MTH 787: Topics in Algebra **Spring 2018**
Lie Theory **Drew Armstrong**

Warning: Since I am learning the material as I go there will be some inevitable lies and mistakes. Hopefully I can come back and polish them later. As always, I received help from generous people on Math StackExchange and Google Plus, especially John B and Allen K. The students in the class[1] have also been helpful.

Contents

Local Classification

Certain problems in mathematics can be completely solved, and this solution becomes a language on which new math is built.

Example. The following problems are completely solved:

- Classify real compact manifolds that are also groups.
- Classify complex affine varieties that are also groups.

These two problems have essentially the same solution, which can described in terms of "root systems."

Definition. Let $V = (\mathbb{R}^n, (-,-))$ be a Euclidean space. A *root system* is a finite set of vectors $\Phi \subseteq V$ satisfying the following four axioms.

(R1) The "roots" are a spanning set:

$$\mathbb{R}\Phi = V.$$

[1] August Hozie, Alex Lazar, Eric Ling, James McKeown, Michael Weiss

(R2) Each root occurs with its negative, and no other multiples:

$$\forall \alpha \in \Phi, \Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}.$$

(R3) For all $\alpha \in V$, let $t_\alpha : V \to V$ be the orthogonal reflection in the hyperplane $\alpha^\perp$. That is, for all $\beta \in V$ we define

$$t_\alpha(\beta) = \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha.$$

Given this defintion, a root system must be closed under reflections:

$$\forall \alpha, \beta \in \Phi, t_\alpha(\beta) \in \Phi.$$

(R4) Furthermore, there is a discreteness ("crystallographic") condition:

$$\forall \alpha, \beta \in \Phi, 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

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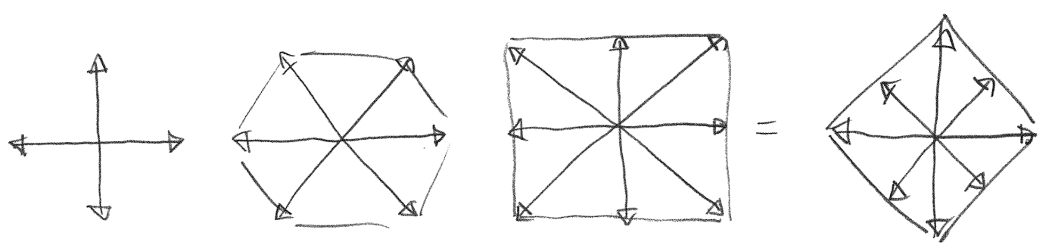
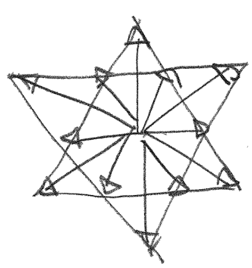
Example. Up to scaling and rotation, there are only four root systems in the plane:



$A_1 \times A_1$ A_2 B_2 C_2



G_2

Definition. We say that the root system (Φ, V) is *reducible* if there exists a non-trivial partition $\Phi = \Phi_1 \sqcup \Phi_2$ such that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. In this case we will write

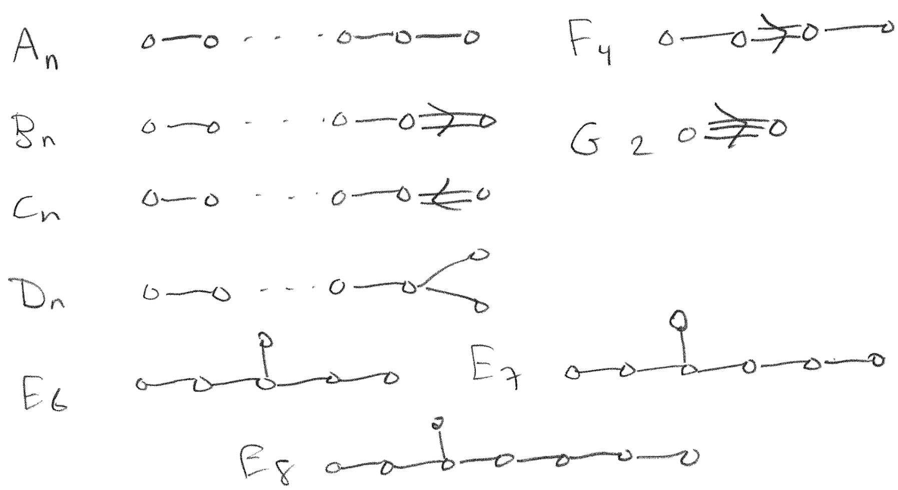
$$(\Phi, V) = (\Phi_1, V_1) \times (\Phi_2, V_2),$$

where $V_1 = \mathbb{R}\Phi_1$ and $V_2 = \mathbb{R}\Phi_2$. For example, see the root system $A_1 \times A_1$ above. ///

The definition of a root systems is not obvious. There are two motivations:

- Many structures in mathematics are surprisingly equivalent to root systems.
- Root systems have a complete classification.

This classification of irreducible roots systems is described by *Dynkin diagrams*:



I'll prove this classification later. For now let me just mention that it explains the accidental coincidence between types B_2 and C_2:

Root systems were invented around 1890 by Wilhelm Killing and Élie Cartan. They succeeded where Sophus Lie had failed to classify the simplest kinds of complex manifolds that are also groups. Types A, B, C, D correspond to obvious kinds of matrix groups. Types E, F, G were discovered by accident as a by-product of the classification. They are related to lots of exceptional structures like the "Monster group" and are still not completely understood.

Here are the classifications of simple compact real manifold groups and simple complex[2] affine

[2]This combination of "simple" and "complex" is just one of the notational joys of the subject.

variety groups:

type	simple compact real	simple complex affine
A_n	$\mathrm{PSU}(n+1)$	$\mathrm{PSL}_{n+1}(\mathbb{C})$
B_n	$\mathrm{SO}(2n+1)$	$\mathrm{SO}_{n+1}(\mathbb{C})$
C_n	$\mathrm{PSp}(n)$	$\mathrm{PSp}_n(\mathbb{C})$
D_n	$\mathrm{PSO}(2n)$	$\mathrm{PSO}_{2n}(\mathbb{C})$

Remarks.

- Types A, B, C, D are called the *classical types*. Hermann Weyl wrote a famous book about them.
- The classical types A, B, C, D exist because of the associative real division algebras $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions). The exceptional types E, F, G exist because of the non-associative real division algebra $\mathbb{O}$ (octonions).
- Yes, the unitary group $\mathrm{U}(n)$ is defined in terms of the complex numbers. However, in the classification above we think of it as a **real manifold** instead of a **complex manifold**.

Global Classification

Actually, the table above is only a local classification. On top of this there is a finite amount of global/topological information that we can add.

Definition. Let $\Phi \subseteq V \cong \mathbb{R}^n$ be a root system. The "crystallographic" condition $2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$ guarantees that the integer span of Φ is a discrete subgroup, called the *root lattice*:

$$\Lambda := \mathbb{Z}\Phi \subseteq V.$$

Since $\mathbb{R}\Phi = V \cong \mathbb{R}^n$ we see that $\Lambda \cong \mathbb{Z}^n$. The root lattice is contained in the larger *weight lattice*:

$$\Omega := \{\omega \in V : (\omega, \alpha) = 0 \text{ for all } \alpha \in \Phi\},$$

which is also isomorphic to $\mathbb{Z}^n$, therefore the quotient Ω/Λ is a finite abelian group. ///

I claim that this finite abelian group controls of the topology of real compact groups.

Theorem (Galois Correspondence). Given a root system Φ, there is a unique real compact simple group G (called the *adjoint form*), with fundamental group

$$\pi_1(G) \cong \Omega/\Lambda.$$

There is also a unique simply connected real compact group $\tilde{G}$, which is the universal cover of G. The simply connected form $\tilde{G}$ is **not** simple, but all of its normal subgroups are contained in the center

$$Z(\tilde{G}) \cong \Omega/\Lambda,$$

and we have $G = \tilde{G}/Z(\tilde{G})$. Furthermore, the identification

$$\Gamma \subseteq \Omega/\Lambda \quad \longleftrightarrow \quad \tilde{G}/\Gamma$$

gives us a bijection

$$\{\text{ subgroups of } \Omega/\Lambda \text{ }\} \quad \longleftrightarrow \quad \left\{ \text{compact groups locally isomorphic to } \tilde{G} \right\}.$$

///

Proof: Maybe later. For now, some examples.

