be f.d. irreducible $\mathbb{C}K$-modules. Then

$$\dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathbb{C}K}(U,V)\right) = \begin{cases} 1, & U \cong V \\ 0, & \text{else} \end{cases}$$

Maschke's Theorem (K compact): f.d. irreducible $\Rightarrow$ decomposable

$$U \subseteq \overset{\text{f.d.}}{V} \Rightarrow \exists\, U' \text{ s.t. } V = U \oplus U'$$

Pf: Haar measure. Make any inner product K-invariant by integrating over K.

$$\langle u, v\rangle' = \int \langle ku, kv\rangle \, dk$$

$\square$

It follows that any f.d. $\mathbb{C}K$-module has a unique decomposition into irreducibles.

Say $V = U_1 \oplus \cdots \oplus U_r$, and U an irrep.

Then $\dim \underset{\text{cpt}}{\mathrm{Hom}(U,V)} = \dim \bigoplus_i \mathrm{Hom}(U, U_i) = \#\{i \text{ s.t. } U \cong U_i\}$

$\Rightarrow$ rep theory of K is Solved, if we can

1) Classify the irreps of K
2) Compute the dimensions $\mathrm{Hom}(U,V)$

Proposition: Let U be a $\mathbb{C}K$-module

$$\begin{tikzcd} K \arrow[r] \arrow[rr, bend right, "\chi_U"'] & GL_n(\mathbb{C}) \arrow[r, "tr"] & \mathbb{C} \end{tikzcd}$$

"Character of U", constant on conj. classes

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(☆) Theorem: for U, V f.d. $\mathbb{C}K$-modules,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}K}(U,V) = \underbrace{\int_K \overline{\chi_U(k)}\chi_V(k)\,dk}_{\text{inner product in } L^2(K)}$$

Peter-Weyl Theorem: Irreducible characters χ_V are a Hilbert space orthonormal basis in $L^2(K)$.

Proof of (☆): $Tr(\text{projection}) = \dim(\mathrm{im}(\text{proj}))$

Given f.d. $\mathbb{C}K$-module U, consider the fixed subspace

$$U^K := \{u \in U \mid k(u) = u \ \forall k \in K\}$$

$$U^K = \underbrace{\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{\dim U^K}$$

Consider the ("Reynolds") operator
$$U \longrightarrow U$$
$$u \longmapsto \int_K k(u)\,dk$$

This is the orthogonal projection onto U^K

$$\dim \mathrm{Hom}_{\mathbb{C}K}(U,V) = \dim \mathrm{Hom}_{\mathbb{C}K}(U \otimes \overset{\text{triv}}{\mathbb{C}}, V)$$

$$= \dim \mathrm{Hom}_{\mathbb{C}K}(\mathbb{C}, U^* \otimes V) = \dim (U^* \otimes V)^K$$

$$= tr_{U^* \otimes V}\left(\int_K k\,dk\right) = \int_K \chi_{U^* \otimes V}(k)\,dk$$

$$= \int_K \chi_{V^*}(k)\,\chi_V(k)\,dk = \int_K \overline{\chi_V(k)}\,\chi_V(k)\,dk.$$

$\square$

Weyl's idea: Let $T \subseteq K$ be a max'l torus and consider the map

$$\pi : (K/T) \times T \longrightarrow K$$
$$(kT, t) \longmapsto ktk^{-1}$$

Claim that π is generically a $|W|$-fold covering map.

Assuming this, then $\forall\ f \in L^2(K)$, we have $\displaystyle\int_K f\,dk = \frac{1}{|W|}\int_{(K/T)\times T} \pi^* f\, \pi^* dk$

Proof of multiplicity: Every conj. class in K meets T at exactly $|W|$ points

Suppose $x, y \in T$ are conjugate in K. Then x, y are also conjugate in $N_K(T)$, hence $y = w \cdot x$, $w \in W$

Suppose $y = kxk^{-1}$, $k \in K$. $T \subseteq Z_K(y) \subseteq K$
$\quad\quad\quad\quad \cup\!\!\!\mid$
$\quad\quad\quad\quad kTk^{-1}$

Let $H = Z_K(y)_1$. Then $\underbrace{T, kTk^{-1}}_{\text{max'l tori in } H} \subseteq H$

Since maximal tori are conjugate, $\exists\ h \in H$ s.t. $kTk^{-1} = hTh^{-1}$
$\iff (h^{-1}k)T(h^{-1}k)^{-1} = T$