Example (Type A_1). Up to scaling, there is a unique 1-dimensional root system called A_1:

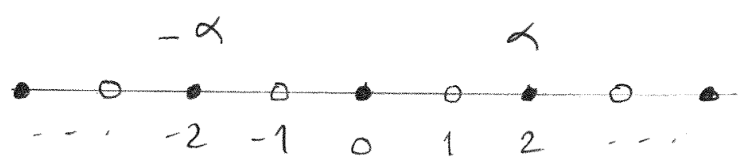
$$\Phi_{A_1} = \{\pm\alpha\}.$$

Let us assume that $\alpha = 2$ so that $(\alpha, \alpha) = |\alpha|^2 = 4$. In the following diagram, the root lattice (black nodes) is defined by

$$\Lambda = \{k\alpha : k \in \mathbb{Z}\} = 2\mathbb{Z},$$

and the weight lattice (black and white nodes) is defined by

$$\begin{aligned}\Omega &= \{\omega \in \mathbb{R} : 2(\alpha, \omega)/(\alpha, \alpha) \in \mathbb{Z}\} \\ &= \{\omega \in \mathbb{R} : \omega \in \mathbb{Z}\} = \mathbb{Z}.\end{aligned}$$

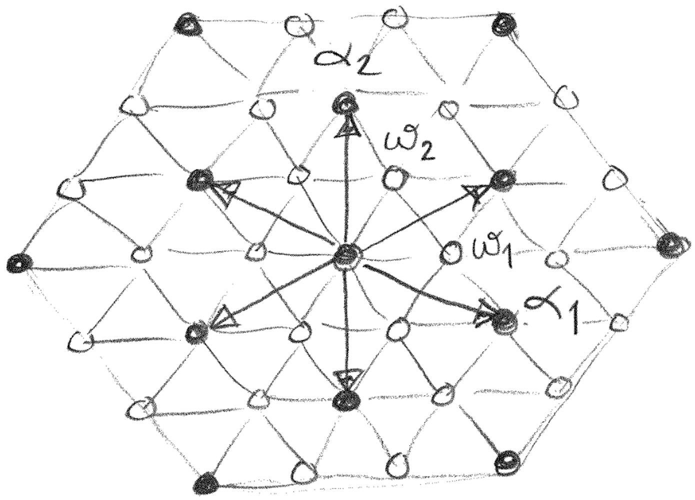


Let me briefly describe the corresponding groups. (We'll fill in the details soon.) The compact **simply-connected** form is $\mathrm{SU}(2) \subseteq \mathrm{Mat}_2(\mathbb{C})$, the group of 2×2 unitary matrices. We view this as a **real** submanifold of $\mathrm{Mat}_2(\mathbb{C}) = \mathbb{C}^4 = \mathbb{R}^8$. In fact, it is homeomorphic to a three-sphere S^3. The center is $Z(\mathrm{SU}(2)) = \{\pm I\} \cong \mathbb{Z}/2$, which agrees with the previous theorem because

$$\Omega/\Lambda = \mathbb{Z}/(2\mathbb{Z}) \cong \mathbb{Z}/2.$$

Since $\{\pm I\}$ are antipodal points of the three-sphere we see that the compact **simple** form $\mathrm{PSU}(2) = \mathrm{SU}(2)/Z(\mathrm{SU}(2))$ is homeomorphic to real projective three-space $\mathbb{R}P^3$. Because of the accidental isomorphism $A_1 \cong B_1$ (the Dynkin diagram of each is a single dot) it turns out that $\mathrm{PSU}(2)$ is also isomorphic to the group $\mathrm{SO}(3)$ of rotations of $\mathbb{R}^3$. The representation theory of these groups is very important to quantum chemistry.

Example (Type A_2). Up to scaling and rotation, the type A_2 root lattice Λ (black nodes) and weight lattice Ω (black and white notes) look like this:



In coordinates, we can let $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$, where e_1, e_2, e_3 is an orthonormal basis for $\mathbb{R}^3$. Then the root system

$$\Phi_{A_2} = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\} = \{e_i - e_j : i, j \in \{1, 2, 3\}, i \neq j\}$$

lives in the two-dimensional hyperplane

$$\mathbb{R}\Phi_{A_2} = V = \{x_1e_1 + x_2e_2 + x_3e_3 : x_1 + x_2 + x_3 = 0\} = (e_1 + e_2 + e_3)^{\perp}.$$

The root lattice is

$$\Lambda = \mathbb{Z}\Phi_{A_2} = \{k_1\alpha_1 + k_2\alpha_2 : k_1, k_2 \in \mathbb{Z}\} \cong \mathbb{Z}^2$$

and, since $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$, the weight lattice is defined by

$$\Omega = \{\omega \in V : 2(\omega, \alpha_i)/(\alpha_i, \alpha_i) = (\omega, \alpha_i) \in \mathbb{Z}, i = 1, 2\}.$$

In order to compute Ω, let A be the matrix whose columns are the root basis α_1, α_2 and define the *Cartan matrix* $C = A^TA$, which records the angles between basis vectors:

$$C = A^TA = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

A basic fact from linear algebra says that $\mathrm{rank}(C) = \mathrm{rank}(A)$.

Proof: We will show that $\ker(A^T A) = \ker(A)$ and then use rank-nullity. On the one hand, if $Ax = 0$ then we have $(A^T A)x = A^T(Ax) = 0$. On the other hand if $(A^T A)x = 0$ then we have

$$\|Ax\|^2 = (Ax)^T(Ax) = x^T(A^T A)x = x^T 0 = 0,$$

which implies that $Ax = 0$. $\square$

It follows that the Cartan matrix is invertible; in our case

$$C^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Now let $\omega_1, \omega_2 \in V$ be the vectors defined by the inverse of the Cartan matrix:

$$\omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 \quad \text{and} \quad \omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2.$$

I claim that $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

Proof: By definition we have $\omega_i = \sum_j d_{ij}\alpha_j$ where $C^{-1} = D = (d_{ij})$. It follows that

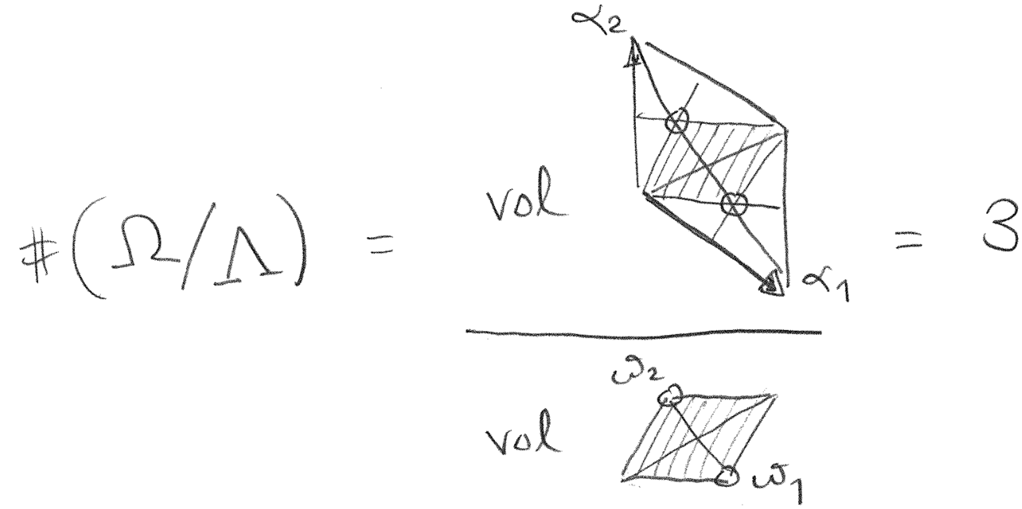
$$(\omega_i, \alpha_k) = \sum_j d_{ij}(\alpha_j, \alpha_k) = \sum_j d_{ij}c_{jk} = (DC)_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}.$$

In any case, we have $(\omega_i, \alpha_k) \in \mathbb{Z}$, which implies that $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subseteq \Omega$. To show the other inclusion, we first note that ω_1, ω_2, being the rows/columns of an invertible matrix, is a real basis for V. Then for any vector $x_1\omega_1 + x_2\omega_2 \in \mathbb{R}\omega_1 + \mathbb{R}\omega_2 = V$ we have

$$\begin{aligned} x_1 &= x_1(\omega_1, \alpha_1) + x_2(\omega_2, \alpha_1) = (x_1\omega_1 + x_2\omega_2, \alpha_1) \in \mathbb{Z}, \\ x_2 &= x_1(\omega_1, \alpha_2) + x_2(\omega_2, \alpha_2) = (x_1\omega_1 + x_2\omega_2, \alpha_2) \in \mathbb{Z}, \end{aligned}$$

which implies that $\Omega \subseteq \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. $\square$

Finally, let us compute the "fundamental group" Ω/Λ. A basic property of lattices (see Michael Artin's *Algebra, 2nd Edition*, section 13.10) tells us that the size of the group Ω/Λ is equal to the volume of V/Λ divided by the volume of V/Ω. In our case:



Algebraically, this is just the determinant of the Cartan matrix:

$$\#(\Omega/\Lambda) = \det C = \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3.$$

Since there is a unique (abelian) group of order 3 we conclude that

$$\Omega/\Lambda \cong \mathbb{Z}/3.$$

What about the groups? The compact **simply-connected** model is the group $\mathrm{SU}(3) \subseteq \mathrm{Mat}_3(\mathbb{C})$ of 3×3 unitary matrices. If $V = \mathbb{R}\Phi_{A_2}$ then it turns out we have a group homomorphism $\exp : (V, +) \to T \subseteq \mathrm{SU}(3)$ onto a maximal abelian subgroup ("maximal torus"):

$$T = \left\{ \begin{pmatrix} e^{2\pi i x_1} & & \\ & e^{2\pi i x_2} & \\ & & e^{2\pi i x_3} \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R}, x_1 + x_2 + x_3 = 0 \right\}.$$

The kernel of this map is the root lattice: $\ker(\exp) = \Lambda$. The center of $\mathrm{SU}(3)$ is the third roots of unity $Z(\mathrm{SU}(3)) = \{e^{2\pi i k/3} I : k = 0, 1, 2\} \cong \mathbb{Z}/3$, which agrees with our computation $\Omega/\Lambda \cong \mathbb{Z}/3$. In fact, the isomorphism

$$\overline{\exp} : V/\Lambda \xrightarrow{\sim} T \subseteq \mathrm{SU}(3)$$

identifies the subgroup $\Omega/\Lambda \subseteq V/\Lambda$ with the center of $\mathrm{SU}(3)$.

The compact **simple** form is $\mathrm{PSU}(3) = \mathrm{SU}(3)/Z(\mathrm{SU}(3))$, which has fundamental group

$$\pi_1(\mathrm{PSU}(3)) \cong Z(\mathrm{SU}(3)) \cong \Omega/\Lambda \cong \mathbb{Z}/3.$$

We'll discuss this more later.

Compact Subgroups

In the last sections we approached compact Lie groups from the point of view of root systems but we didn't prove anything yet. First of all, why do we restrict our attention to **compact** groups?

Theorem (Cartan-Iwasawa-Malcev). Let G be a connected Lie group.

- There exists a maximal compact subgroup $K \subseteq G$.
- Every compact subgroup of G is conjugate to a subgroup of K.
- The quotient space G/K is homeomorphic to $\mathbb{R}^m$ for some m.
- Thus G is homeomorphic to $K \times \mathbb{R}^m$.
- In particular, we see that K is connected.

In the classical types A, B, C, D these theorems can be proved very directly by means of *Gram Schmidt orthogonalization* over the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$. The following classification result then tells us why no other infinite families of Lie groups are possible.

Frobenius Theorem (1877). Up to isomorphism, there are only three finite-dimensional associative real division algebras: $\mathbb{R} = \mathbb{R}^1$, $\mathbb{C} = \mathbb{R}^2$ and $\mathbb{H} = \mathbb{R}^4$. ///

The real and complex numbers need no introduction. The *quaternions* are defined via generators and relations:

$$\mathbb{H} = \{a + ib + jc + kd : a, b, c, d \in \mathbb{R}\}/(i^2 = j^2 = k^2 = ijk = -1).$$

If $q = a + ib + jc + kd \in \mathbb{H}$ then we define the *quaternion conjugate*

$$q^* := a - ib - jc - kd$$

and we observe that $q^*q = (a^2 + b^2 + c^2 + d^2) + 0i + 0j + 0k = |q|^2 \in \mathbb{R}$. This allows us to define *quaternionic space* $\mathbb{H}^n$ with a positive-definite sesquilinear form:

$$(\alpha, \beta) := \alpha^*\beta := \sum_{i=1}^{n} \alpha_i^*\beta_i \in \mathbb{H} \qquad \text{for all column vectors } \alpha, \beta \in \mathbb{H}^n.$$

Quaternion multiplication is not commutative (example: $ij = k \neq -k = ji$), however it is still associative. This means that we can still represent linear maps $\mathbb{H}^n \to \mathbb{H}^n$ by $n \times n$ matrices $\mathrm{GL}_n(\mathbb{H})$ as long as we're a bit careful with definitions.

If we relax the requirement of associativity, then there is one more "normed division algebra" called the *octonions* $\mathbb{O} = \mathbb{R}^8$. Since it is not associative we cannot represent it by matrices and therefore we will not obtain any more infinite families of Lie groups. However if one is very clever one can still squeeze a few Lie groups out of it: F_4, G_2, E_6, E_7, E_8. Google *Freudenthal's magic square.*

Theorem (Existence for Classical Types). Let $G = \mathrm{GL}_n(F)$ where $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and for all $g \in G$ let g^* denote the conjugate transpose matrix (using complex or quaternionic conjugation as necessary). Consider the "orthogonal subgroup"

$$K = \mathrm{O}_n(F) := \{g \in G : g^* g = I\}.$$

I claim that $K \subseteq G$ is a maximal compact subgroup. More generally, if $H \subseteq G$ is any compact subgroup I claim that there exists a group element $g \in G$ such that $gHg^{-1} \subseteq K$. It follows from this that every maximal subgroup of G is isomorphic to K. ///

Remark: We will use the special notations

$$\mathrm{O}_n(F) = \begin{cases} \mathrm{O}(n) & F = \mathbb{R} \\ \mathrm{U}(n) & F = \mathbb{C} \\ \mathrm{Sp}(n) & F = \mathbb{H}. \end{cases}$$

Proof (Haar Measure). Topologically we will view $G = \mathrm{GL}_n(F)$ as an open dense subset of the Euclidean space

$$\mathrm{Mat}_n(F) = \mathbb{R}^{n^2 \dim_{\mathbb{R}}(F)} = \begin{cases} \mathbb{R}^{n^2} & F = \mathbb{R} \\ \mathbb{R}^{2n^2} & F = \mathbb{C} \\ \mathbb{R}^{4n^2} & F = \mathbb{H}. \end{cases}$$

Thus in order to prove that K is compact it suffices (by Heine-Borel) to show that it is closed and bounded. To prove boundedness, we note that the length of a matrix $g \in G$ is the sum of the squares of its real components. If $F \in \{\mathbb{C}, \mathbb{H}\}$ this can be simplified by writing

$$|g|^2 = \sum_{i,j} |g_{ij}|^2,$$

where $|g_{ij}|$ is the complex or quaternionic absolute value of the i, j entry $g_{ij} \in F$. If $g \in K$, i.e., if $g^* g = I$ then the jth column of g has length 1:

$$\sum_i g_{ij}^* g_{ij} = \sum_i |g_{ij}|^2 = 1.$$

Therefore we have

$$|g|^2 = \sum_j \left(\sum_i |g_{ij}|^2 \right) = \sum_j 1 = n,$$

and it follows that every element of K has length $\sqrt{n}$. To prove closedness, we note that the the polynomial equations $g^* g = I$ are continuous, hence preserved under limits.