$\Rightarrow h^{-1}k \in N_K(T)$, and $(h^{-1}k)x(h^{-1}k)^{-1} = h^{-1}kxk^{-1}h = h^{-1}yh = y$

$\square$

4/16/18 Let K be a Compact Lie group. We can assume by Haar measure that any f.d. $\mathbb{C}K$-module is Unitary:

$$G \circlearrowright V$$
$$\varphi: G \longrightarrow U(V)$$

Let U, V be f.d. $\mathbb{C}K$-modules, and recall from last time

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}K}(U,V)) = \int_K \overline{\chi_U(k)}\chi_V(k)\,dk$$

$$\begin{tikzcd} \varphi: G \arrow[r] \arrow[rr, bend right, "\chi_U"'] & GL(U) \arrow[r, "tr"] & \mathbb{C} \end{tikzcd}$$

If U, V are both irreducible, $\dim \mathrm{Hom}_{\mathbb{C}K}(U,V) = \int \overline{\chi}_U \chi_V\, dk \begin{cases} 1, & U \cong V \\ 0 & \text{else} \end{cases}$

If V not irred., $\dim \mathrm{Hom}_{\mathbb{C}K}(U,V) =$ multiplicity of U in the unique prime factorization of V

A f.d. $\mathbb{C}K$-module is determined up to isomorphism by its trace

$tr(AB) = tr(BA)$, so $tr(BAB^{-1}) = tr(B^{-1}BA) = Tr(A)$

$\therefore$ Characters of representations are (square-integrable) Class functions

$$\chi: K \longrightarrow \mathbb{C}, \text{ and } \int \overline{\chi}\chi < \infty$$

Peter-Weyl: Irred. Chars are a dense orthonormal basis for $L^2_c(K)$, Matrix elts are a dense orthonormal basis for $L^2(K)$

To solve rep. theory we need:

1) A method to compute $\int f \, dk$ (Weyl integral formula)

2) to identify the irreps. (Weyl character formula)

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1) Consider the map

$$\pi : (K/T) \times T \longrightarrow K$$
$$(kT, t) \longmapsto k t k^{-1}$$

This is generically a $|W|$-fold covering map.

Given $g \in K$, we consider the fiber

$\# \pi^{-1}(g) = \# \{ (kT, t) \mid k t k^{-1} = g \Leftrightarrow t = k^{-1} g k \}$

Claim $= \# (T \cap \text{Conj. class of } g \in K)$

Check: $k_1 g k_1^{-1} = k_2 g k_2^{-1} = t \in T$

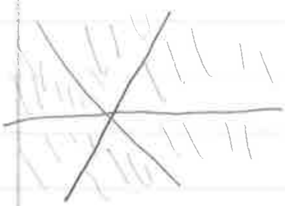
$g = k_1^{-1} t k_1 \Rightarrow k_2 k_1^{-1} t (k_2 k_1^{-1})^{-1} = t$

$\Rightarrow ???? \Rightarrow k_2 k_1^{-1} \in Z_K(T) = T \Rightarrow k_1 T = k_2 T.$

Last thing we showed that $\forall x \in T$, $T \cap \text{Conj. Class of } x = \{ w \cdot x \mid w \in W \}$

"Generically" it is a set of size $|W|$.

$$W \circlearrowright \mathfrak{h} \xrightarrow{\exp} W \circlearrowright T$$



Smooth Class functions : $K \longrightarrow \mathbb{C}$

$\shortparallel$

W-invariant functions : $T \longrightarrow \mathbb{C}$

$\shortparallel$

W-invariant Polys functions : $\mathfrak{h} \longrightarrow \mathbb{C}$

In type A, Class functions $SU(n+1) \longrightarrow \mathbb{C}$ are in correspondence with $\mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$

Weyl Integral Formula: $\chi : K \longrightarrow \mathbb{C}$ class function

$$\int_K \chi(k)\,dk = \frac{1}{|W|}\int \pi^*\chi \; \pi^* dk = \frac{1}{|W|}\int_T \chi(t) \prod_{\alpha \in R} \left(1 - e^{2\pi i \alpha}\right) dt$$

$\pi^*\chi(kT, t) = \chi(ktk^{-1}) = \chi(t)$

Where $\alpha : \mathfrak{h} \longrightarrow \mathbb{C}$ are functions on the Lie algebra of T.

$\Rightarrow e^{2\pi i \alpha} : T \to \mathbb{C}$ are functions on the torus.