Thus $K \subseteq G$ is a compact subgroup. Now let $H \subseteq G$ be any other compact subgroup. Recall that the elements of K are characterized by preserving the standard Hermitian form. Indeed, for $k \in K$ and all $x, y \in F^n$ we have

$$(kx, ky) = (kx)^*(ky) = x^*(k^*k)y = x^*Iy = x^*y = (x, y).$$

Conversely, if k is any matrix satisfying these equations then by substituting $x = e_i$ and $y = e_j$ we find that the i, j entry of k is given by

$$e_i^*(k^*k)e_j = e_i^*e_j = \delta_{ij},$$

and hence $k^*k = I$. If H is not contained in K then there must exist some element $h \in H$ and some vectors $x, y \in F^n$ such that

$$(hx, hy) \neq (x, y).$$

In order to fix this we will define a new Hermitian inner product by "averaging" over the group. Since H is compact there exists a unique translation invariant measure called *Haar measure* which, for any function $f : H \to \mathbb{C}$ and for any group element $h' \in H$ satisfies

$$\int_{h\in H} f(h)\, dh = \int_{h\in H} f(hh')\, dh = \int_{h\in H} f(h'h)\, dh.$$

For example, f could be the characteristic function of a subset $U \subseteq H$. Then the volumes of the blobs U and $h'U$ are the same:

So we define the new inner product as follows:

$$(x, y)' := \int_{h\in H} (hx, hy)\, dh.$$

Then by translation invariance we have for all h' that

$$(h'x, h'y)' = \int_{h\in H} (h'hx, h'hy)\, dh = \int_{h\in H} (hx, hy)\, dh = (x, y)'.$$

Finally, let $g \in G$ the change of basis matrix from the standard basis for F^n to some orthonormal basis for the new Hermitian form. It follows that $gHg^{-1} \subseteq K$, as desired.

In particular, suppose that $H \subseteq G$ is a **maximal** compact subgroup. We just showed that there exists $g \in G$ such that

$$\begin{aligned} gHg^{-1} &\subseteq K, \\ H &\subseteq g^{-1}Kg. \end{aligned}$$

Since $g^{-1}Kg \subseteq G$ is a compact subgroup this implies that $H = g^{-1}Kg$ and it follows that every maximal subgroup of G is isomorphic (in fact, conjugate) to K. $\square$

It remains to show that G/K is contractible. In general this is described by the so-called *Iwasawa decomposition* $G = KAN$. In the classical types this is just the process of Gram-Schmidt orthogonalization.

Theorem (Iwasawa Decomposition for Classical Types). Let $G = \mathrm{GL}_n(F)$ where $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and consider again the standard maximal compact subgroup

$$K = \{g \in G : g^* g = I\}.$$

Let us also consider the group of diagonal matrices with positive real entries

$$A = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} : 0 < a_i \in \mathbb{R} \text{ for all } i \right\},$$

and the group of upper-triangular matrices with ones on the diagonal

$$N = \left\{ \begin{pmatrix} 1 & & n_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : n_{ij} \in F \text{ for all } 1 \leqslant i < j \leqslant n \right\}.$$

Then I claim that the multiplication map

$$\begin{aligned} K \times A \times N &\to G \\ (k, a, n) &\mapsto kan \end{aligned}$$

is a diffeomorphism of real manifolds. ///

Corollary. To be more specific, we have the following diffeomorphisms:

$$\begin{aligned} \mathrm{GL}_n(\mathbb{R}) &= \mathrm{O}(n) \times \mathbb{R}_+^n \times \mathbb{R}^{n(n-1)/2}, && \text{(types } B \text{ and } D) \\ \mathrm{GL}_n(\mathbb{C}) &= \mathrm{U}(n) \times \mathbb{R}_+^n \times \mathbb{R}^{n(n-1)}, && \text{(type } A) \end{aligned}$$

$$\mathrm{GL}_n(\mathbb{H}) = \mathrm{Sp}(n) \times \mathbb{R}_+^n \times \mathbb{R}^{2n(n-1)}. \qquad \text{(type } C\text{)}$$

And from this we can compute the dimensions:

$$\begin{aligned}
\dim_{\mathbb{R}} \mathrm{O}(n) &= \dim_{\mathbb{R}} \mathrm{GL}_n(\mathbb{R}) - n - n(n-1)/2 = n(n-1)/2, \\
\dim_{\mathbb{R}} \mathrm{U}(n) &= \dim_{\mathbb{R}} \mathrm{GL}_n(\mathbb{C}) - n - n(n-1) = n^2, \\
\dim_{\mathbb{R}} \mathrm{Sp}(n) &= \dim_{\mathbb{R}} \mathrm{GL}_n(\mathbb{H}) - n - 2n(n-1) = n(2n+1).
\end{aligned}$$

Proof of Corollary. As real manifolds we have

$$\begin{aligned}
\mathrm{Mat}_n(F) &= \mathbb{R}^{\dim_{\mathbb{R}}(F)n^2}, \\
A &= \mathbb{R}_+^n, \\
N &= \mathbb{R}^{\dim_{\mathbb{R}}(F)n(n-1)/2}.
\end{aligned}$$

Since $\mathrm{GL}_n(F)$ is open in $\mathrm{Mat}_n(F)$ it has the same dimension. $\square$

Proof of Iwasawa (Gram-Schmidt Algorithm). From the definition of matrix multiplication we see that the map

$$\begin{aligned}
K \times A \times N &\to G \\
(k, a, n) &\mapsto kan
\end{aligned}$$

is smooth. Let us prove that it is bijective. To show injectivity it is enough to prove that $A \cap N = \{I\}$ and $K \cap AN = \{I\}$, since then for $a_1, a_2 \in A$ and $n_1, n_2 \in N$ we will have

$$a_1 n_1 = a_2 n_2 \Rightarrow a_2^{-1} a_1 = n_2 n_1^{-1} \in A \cap N \Rightarrow a_2^{-1} a_1 = n_2 n_1^{-1} = I \Rightarrow (a_1, n_1) = (a_2, n_2),$$

and for $k_1, k_2 \in K$ and $b_1, b_2 \in AN$ we will have

$$k_1 b_1 = k_2 b_2 \Rightarrow k_2^{-1} k_1 = b_2 b_1^{-1} \in K \cap AN \Rightarrow k_2^{-1} k_1 = b_2 b_1^{-1} = I \Rightarrow (k_1, b_1) = (k_2, b_2).$$

The fact $A \cap N = \{I\}$ is immediate. To see that $K \cap AN = \{I\}$ consider any matrix $g \in K \cap AN$. Since $g \in AN$ we have

$$g = \begin{pmatrix} a_1 & & n_{ij} \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

where $0 < a_1, \ldots, a_n \in \mathbb{R}$ and $n_{ij} \in F$ for all $1 \leqslant i < j \leqslant n$. And since $g \in K$ we know that $(g_i, g_j) = g_i^* g_j = \delta_{ij}$, where $g_i \in F^n$ is the ith column of g. To see that this implies $g = I$, note that

- For $1 < j$, the equality $(g_1, g_j) = 0$ implies that $n_{1j} = 0$.
- Then for $2 < j$, the equality $(g_2, g_j) = 0$ implies that $n_{2j} = 0$.

- Continuing in this way gives $n_{ij} = 0$ for all $1 \leqslant i < j \leqslant n$.
- Then for all i, the equality $(g_i, g_i) = 1$ implies $|a_i|^2 = 1$, which since a_i is real and positive implies that $a_i = 1$.

To show surjectivity of $(k, a, n) \mapsto kan$, consider an arbitrary element $g \in \mathrm{GL}_n(F)$. Since g is invertible we know that its columns g_i form a basis for F^n. We will use the *Gram-Schmidt procedure* to transform the arbitrary basis $g_i \in F^n$ into an **orthonormal** basis $k_i \in F^n$, i.e., such that $(k_i, k_j) = k_i^* k_j = \delta_{ij}$. The idea is to define k_d inductively by subtracting from g_d the projection of g_d onto the subspace orthogonal to the previous vectors $k_1, \ldots, k_{d-1}$. To be precise, we define

$$
\begin{aligned}
k_1 &:= g_1/\|g_1\|, \\
k_d &:= \frac{g_d - \sum_{i=1}^{d-1}(k_i, g_d)k_i}{\|g_d - \sum_{i=1}^{d-1}(k_i, g_d)k_i\|} \qquad \text{for } 1 < d \leqslant n.
\end{aligned}
$$