$\mathfrak{h}$ vs. $\mathfrak{h}^*$

Weyl Character Formula: for each positive $\lambda \in \Omega$, we have an irreducible char:

$$\chi_\lambda = \frac{\sum_w (-1)^{\det(w)} e^{2\pi i(\lambda+\rho)}}{\sum_w (-1)^{\det(w)} e^{2\pi i \rho}},$$

where $\rho = \sum$ fund. weights $= \frac{1}{2}\sum$ pos. roots.

4/19/18

Weyl Integral Formula: $T \subseteq K$

For any $(\cdot\cdot)$ class function $f : K \to \mathbb{R}$

$$\int_K f(k)\,dk = \frac{1}{|W|}\int_T f(t)\,\Delta(t)\,dt$$

Where $\boxed{\Delta(t) = \prod_\alpha (1 - e^{\alpha})}$

What is this?

Recall "quantization of charge"

Continuous group hom is of the form

$$(\mathbb{R}, +) \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$
$$t \longmapsto e^{t\mu} \quad \text{for some } \mu \in \mathbb{C}$$

$$(\mathbb{R}/\mathbb{Z}, +) \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}$$
$$t \longmapsto e^{t\mu} \quad \mu \in 2\pi i \mathbb{Z}$$
$$t \longmapsto (e^{2\pi i \mu t}, \quad \mu \in \mathbb{Z}$$

$$U(1) \longrightarrow U(1)$$
$$t \longmapsto t^{\mu} \quad \text{for } \mu \in \mathbb{Z}$$

Now let $T = U(1)^r$

Continuous homoms

$$T \xrightarrow{V_\mu} \mathbb{C}^\times$$
$$(t_1,\dots,t_r) \longmapsto t_1^{\mu_1} t_2^{\mu_2} \cdots t_r^{\mu_r} \quad \text{for some } (\mu_1,\dots,\mu_r) \in \mathbb{Z}^n$$

(accidentally, we land in $U(1)$)

These $(\mu_1,\dots,\mu_r) \in \mathbb{Z}^r$ are called "weights"

$T \curvearrowright V_\mu \ (= \mathbb{C})$

$$(t_1,\dots,t_r) \cdot \alpha = (t_1^{\mu_1} \cdots t_r^{\mu_r})\alpha$$

Now consider any f.d. $\mathbb{C}K$-module $K \curvearrowright V$

$$\Downarrow$$

$$T \curvearrowright V = \bigoplus_{\mu} V_\mu$$

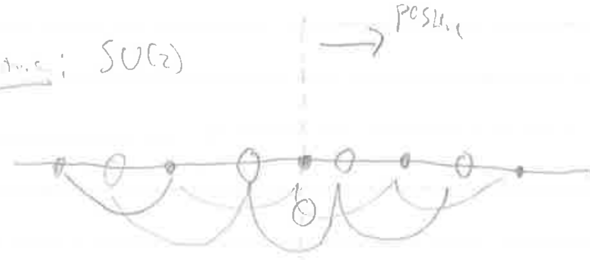
for some weights $\mu \in \mathbb{Z}^r$ (with multiplicity)

Remarks: Weights of any rep are permuted by the Weyl group

Weights live in the weight lattice

(in fact, in a coset of the root lattice)

Picture: $SU(2)$ $\longrightarrow$ positive



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How does this relate to exponential map?

$$\mathfrak{t} \xrightarrow{\exp} T \curvearrowright V = \bigoplus_{\mu} V_\mu$$

$$\underset{\mathbb{R}^r}{(x_1, \ldots, x_r)} \longmapsto (e^{2\pi i x_1}, \ldots, e^{2\pi i x_r}) \curvearrowright V_\mu$$

$$v \longmapsto \left(\prod e^{2\pi i x_j \mu_j}\right) v$$

$$v \longmapsto e^{2\pi i (x, \mu)} v$$

$$\ker(\exp) = \{ x \in \mathbb{R}^r \mid (x, \mu) \in \mathbb{Z} \ \forall \text{ weights } \mu \text{ of } V \}$$

4/20/18 Recall: $\displaystyle\int_K f(k)\,dk = \frac{1}{|W|}\int_T f(t)\Delta(t)\,dt$

$$\Delta(t) = \det(d\pi)$$

$$\pi : (K/T) \times T \longrightarrow K$$
$$(kT, t) \longmapsto ktk^{-1}$$

Turns out: $\Delta(t) = \det(I - \mathrm{Ad}(t^{-1}))$

$\mathrm{Ad}(t^{-1})$ = derivative of conjugation by t^{-1}

$$K/T \longrightarrow K/T$$
$$kT \longmapsto t^{-1}kT$$

The eigenvalues of $\mathrm{Ad}(t^{-1})$ depend continuously on $t \in T$.

th $\quad T \xrightarrow{\alpha} U(1)$

$(t_1, \ldots, t_r) \longmapsto t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_r^{\alpha_r}$, $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ is called a root.

$\Delta(t) = \det(I - Ad(t^{-1})) = \prod(1 - e^{2\pi i \alpha})$

Weight lattice: $X^*(T) = \mathrm{Hom}(T, U(1))$
Coweight lattice: $X_*(T) = \mathrm{Hom}(U(1), T)$

$\langle\, , \,\rangle : X^*(T) \otimes_{\mathbb{Z}} X_*(T) \longrightarrow X^*(U(1)) = \mathbb{Z}$

$\exp : \mathfrak{t} \twoheadrightarrow T$
$\Lambda = \ker(\exp)$

<u>Claim:</u> $\Lambda \cong X_*(T)$

$$\lambda(\exp(t)) = e^{2\pi i \langle \lambda, t\rangle}$$

$$\begin{tikzcd} \lambda : T \arrow[r, "\exp(t)"] & \mathbb{C}^\times \\ D\lambda : \mathfrak{t} \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \\ t & 2\pi i \langle \lambda, t\rangle \end{tikzcd}$$

$\frac{D\lambda}{2\pi i} : \mathfrak{t} \longrightarrow \mathbb{R}$
$\qquad t \longmapsto \langle \lambda, t\rangle$

$\lambda \longmapsto \frac{D\lambda}{2\pi i}$

$X^*(T) \hookrightarrow \mathfrak{t}^*$

$T \subseteq K \rightsquigarrow H \subseteq G$

$\Lambda = \mathrm{Hom}(H, \mathbb{C}^\times)$
$\lambda \longmapsto \frac{D\lambda}{2\pi i}$
$\Lambda \hookrightarrow \mathfrak{h}^*$, where $\exp : \mathfrak{h} \to H$

$\ker(\exp) \cong \Lambda^\vee$, $\quad H \cong \mathfrak{h}/\Lambda^\vee$

$$\begin{tikzcd} G \curvearrowright V \arrow[d] \\ H \curvearrowright V = \bigoplus V_\mu, \quad \mu \in \Lambda \end{tikzcd}$$

$$\begin{tikzcd} G \curvearrowright \mathfrak{g} \arrow[d] \\ H \curvearrowright \mathfrak{g} = \overset{\text{Cartan Subalgebra}}{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \Phi} V_\alpha \end{tikzcd}$$ rootspaces

$\Phi \subseteq \Lambda$ finite set of "roots"

$$\alpha \in \Phi \implies \alpha \in \Lambda$$
$$\alpha : H \longrightarrow \mathbb{C}^\times$$
$$\begin{array}{c} H \\ \cup\!| \\ \operatorname{Ker}(\alpha) \end{array}$$
root subgroup

$$Z(G) = \bigcap_{\alpha \in \Phi} \operatorname{Ker}(\alpha) \subseteq H$$

Let $\langle \Phi \rangle = \mathbb{Z}\Phi \subseteq \Lambda$, and assume $\Lambda/\langle \Phi \rangle$ is finite "Semi-simple"

$$\begin{array}{ccc} H & \xrightarrow{\cong} & \mathfrak{h}/\Lambda^\vee \\ \cup\!| & & \\ \operatorname{Ker}(\alpha) & \longrightarrow & \{h \in \mathfrak{h} : \langle \alpha, h \rangle \in \mathbb{Z}\}/\Lambda^\vee \end{array}$$

$$Z(G) \xrightarrow{\cong} \{h \in \mathfrak{h} : \langle \alpha, h \rangle \in \mathbb{Z} \ \forall \alpha \in \Phi\}/\Lambda^\vee$$

$$\Rightarrow Z(G) \cong \langle \Phi \rangle^\vee / \Lambda^\vee$$

265 Also, $\Lambda/\langle \Phi \rangle \cong \widehat{Z(G)} = \operatorname{Hom}(Z(G), \mathbb{C}^\times) \cong \mathbb{Z}(G)$

$$\begin{tikzcd} \tilde{G} \arrow[r] & G \\ Z(\tilde{G}) \arrow[u] \arrow[r] & Z(G) \arrow[u] \end{tikzcd}$$