To see that this does the trick, first note that we have $(k_i, k_i) = \|k_i\|^2 = 1$ for all i by definition. Now assume for induction on d that $(k_i, k_j) = 0$ for all $1 \leqslant i < j < d$. It follows that the dth vector is perpendicular to the previous vectors $k_1, \ldots, k_{d-1}$ since for all $1 \leqslant j < d$ we have[3]

$$
\begin{aligned}
(k_j, k_d) &= \frac{1}{\|k_d\|}\left[(k_j, g_d) - \sum_{i=1}^{d-1}(k_i, g_d)(k_j, k_i)\right] \\
&= \frac{1}{\|k_d\|}\left[(k_j, g_d) - \sum_{i=1}^{d-1}(k_i, g_d)\delta_{ij}\right] \\
&= \frac{1}{\|k_d\|}\left[(k_j, g_d) - (k_j, g_d)\right] = 0,
\end{aligned}
$$

as desired. In summary we have shown that

$$
gna = \begin{pmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{pmatrix} \begin{pmatrix} 1 & & n_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = \begin{pmatrix} | & & | \\ k_1 & \cdots & k_n \\ | & & | \end{pmatrix} = k
$$

where $n_{ij} = -(k_i, g_j) \in F$ for all $1 \leqslant i < j \leqslant n$ and where $0 < a_i = 1/\|k_i\| \in \mathbb{R}$ for all i. Note that since $k_i^* k_j = (k_i, k_j) = \delta_{ij}$ we have $k^*k = I$ and hence $k \in K$. Then since A and N (being subgroups) are each closed under inversion we conclude that

$$
g = ka^{-1}n^{-1} \in KAN,
$$

as desired. We have show that the map $(k, a, n) \mapsto kan$ is smooth and bijective. Finally, since the matrices (k, a^{-1}, n^{-1}) in the previous formula have entries that are (complicated) rational expressions in the entries of g, we conclude that the inverse is also smooth. $\square$

[3]Recall that the sesquilinear form $(\alpha, \beta) = \alpha^* \beta$ is linear in the second coordiante and conjugate-linear in the first.

Note that our proofs were based very much on matrix arithmetic. Here's a sketch of how the theorem might be proved in general.

Sketch Proof of Cartan-Iwasawa-Malcev. Under nice conditions (when G is semisimple), Élie Cartan used a generalization of the polar decomposition to show that there exists a maximal compact subgroup $K \subseteq G$ and furthermore that the coset space G/K is simply connected with non-positive curvature. If $H \subseteq G$ is any compact group, note that H acts on G/K by left multiplication. Then *Cartan's fixed point theorem* shows that this action has a global fixed point gK. That is, for all $h \in H$ we have

$$\begin{aligned} h(gK) &= gK \\ (hg)K &= gK \\ (g^{-1}hg)K &= K \\ g^{-1}hg &\in K, \end{aligned}$$

and hence $g^{-1}Hg \subseteq K$. It follows that K is unique up to conjugation. Iwasawa and Malcev extended this result from semisimple groups to general connected Lie groups. □

Abelian Subgroups

Secondly, we will restrict our attention to **compact** and **abelian** subgroups, called *tori*. First let me state two general results without proof. The first result is deep.

Theorem (Chevalley's Structure Theorem). Let G be a smooth connected complex algebraic group. Then there exists a unique smooth connected normal subgroup H such that

- H is isomorphic to a group of complex matrices,
- G/H is compact and abelian.

For general reasons (see below) this implies that $G/H \cong \mathbb{C}^n/\Lambda$ where $\Lambda \subseteq \mathbb{C}^n$ is a lattice of full rank $2n$. Such groups are called *abelian varieties*. ///

The next result is not deep. We will prove it later when we discuss Lie algebras.

Theorem (Abelian Lie Groups). Let G be an n-dimensional connected abelian Lie group. Then we must have

$$G \cong \mathbb{R}^n/\Lambda$$

where $\Lambda \subseteq \mathbb{R}^n$ is a lattice. If Λ has rank k, then by choosing a basis we obtain

$$G \cong (\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-k}.$$

///

The group $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ can be thought of in several equivalent ways:

- Topologically as a circle S^1.
- As an additive group $\mathbb{R}$ modulo the subgroup $\mathbb{Z}$.
- As a multiplicative group of complex numbers $\mathrm{U}(1)$.
- As the multiplicative group of 2×2 special orthogonal matrices $\mathrm{SO}(2)$.

Definition. The direct product of circle groups is called a *torus*:

$$\mathbb{T}^k = \mathbb{T} \times \mathbb{T} \times \cdots \times \mathbb{T}.$$

By the above theorem, a torus is the same thing as a connected, compact abelian Lie group.

Because tori are abelian their representation theory and structure theory is straightforward. One of the most powerful techniques in Lie theory is to refer all information about a connected Lie group G to a fixed choice of maximal torus $T \subseteq G$.

Theorem (Maximal Tori). Let G be a connected Lie group with maximal compact subgroup $K \subseteq G$. There exists a torus $T \subseteq K$ with the following properties:

- For any $\ell \in K$ there exists $k \in K$ such that $k\ell k^{-1} \in T$.[4]
- For any torus $H \subseteq G$ there exists $g \in G$ such that $gHg^{-1} \subseteq T$.

It follows that T is a maximal torus in G and that any two maximal tori are conjugate. If $T \cong \mathbb{T}^r$ then we say that K and G are Lie groups of *rank* r, where

$$r \leqslant \dim_{\mathbb{R}} K \leqslant \dim_{\mathbb{R}} G.$$

///

Corollary. Let G be a connected Lie group with maximal compact subgroup $K \subseteq G$ and maximal torus $T \subseteq K$. Then we have the following:

- The group K is a union of maximal tori:

$$K = \bigcup_{\ell \in K} \ell^{-1} T \ell.$$

- The center of K is an intersection of maximal tori:

$$Z(K) = \bigcap_{\ell \in K} \ell^{-1} T \ell.$$

[4]This is also called the *Principal Axis Theorem*, referring to Euler's proof in the case $K = \mathrm{SO}(3)$. See below.

Proof of Corollary. From the theorem we know that gTg^{-1} is a maximal torus for all $g \in G$. For any $k \in K$ we also know that there exists $\ell \in K$ such that $\ell k \ell^{-1} \in T$ and hence $k \in \ell^{-1}T\ell$. This proves the first statement.

For the second statement, let $k \in Z(K)$. Then there exists $u \in K$ such that $k = uku^{-1} \in T$. It follows that for all $\ell \in K$ we have $k = \ell^{-1}k\ell \in \ell^{-1}T\ell$. Conversely, suppose that we have $k \in \ell^{-1}T\ell$ for all $\ell \in K$. Now consider any other element $k' \in K$ and recall that there exists $u \in K$ such that $uk'u^{-1} \in T$, hence $k' \in u^{-1}Tu$. It follows that k and k' commute because they are both members of the abelian group $u^{-1}Tu$. □

As with our discussion of compact subgroups, I will give a complete proof of a classical case and then I will sketch how one might prove the general case. This time I will postpone types B, C, D and show you the complete proof in type A. This case is about the simultaneous diagonalization of unitary matrices.

Proof of Maximal Tori in Type A (Diagonalization). Let $G = \mathrm{GL}_n(\mathbb{C})$ with maximal compact subgroup $K = U(n)$. I claim that the group of unitary diagonal matrices satisfies the conditions of the theorem:

$$T := \left\{ \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{pmatrix} : |u_i| = 1 \right\} = \left\{ \begin{pmatrix} e^{2\pi i x_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i x_n} \end{pmatrix} : x_i \in \mathbb{R} \right\}.$$

Since this $T \cong \mathbb{T}^n$ it will follow that $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{U}(n)$ are Lie groups of rank n.

To prove the first part of the theorem we need to show that every unitary matrix is "unitarily diagonalizable." That is, given any unitary matrix $g \in \mathrm{U}(n)$ we want to find a unitary matrix $u \in \mathrm{U}(n)$ such that ugu^{-1} is diagonal. Since ugu^{-1} is also unitary, it will follow that $ugu^{-1} \in T$.

We will prove this by induction. For the induction step, consider any matrix $g \in \mathrm{GL}_n(\mathbb{C})$ and any subspace $V \subseteq \mathbb{C}^n$. Let g^* be the conjugate transpose matrix defined by

$$(gx, y) = (x, g^* y) \qquad \text{for all } x, y \in \mathbb{C}^n,$$

where $(,)$ is the standard Hermitian inner product. It follows from the definitions that

$$g \text{ stabilizes } V \qquad \Longleftrightarrow \qquad g^* \text{ stabilizes } V^\perp.$$

Indeed, suppose that g stabilizes V and consider any $y \in V^\perp$. Then for all $x \in V$ we have $gx \in V$ and hence $(x, g^*y) = (gx, y) = 0$. By non-degenracy of the inner product it follows that $g^*y \in V^\perp$. Conversely, suppose that g^* stabilizes $V^\perp$ and consider any $x \in V$. Then for all $y \in V^\perp$ we have $g^*y \in V^\perp$ and hence $(gx, y) = (x, g^*y) = 0$. By non-degeracy we have $gx \in (V^\perp)^\perp$ and by finite-dimensionality we have $(V^\perp)^\perp = V$, hence $gx \in V$. Furthermore, for any matrix $h \in \mathrm{GL}_n(\mathbb{C})$ and any subspace $U \subseteq \mathbb{C}^n$ we have

$$h \text{ stabilizes } U \qquad \Longleftrightarrow \qquad h^{-1} \text{ stabilizes } U.$$

Indeed, suppose that h stabilizes U and consider the linear map $h : U \to U$. Since the kernel is trivial (h is invertible) we conclude from finite-dimensionality that $h(U) = U$. Thus for any $y \in U$ we have $y = gx$ for some $x \in U$, and it follows that $g^{-1}y = x \in U$. By putting these two results together we conclude for any **unitary matrix** $g^* = g^{-1}$ and any subspace $V \subseteq \mathbb{C}^n$ that

$$g \text{ stabilizes } V \qquad \Longleftrightarrow \qquad g \text{ stabilizes } V^\perp.$$

Now consider an arbitrary unitary matrix $g \in \mathrm{U}(n)$. Since $\mathbb{C}$ is algebraically closed there exists an eigenvector $gv = \lambda v$. We complete v to an orthonormal basis of $\mathbb{C}^n$ via Gram-Schmidt and let $u \in \mathrm{U}(n)$ be the change of basis matrix (which is unitary because the basis is orthonormal). Since g is unitary and stabilizes the line $\mathbb{C}v$ it must also stabilize the orthogonal hyperplane $(\mathbb{C}v)^\perp$. Thus we have

$$ugu^{-1} = \left(\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & g' & \\ 0 & & & \end{array}\right).$$

Since ugu^{-1} is unitary we must have $I = (ugu^{-1})^*(ugu^{-1})$, or

$$I = \left(\begin{array}{c|ccc} \lambda^* & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & (g')^* & \\ 0 & & & \end{array}\right)\left(\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & g' & \\ 0 & & & \end{array}\right) = \left(\begin{array}{c|ccc} \lambda^*\lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & (g')^*g' & \\ 0 & & & \end{array}\right),$$

from which is follows that $\lambda \in \mathrm{U}(1)$ and $g' \in \mathrm{U}(n-1)$. Now by induction the matrix g' is unitarily diagonalizable, say $w'g'(w')^{-1} = t$ for some $w' \in \mathrm{U}(n-1)$ and diagonal $t \in \mathrm{U}(n-1)$. We define the unitary matrix

$$w := \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & w' & \\ 0 & & & \end{array}\right) \in \mathrm{U}(n)$$

so that wu is also unitary. Finally, we observe that

$$(wu)g(wu)^{-1} = w(ugu^{-1})w^{-1} = \left(\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & t & \\ 0 & & & \end{array}\right)$$

is diagonal. This completes the proof of the first statement.

For the second statement, let $H \subseteq \mathrm{GL}_n(\mathbb{C})$ be any connected, compact abelian subgroup. By averaging over the group using Haar measure[5] we find a matrix $g \in \mathrm{GL}_n(\mathbb{C})$ such that $gHg^{-1} \subseteq \mathrm{U}(n)$. Now we have a group of commuting unitary matrices. From the above argument we know that each element $h \in gHg^{-1}$ can be unitarily diagonalized. Our goal is to show that all of these diagonalizations can be done **simultaneously**. So consider any two elements $h_1, h_2 \in gHg^{-1} \subseteq \mathrm{U}(n)$ and suppose that $v \in \mathbb{C}^n$ is an eigenvector of h_1, say $h_1 v = \lambda v$. Then since $h_1 h_2 = h_2 h_1$ we have

$$h_1(h_2 v) = h_2(h_1 v) = h_2(\lambda v) = \lambda(h_2 v)$$

and it follows that $h_2 v \in \mathbb{C}^n$ is an eigenvector of h_1 for the same eigenvalue. In other words, h_2 stabilizes each eigenspace of h_1. From the previous step there exists a unitary matrix $u \in \mathrm{U}(n)$ that diagonalizes h_1. Since h_2 stabilizes the eigenspaces of h_1 we obtain

$$uh_1u^{-1} = \begin{pmatrix} \boxed{\lambda_1 I} & & & 0 \\ & \boxed{\lambda_2 I} & & \\ & & \ddots & \\ 0 & & & \boxed{\lambda_d I} \end{pmatrix} \quad \text{and} \quad uh_2u^{-1} = \begin{pmatrix} \boxed{s_1} & & & 0 \\ & \boxed{s_2} & & \\ & & \ddots & \\ 0 & & & \boxed{s_d} \end{pmatrix}$$

for some matrices s_i. Since uh_2u^{-1} is unitary we see that each submatrix s_i is unitary, hence there exist unitary matrices w_i such that $w_i s_i w_i^{-1} = t_i$, with t_i diagonal. Now consider the unitary matrix

$$w := \begin{pmatrix} \boxed{w_1} & & & 0 \\ & \boxed{w_2} & & \\ & & \ddots & \\ 0 & & & \boxed{w_d} \end{pmatrix} \in \mathrm{U}(n),$$

and note that $wu \in \mathrm{U}(n)$. Finally, we see that $(wu)h_1(wu)^{-1} = w(uh_1u^{-1})w^{-1}$ is given by

$$\begin{pmatrix} \boxed{w_1(\lambda_1 I)w_1^{-1}} & & & 0 \\ & \boxed{w_2(\lambda_2 I)w_2^{-1}} & & \\ & & \ddots & \\ 0 & & & \boxed{w_d(\lambda_d I)w_d^{-1}} \end{pmatrix} = \begin{pmatrix} \boxed{\lambda_1 I} & & & 0 \\ & \boxed{\lambda_2 I} & & \\ & & \ddots & \\ 0 & & & \boxed{\lambda_d I} \end{pmatrix}$$

[5] See the proof of Iwasawa above.

and that $(wu)h_2(wu)^{-1} = w(uh_2u^{-1})w^{-1}$ is given by

$$\begin{pmatrix} \boxed{w_1s_1w_1^{-1}} & & & 0 \\ & \boxed{w_2s_2w_2^{-1}} & & \\ & & \boxed{\ddots} & \\ 0 & & & \boxed{w_ds_dw_d^{-1}} \end{pmatrix} = \begin{pmatrix} \boxed{t_1} & & & 0 \\ & \boxed{t_2} & & \\ & & \boxed{\ddots} & \\ 0 & & & \boxed{t_d} \end{pmatrix} \in T,$$

so that h_1 and h_2 are simultaneously diagonalized by the unitary matrix $wu \in \mathrm{U}(n)$.

Now by induction we can simultaneously diagonalize any countable subset of gHg^{-1}. To complete the proof it is enough to show that gHg^{-1} contains a dense countable subset, and for this purpose we can choose the matrices whose entries have real and imaginary parts in $\mathbb{Q}$. We conclude that there exists a (unitary) matrix $u \in \mathrm{GL}_n(\mathbb{C})$ such that

$$(ug)H(ug)^{-1} = u(gHg^{-1})u^{-1} \subseteq T,$$

as desired. $\square$

We will dive into the consequences of this proof in the next section. For now, here is a sketch of the general case.