$\pi_1(G) = \ker(Z(\tilde{G}) \longrightarrow Z(G))$

$\tilde{\Lambda} = \mathrm{Hom}(\tilde{H}, \mathbb{C}^\times)$

As before, $Z(\tilde{G}) = \langle \Phi \rangle^\vee / \tilde{\Lambda}^\vee$

defined only locally, so don't need to consider $\tilde{\Phi}$ — $\Phi = \Phi$

Claim: For some reason (simple-connectedness), $\tilde{\Lambda}^\vee = \langle \Phi^\vee \rangle$

Hence, $\pi_1(G) = \ker\big(Z(\tilde{G}) \longrightarrow Z(G)\big) \cong \boxed{\Lambda^\vee / \langle \Phi^\vee \rangle}$

$$\langle \Phi \rangle^\vee / \langle \Phi^\vee \rangle \longrightarrow \langle \Phi \rangle^\vee / \Lambda^\vee$$

$$\langle \Phi^\vee \rangle \overset{\pi_1}{\subseteq} \Lambda^\vee \overset{Z}{\subseteq} \langle \Phi \rangle^\vee$$

$$\langle \Phi \rangle \underset{Z}{\subseteq} \Lambda \underset{\pi_1}{\subseteq} \langle \Phi^\vee \rangle^\vee$$

Simple: $\Lambda = \langle \Phi \rangle$

Simply-connected: $\Lambda^\vee = \langle \Phi^\vee \rangle$

4/23/18 Roughly Speaking!

For every root system Φ, we have

$$\underset{\text{root lattice}}{\Lambda} \overset{Z}{\subseteq} X(G) \overset{\pi_1}{\subseteq} \underset{\text{weight lattice}}{\Omega}$$

Every Compact Lie group has a "Character lattice" $X(G)$

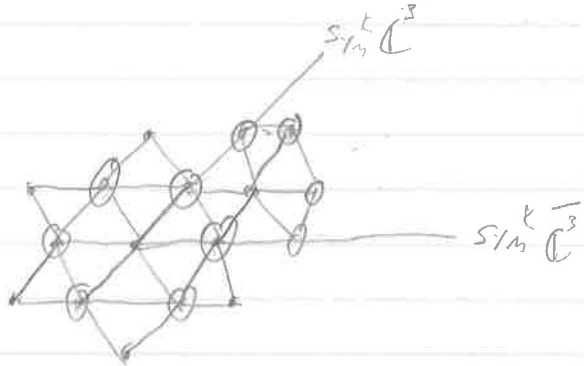
$$Z(G) \cong X(G)/\Lambda, \quad \pi_1(G) \cong \Omega/X(G)$$

Weyl group acts on $\Lambda, X(G), \Omega$

"Highest Weight Theorem": $\mathrm{Rep}(G) \cong \mathbb{Z}[X(G)]^W$

basis of irr. Chars $\longleftrightarrow$ "Dominant weights" $X_+ = X/W$

Example: A_2



$\bullet$ = irreps of $PSU(3)$
$\bullet, \circ$ = irreps of $SU(3)$

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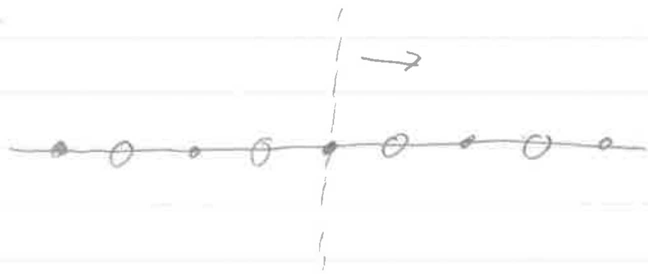
Example: A1



$\bullet$ = irreps of $SO(3)$
$\bullet$, O = irreps of $SU(2)$

$$\begin{tikzcd} SU(2) \arrow[r, "\pi"] \arrow[rr, bend right, "\tilde{\rho}"'] & SO(3) \arrow[r, "\rho"] & GL(V) \end{tikzcd}$$

Claim: ρ irrep $\iff$ $\tilde{\rho}$ irrep

Pf: Suppose $U \subseteq V$ is invariant under $SU(2)$: $\tilde{\rho}(\tilde{g})U \subseteq U$ $\forall\, \tilde{g} \in SU(2)$

WTS $\rho(g)U \subseteq U$ $\forall\, g \in SO(3)$

Consider any $g \in SO(3)$. By surjectivity, we have $g = \pi(\tilde{g})$
Hence, $\boxed{\rho(g) = \rho(\pi(\tilde{g})) = \tilde{\rho}(\tilde{g})}$

$\square$

Irreps of $SU(2)$: $SU(2) \curvearrowright \mathrm{Sym}^n \mathbb{C}^2 = \mathbb{C}\{X^n, X^{n-1}Y, X^{n-2}Y^2, \dots, Y^n\}$