Sketch Proof of Maximal Tori (General Case). Let us assume that a maximal torus $T \subseteq K \subseteq G$ exists. First we will show that any single element $\ell \in K$ can be conjugated into T. To do this we consider the action of u on the coset space K/T by left multiplication:

$$\begin{aligned} \mu_\ell : K/T &\to K/T \\ kT &\mapsto \ell kT. \end{aligned}$$

Note that this is a smooth map of real manifolds. We will be done if we can find a fixed point $\ell kT = kT$ since then it will follow that $k^{-1}\ell kT = kT$ and hence $k^{-1}\ell k \in T$. A. Weil gave a classic proof of this by using the *Lefschetz fixed point theorem*, which says the following:

> *Let X be a manifold with nonzero Euler characteristic $\chi(X) \neq 0$ and let $f : X \to X$ be a smooth map that is homotopic to the identity* $\mathrm{id} : X \to X$*. Then f has a fixed point.*

Since K is connected there erxists a path $\varphi : [0,1] \to K$ with $\varphi(0) = I$ and $\varphi(1) = \ell$. Then the composite map

$$\begin{aligned} \mu_\varphi : [0,1] \times K/T &\to K/T \\ (t, kT) &\mapsto \varphi(t)kT \end{aligned}$$

is a homotopy between $\mu_{\varphi(0)} = \mu_I = \mathrm{id}$ and $\mu_{\varphi(1)} = \mu_\ell$. To complete the proof I will just ask you to believe that $\chi(K/T) \neq 0$.[6]

[6] We will return to this argument when we discuss Weyl groups. If $N_K(T) \subseteq K$ is the normalizer of the maximal torus $T \subseteq K$ it turns out that $N_K(T)/T$ is a finite group and that $\chi(K/T) = \#N_K(T)/T$.

For the second statement, I will assume without proof that every torus has a *topological generator*. That is, if H is a torus (compact, connected, abelian group) then there exists an element $h \in H$ such that the cyclic subgroup $\langle h \rangle \subseteq H$ is dense. Now let $H \subseteq G$ be any torus. From Cartan-Iwasawa-Malcev above we know that there exists $g \in G$ such that $gHg^{-1} \subseteq K$. Since gHg^{-1} is also a torus we know that it has a topological generator $\ell \in K$. Then from the previous argument there exists $k \in K$ such that $k\ell k^{-1} \in T$. Finally, since T is a closed topological group we have

$$\begin{aligned} k\ell k^{-1} &\in T \\ \langle k\ell k^{-1} \rangle &\subseteq T \\ \overline{\langle k\ell k^{-1} \rangle} &\subseteq T \\ k\overline{\langle \ell \rangle}k^{-1} &\subseteq T \\ k\left(gHg^{-1}\right)k^{-1} &\subseteq T \\ (kg)H(kg)^{-1} &\subseteq T, \end{aligned}$$

as desired. □

Remark: As always, I am happy to accept topological theorems (such as the Leftschetz fixed point theorem and the existence of topological generators) without proof.

Type A

Now let's work out the details of these contructions in type A. All Lie groups of type A are subgroups of "her all-embracing majesty" $\mathrm{GL}_n(\mathbb{C})$. We have seen that a standard choice of maximal torus and compact subgroup is given by the unitary matrices

$$\mathrm{U}(n) = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^*g = I\}$$

and the diagonal unitary matrices

$$T = \left\{ \begin{pmatrix} e^{2\pi i x_i} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i x_n} \end{pmatrix} : x_i \in \mathbb{R} \right\}.$$

Since $T \cong \mathrm{U}(1)^n$ we conclude that $\mathrm{GL}_n(\mathbb{C})$ (of real dimension $2n^2$) and $\mathrm{U}(n)$ (of real dimension n^2) are both Lie groups of rank n. What about the special linear group?

Theorem (Special Linear Group). Recall that

$$\mathrm{SL}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) : \det(g) = 1\}.$$

I claim that a maximal compact subgroup in $\mathrm{SL}_n(\mathbb{C})$ is given by

$$\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}_n(\mathbb{C})$$

and a maximal torus in $\mathrm{SL}_n(\mathbb{C})$ is given by

$$T_0 = T \cap \mathrm{SL}_n(\mathbb{C}).$$

Proof. Let $H \subseteq \mathrm{SL}_n(\mathbb{C})$ be any compact subgroup. From previous results we know that there exists $g \in \mathrm{GL}_n(\mathbb{C})$ such that $gHg^{-1} \subseteq \mathrm{U}(n)$. In fact, we can assume that $g \in \mathrm{SL}_n(\mathbb{C})$ by replacing g with g/λ, where $\lambda \in \mathbb{C}$ is any fixed n-th root of $\det(g) \in \mathbb{C}^\times$, so that

$$\det(g/\lambda) = \det(g)/\lambda^n = \det(g)/\deg(g) = 1.$$

Since determinant is preserved under conjugation we must also have $gHg^{-1} \subseteq \mathrm{SL}_n(\mathbb{C})$ and hence $gHg^{-1} \subseteq \mathrm{SU}(n)$. It follows that $\mathrm{SU}(n)$ is maximal compact since if $H \subseteq \mathrm{SL}_n(\mathbb{C})$ is any compact subgroup properly containing $\mathrm{SU}(n)$ then there exists $g \in \mathrm{SL}_n(\mathbb{C})$ such that

$$gHg^{-1} \subseteq \mathrm{SU}(n) \subsetneq H,$$

which is a contradiction. Next, let $A \subseteq \mathrm{SL}_n(\mathbb{C})$ be any torus. By the same argument as above, there exists $g \in \mathrm{SL}_n(\mathbb{C})$ such that $gAg^{-1} \subseteq T$ and since conjugation preserves the determinant we also have $gAg^{-1} \subseteq \mathrm{SL}_n(\mathbb{C})$, hence $gAg^{-1} \subseteq T_0 = T \cap \mathrm{SL}_n(\mathbb{C})$. It follows that T_0 is a maximal torus since if T_0 is properly contained in A then we obtain the contradiction

$$gAg^{-1} \subseteq T_0 \subsetneq A.$$

$\square$

Let us examine the maximal torus $T_0 \subseteq \mathrm{SL}_n(\mathbb{C})$ more closely. The determinant of a general element of T is

$$e^{2\pi i x_1} e^{2\pi i x_2} \cdots e^{2\pi i x_n} = e^{2\pi i (x_1 + x_2 + \cdots + x_n)},$$

which equals 1 if and only if $x_1 + x_2 + \cdots + x_n$ is an integer. Therefore by definition we have

$$T_0 = T \cap \mathrm{SL}_n(\mathbb{C}) \left\{ \begin{pmatrix} e^{2\pi i} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i x_n} \end{pmatrix} : x_i \in \mathbb{R} \text{ and } x_1 + x_2 + \cdots + x_n \in \mathbb{Z} \right\}.$$

But I claim that we can write this more simply as

$$T_0 = \left\{ \begin{pmatrix} e^{2\pi i x_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i x_n} \end{pmatrix} : x_i \in \mathbb{R} \text{ for all } i \text{ and } x_1 + x_2 + \cdots + x_n = 0 \right\}.$$

Indeed, given any element of T_0 with $x_1 + x_2 + \cdots + x_n = s \in \mathbb{Z}$ we can (for example) replace x_1 by $x_1 - s$ without changing the matrix:

$$\begin{pmatrix} e^{2\pi i(x_1-s)} & & & 0 \\ & e^{2\pi i x_2} & & \\ & & \ddots & \\ 0 & & & e^{2\pi i x_n} \end{pmatrix} = \begin{pmatrix} e^{2\pi i x_1} & & & 0 \\ & e^{2\pi i x_2} & & \\ & & \ddots & \\ 0 & & & e^{2\pi i x_n} \end{pmatrix}.$$

This is a good time to introduce the exponential homomorphism.

Definition (Exponential Homomorphism and Root Lattice of Type A). We have a surjective group homomorphism $\exp : (\mathbb{R}^n, +) \to T \subseteq \mathrm{GL}_n(\mathbb{C})$ defined by

$$\exp(x_1, \ldots, x_n) = \begin{pmatrix} e^{2\pi i x_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i x_n} \end{pmatrix}$$

The kernel is $\mathbb{Z}^n \subseteq \mathbb{R}^n$ so by the First Isomorphism Theorem we have $\mathbb{R}^n/\mathbb{Z}^n \cong T$. If we define

$$\mathbb{R}^n_0 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$$

then by restricting to this subgroup we obtain a surjective group homomorphism onto the maximal torus in $\mathrm{SL}_n(\mathbb{C})$:

$$\exp : (\mathbb{R}^n_0, +) \to T_0 \subseteq \mathrm{SL}_n(\mathbb{C}).$$

Let $\Lambda \subseteq \mathbb{R}^n_0$ be the kenel, which we call the *root lattice of type A*. Then we obtain a group isomorphism $\mathbb{R}^n_0/\Lambda \cong T_0$. ///

Remark: This is a very special case of the exponential map $\exp : \mathfrak{g} \to G$ from a Lie algebra $\mathfrak{g}$ onto a neighborhood of a Lie group G. In general it will not be a group homomorphism, but will have some more complicated structure.