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C} \right\}$$

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y \\ -\bar{\beta} x + \bar{\alpha} y \end{pmatrix} \Rightarrow X^{n-k}Y^k \longmapsto (\alpha x + \beta y)^{n-k}(-\bar{\beta} x + \bar{\alpha} y)^k$$

Claim that $SU(2) \curvearrowright Sym^n \mathbb{C}^2$ is Irreducible

To see this, we compute the Character

$$T = diag(e^{i\theta}, e^{-i\theta}) \leq SU(2)$$
$$t_\theta$$

$$t_\theta(x^{n-k}y^k) = (e^{i\theta}x)^{n-k}(e^{-i\theta}y)^k = e^{i(n-2k)\theta}(x^{n-k}y^k)$$

$$t_\theta = diag(e^{in\theta}, e^{i(n-2)\theta}, \dots, e^{-in\theta})$$

$$\chi_n(\theta) = tr(t_\theta) = e^{in\theta} + e^{i(n-2)\theta} + \dots + e^{i(-n)\theta}$$
$$= e^{-in\theta}(1 + e^{i2\theta} + e^{i4\theta} + \dots + e^{i2n\theta})$$
$$= e^{-in\theta}\frac{1 - e^{i2\theta(n+1)}}{1 - e^{i2\theta}} = \frac{e^{-in\theta} - e^{i(n+2)\theta}}{1 - e^{i2\theta}} \cdot \frac{e^{-in\theta}}{e^{-in\theta}}$$
$$= \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}}$$

$$\chi_n(\theta) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

To show that $SU(2) \curvearrowright Sym^n \mathbb{C}^2$ is Irreducible, we will show

$$\int_{SU(2)} |\chi_n|^2 = 1$$

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$\{x_i - y_j \mid i \leq j,\ 1 \leq i \leq n-2,\ 2 \leq j \leq n-1\}$

Weyl Integral Formula:

$$\int_{SU(2)} |\chi_n(g)|^2 dg = \frac{1}{2}\int_{\Theta} |\chi_n(\theta)|^2 \prod_{\alpha \in \Phi} (1-e^{\alpha}) \frac{d\theta}{2\pi} \qquad (\star)$$

$$(1-e^{i2\theta})(1-e^{-i2\theta})$$

$$= (1-e^{i2\theta})e^{-i\theta}e^{i\theta}(1-e^{-i2\theta})$$

$$= (e^{-i\theta} - e^{i\theta})(e^{i\theta} - e^{-i\theta})$$

$$= |e^{i\theta} - e^{-i\theta}|^2 = 4\sin^2\theta$$

$$\star = \frac{1}{2}\int \frac{\sin^2((n+1)\theta)}{\sin^2\theta} \cdot 4\sin^2\theta \frac{d\theta}{2\pi} = \frac{1}{\pi}\int_0^{2\pi} \sin^2((n+1)\theta)\, d\theta = 1$$

4/25/18

The irreducible Chars of $SU(2)$ are $\chi_n(t_\theta) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$

$$\dim = \chi_n(I) = \lim_{\theta \to 0} \frac{\sin((n+1)\theta)}{\sin(\theta)} = n+1$$

Proof that χ_n are irreducible

$$\int_T \frac{\sin((n+1)\theta)}{\pi} d\theta = 1$$

$$\frac{\sin((n+1)\theta)}{\sin(\theta)} = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{i(-n)\theta}$$

In general, let $\lambda \in X_+$. Then $\chi_\lambda(\vec{\theta}) = \dfrac{\sum (-1)^{\ell(w)} e^{\pi i (w(\lambda+\rho))\vec{\theta}}}{\sum (-1)^{\ell(w)} e^{\pi i (w(\rho))\vec{\theta}}}$

$x_1 - y_2$ $\qquad a_1 = x_1$

$x_1 - y_3$ $\qquad a_2 = x_1$

$x_2 - y_2$ $\qquad b_1 = y_2$

$x_2 - y_3$ $\qquad b_2 = x_2 - y_3 \longleftrightarrow y_3' = b_2 - a_2$

Where ρ = Sum of fund. weights $= \frac{1}{2}$ Sum of positive roots

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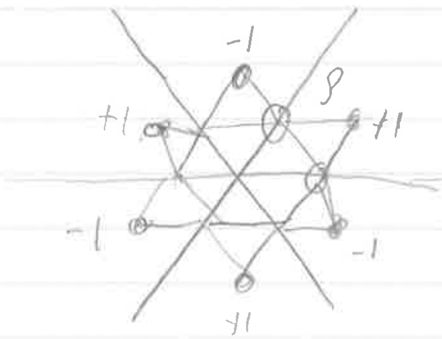
ex: $SU(3)$



$$A_\mu = \sum (-1)^{\ell(w)} e(w(\mu)), \text{ where } e^{\pi i \alpha} = e(\alpha)$$

Claim: $A_\rho = \prod_{\alpha \in \Phi^+} \left(e\left(\frac{\alpha}{2}\right) - e\left(-\frac{\alpha}{2}\right) \right)$

Define $$WCF_\lambda = \frac{\sum_w (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})}$$

$$= \left(\sum_W (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho} \right) \left(\sum_{\mu \in Q^+} \text{kpf}(\mu) e^{-\mu} \right)$$

~~Pf:~~ Lemma: • WCF_λ is W-invariant ✓

• Highest weight λ

• $WCF_0 = 1$

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$$1 = WCF_\lambda(0) = \frac{\sum (-1)^{\ell(w)} e^{w(\rho)-\rho}}{e^\rho \prod (1-e^{-\alpha})}$$

$$\Rightarrow \sum (-1)^{\ell(w)} e^{w(\rho)} = e^\rho \prod (1-e^{-\alpha})$$

$$WCF_\lambda = \frac{A_{\lambda+\rho}}{A_\rho}$$

Theorem: $\{WCF_\lambda : \lambda \in X_+\}$ is the set of irred. chars.

Hardest part $\int_K \overline{WCF_\lambda}\, WCF_\mu = \delta_{\lambda\mu}$

$$\int_K \overline{WCF_\lambda}\, WCF_\mu = \frac{1}{|W|}\int_T \overline{WCF_\lambda}\, WCF_\mu \prod_{\alpha\in\Phi}(1-e^\alpha)$$

$$= \frac{1}{|W|}\int_T \left(\sum (-1)^{\ell(w)} \overline{e^{w(\lambda+\rho)-\rho}}\right)\left(\sum (-1)^{\ell(w)} e^{w(\mu+\rho)-\rho}\right)$$

$$= \frac{1}{|W|}\sum_{w,w'} (-1)^{\ell(w)+\ell(w')} \int_T \overline{e^{w(\lambda+\rho)-\rho}}\, e^{w'(\mu+\rho)-\rho}$$

$$(\ast) \qquad = \frac{1}{|W|}\sum_{w,w'} (-1)^{\ell(w)+\ell(w')}\, \delta_{w(\lambda+\rho)-\rho,\, w'(\mu+\rho)-\rho}$$

When does $w(\lambda+\rho) = w'(\mu+\rho)$? Need $w = w'$

Then $\delta_{w(\lambda+\rho),\, w(\mu+\rho)} = \delta_{\lambda,\mu}$, so

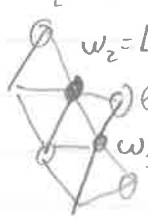
$$\bigstar = \frac{1}{|W|}\sum_w \delta_{\lambda\mu} = \delta_{\lambda,\mu}.$$

$\square$

4/27/18 $$WCF_\lambda = \frac{\sum (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod(1-e^{-\alpha})} = \frac{\sum (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{e^{\rho}\prod(1-e^{-\alpha})}$$

$$= \frac{\sum (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum (-1)^{\ell(w)} e^{w(\rho)}} = \frac{A_{\lambda+\rho}}{A_\rho}$$

SU(3):



, where $L_i = e_i - \frac{1}{n}(e_1 + \dots + e_n)$

$L_1 + L_2 + L_3 = 0$

$\alpha_i = e_i - e_{i+1}$, $\omega_i = L_1 + L_2 + \dots + L_i$

<u>Recall</u>: $(\omega_i, \alpha_j) = \delta_{i,j}$

$$\rho = \sum_{i=1}^{n-1} w_i = (n-1)L_1 + (n-2)L_2 + \dots + 1L_{n-1}$$

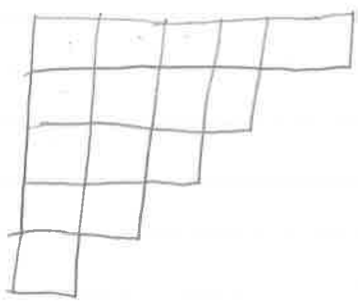
$$= \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha_i = \frac{1}{2}\sum_{1\le i<j\le n} (L_i - L_j)$$

Use the notation

$$e^{L_i} = x_i$$

$$\lambda = \sum \lambda_i L_i \qquad \begin{array}{c|cccccc} & \lambda_1' & \lambda_2' & \lambda_3' & \lambda_4' & \lambda_5' & \lambda_6' \\ \hline \lambda_1 & L_1 & L_1 & L_1 & L_1 & L_1 & L_1 \\ \lambda_2 & L_2 & L_2 & L_2 & L_2 & & \\ \lambda_3 & L_3 & L_3 & L_3 & & & \end{array}$$

$$= \sum \lambda_i' w_i$$

$\rho =$ 

Weyl group $\mathfrak{S}_n$ acts by permuting L_i (and hence x_i)

$$A_\mu = \sum (-1)^{\ell(w)} e^{w(\mu)}$$

$$= \sum (-1)^{\ell(w)} e^{w(\sum \mu_i L_i)} = \sum \operatorname{sgn}(w) \prod \left(e^{L_{w(i)}}\right)^{\mu_i}$$

$$= \sum \operatorname{sgn}(w)\, x_{w(1)}^{\mu_1} x_{w(2)}^{\mu_2} \cdots x_{w(n)}^{\mu_n}$$

$$= \det \begin{pmatrix} x_1^{\mu_1} & x_1^{\mu_2} & \cdots & x_1^{\mu_n} \\ x_2^{\mu_1} & x_2^{\mu_2} & \cdots & x_2^{\mu_n} \\ \vdots & \vdots & & \vdots \\ x_n^{\mu_1} & x_n^{\mu_2} & \cdots & x_n^{\mu_n} \end{pmatrix}$$

$$A_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^1 & x_1^0 \\ \vdots & \vdots & & & \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n^1 & x_n^0 \end{pmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j) = \prod_{\alpha \in \Phi^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right)$$

Example: $SU(2)$ $\quad L_1+L_2=0,\ x_1x_2=1$

$\rho = 1L_1 + 0L_2$

$\lambda = nL_1 + 0L_2$

$(x_1x_2=1)$

$$WCF_\lambda = \frac{\det\begin{pmatrix} x_1^{n+1} & x_1^0 \\ x_2^{n+1} & x_2^0\end{pmatrix}}{\det\begin{pmatrix} x_1^1 & x_1^0 \\ x_2^1 & x_2^0\end{pmatrix}} = \frac{x_1^{n+1}-x_2^{n+1}}{x_1-x_2} = \frac{x_1^{n+1}-x_1^{-(n+1)}}{x_1-x_1^{-1}}$$

(Set $x_1=e^{i\theta}$)

$SU(3)$: $\quad L_1+L_2+L_3=0,\ x_1x_2x_3=1$

$\rho = 2L_1 + 1L_2 + 0L_3$

$\lambda = (a+b)L_1 + bL_2 + 0L_3$

$\quad aw_1 + bw_2$

$$WCF_\lambda = \frac{\det\begin{pmatrix} x_1^{a+b+2} & x_1^{b+1} & 1 \\ x_2^{a+b+2} & x_2^{b+1} & 1 \\ x_3^{a+b+2} & x_3^{b+1} & 1\end{pmatrix}}{\det\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1\end{pmatrix}}$$

$\lambda = w_1, \quad a=1, b=0$

$WCF_\lambda = x_1+x_2+x_3$

"Quarks"

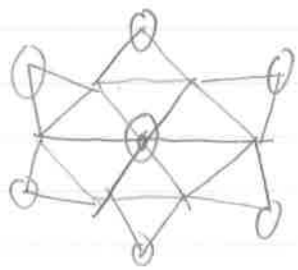
$\lambda = w_2, \quad a=0 \quad b=1$

$WCF_\lambda = x_1x_2 + x_2x_3 + x_1x_3$

$= \frac{1}{x_3} + \frac{1}{x_1} + \frac{1}{x_2}$ "Antiquarks"

$X_1 X_2 X_3 = 1 = WCF_0$

$WCF_{\omega_1} \cdot WCF_{\omega_2} = X_1^2 x_2 + x_1 x_2^2 + X_2^2 X_3 + x_2 x_3^2 + x_1^2 x_3 + X_1 X_3^2 + 3 X_1 X_2 X_3$



$WCF_{\vartheta} = \left(\sum_{i \neq j} x_i x_j^2 \right) + 2$

$WCF_{\omega_1} \cdot WCF_{\omega_2} = WCF_{\vartheta} + WCF_0$

"$3 \cdot \bar{3} = 8 + 1$"