It seems clear that $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{SU}(n)$ are Lie groups of rank $n-1$. Now let me show you an explicit isomorphism $T_0 \cong \mathrm{U}(1)^{n-1}$. If $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$ then we define the *standard root basis* $\alpha_1, \ldots, \alpha_{n-1}$ for $\mathbb{R}^n_0$ by

$$\begin{aligned} \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= e_2 - e_3 \\ &\vdots \\ \alpha_{n-1} &= e_{n-1} - e_n. \end{aligned}$$

For all α_j let us define the subgroup

$$\mathrm{U}(1)_j = \left\{ \exp(\theta\alpha_j) = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & e^{2\pi i\theta} & & & \\ & & & e^{-2\pi i\theta} & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\} \subseteq T_0,$$

which is clearly isomorphic to $\mathrm{U}(1)$. I claim that the multipliction map

$$\mathrm{U}(1)^{n-1} \cong \mathrm{U}(1)_1 \times \cdots \times \mathrm{U}(1)_{n-1} \to T_0$$

is a group isomorphism. Indeed, the map is injective since the subgroups intersect trivially. To see that the map is surjective, note that a general element of $T_0 = \exp(\mathbb{R}_0^n)$ looks like $\exp(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}_0^n$. By expressing this in the root basis we have

$$\begin{aligned} \mathbf{x} &= \theta_1\alpha_1 + \theta_2\alpha_2 + \cdots + \theta_{n-1}\alpha_{n-1} \\ \exp(\mathbf{x}) &= \exp(\theta_1\alpha_1 + \theta_2\alpha_2 + \cdots + \theta_{n-1}\alpha_{n-1}) \\ \exp(\mathbf{x}) &= \exp(\theta_1\alpha_1)\exp(\theta_2\alpha_2)\cdots\exp(\theta_{n-1}\alpha_{n-1}), \end{aligned}$$

for some real numbers $\theta_1, \ldots, \theta_{n-1} \in \mathbb{R}$, as desired.

Apart from the groups $\mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C}), \mathrm{U}(n), \mathrm{SU}(n)$, we might also consider quotients of these. It turns out that **discrete** normal subgroups are contained in the center.

Theorem (Centrality of Discrete Normal Subgroups). Let G be a connected Lie group and let $H \subseteq G$ be a discrete normal subgroup. Then H is contained in the center: $H \subseteq Z(G)$.

Proof. Fix an element $h \in H$. Then for all $g \in G$ we have a continuous map $G \to H$ defined by $g \mapsto ghg^{-1}$. As g varies continuously the image ghg^{-1} varies continuously. However, since H is discrete the image must be constant. That is, we must have $ghg^{-1} = g'h(g')^{-1}$ for all $g' \in G$ connected to g by a continuous path. Since G is assumed to be path connected we may take $g' = I$ to obtain

$$ghg^{-1} = IhI^{-1} = h.$$

□

This is why we want to investigate the centers of the type A Lie groups. It turns out that they are just scalar matrices.

Theorem (Centers of Type A Lie Groups). We have

$$Z(\mathrm{GL}_n(\mathbb{C})) = \{\lambda I : \lambda \in \mathbb{C}^\times\},$$

$$\begin{aligned} Z(\mathrm{SL}_n(\mathbb{C})) &= \{\lambda I : \lambda^n = 1\}, \\ Z(\mathrm{U}(n)) &= \{\lambda I : |\lambda| = 1\}, \\ Z(\mathrm{SU}(n)) &= \{\lambda I : \lambda^n = 1\}. \end{aligned}$$

Proof. Let $g \in Z(\mathrm{GL}_n(\mathbb{C}))$ and let $e_{ij} \in \mathrm{Mat}_n(\mathbb{C})$ be the "matrix unit" with 1 in position i, j and zeroes elsewhere. If $i \neq j$ then we observe that $I + e_{ij}$ is an invertible matrix with

$$(I + e_{ij})^{-1} = I - e_{ij}.$$

Since g commutes with all intvertible matrices we have

$$\begin{aligned} g(I + e_{ij}) &= (I + e_{ij})g \\ g + ge_{ij} &= g + e_{ij}g \\ ge_{ij} &= e_{ij}g. \end{aligned}$$

If g_{ij} is the i, j entry of the matrix g then this last equation says explicitly that

$$\begin{array}{cc} & \begin{array}{cccccc} & & & j & & \end{array} \\ i & \begin{pmatrix} & & & g_{1i} & & \\ & & & \vdots & & \\ 0 & \cdots & \cdots & g_{ii} & \cdots & 0 \\ & & & \vdots & & \\ & & & \vdots & & \\ & & & g_{ni} & & \end{pmatrix} \end{array} = \begin{array}{cc} & \begin{array}{cccccc} \phantom{g_{j1}} & & & j & & \phantom{g_{jn}} \end{array} \\ i & \begin{pmatrix} & & & 0 & & \\ & & & \vdots & & \\ g_{j1} & \cdots & \cdots & g_{jj} & \cdots & g_{jn} \\ & & & \vdots & & \\ & & & \vdots & & \\ & & & 0 & & \end{pmatrix} \end{array}.$$

It follows that $g_{ki} = g_{jk} = 0$ for all k and that $g_{ii} = g_{jj}$. By repeating this argument for all $i \neq j$ we see that g has the form λI for some $\lambda \in \mathbb{C}$. Finally, since $\lambda^n = \det(\lambda I) = \det(g) \neq 0$ we conclude that $\lambda \neq 0$.

Next let $g \in Z(\mathrm{SL}_n(\mathbb{C}))$. If $i \neq j$ then luckily we have $\det(I + e_{ij}) = 1$, so the same argument as before shows that $g = \lambda I$ for some $\lambda \in \mathbb{C}$. Finally, we have $\lambda^n = \det(\lambda I) = \deg(g) = 1$.

Next let $g \in Z(\mathrm{U}(n))$. Sadly the matrix $I + e_{ij}$ is not unitary so we need a new trick. We know from the theorem on maximal tori that there exists $u \in \mathrm{U}(n)$ with $g = ugu^{-1} \in T$, so that g must be diagonal. Next we observe that permutation matrices are unitary. For all $i < j$ let π_{ij} be the *transposition matrix*:

$$\pi_{ij} = \begin{array}{cc} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} I & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & I & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & I \end{pmatrix} \end{array}$$

Then the equation $g = \pi_{ij} g \pi_{ij}^{-1}$ tells us that $g_{ii} = g_{jj}$. By repeating the argument for all $i < j$ we find that $g = \lambda I$ for some $\lambda \in \mathbb{C}$. Finally, since each column and row of g has length 1 we conclude that $|\lambda| = 1$.

Next let $g \in Z(\mathrm{SU}(n))$. As before, there exists $u \in \mathrm{U}(n)$ such that $ugu^{-1} \in T$. Since u is unitary we have $1 = \det(I) = \det(u^*u) = \det(u)^* \det(u) = |\det(u)|^2$, from which it follows that $|\det(u)| = 1$. Let λ be any fixed n-th root of this determinant. Then $|\lambda|^n = |\lambda^n| = |\det(u)| = 1$ implies that $\lambda^*\lambda = |\lambda|^2 = 1$. Now define the matrix $w = u/\lambda$, which satisfies

$$\det(w) = \det(u/\lambda) = \det(u)/\lambda^n = \det(u)/\det(u) = 1$$

and

$$w^*w = (u/\lambda)^*(u/\lambda) = \frac{u^*u}{\lambda^*\lambda} = \frac{I}{|\lambda|^2} = I.$$

We conclude that $w \in \mathrm{SU}(n)$ and since g is in the center of $\mathrm{SU}(n)$ we have

$$g = wgw^{-1} = (u/\lambda)g(u/\lambda)^{-1} = ugu^{-1} \in T.$$

Thus we see that g is diagonal. Now let us consider the family of 3-cycles $\pi_{ijk} := \pi_{ij}\pi_{jk}$ which are special unitary because $\pi_{ij}^{-1} = \pi_{ij}$ and because

$$\det(\pi_{ijk}) = \det(\pi_{ij})\det(\pi_{jk}) = (-1)(-1) = 1.$$

Since g commutes with all special unitary matrices we have $g = \pi_{ijk} g \pi_{ijk}^{-1}$, which implies that $g_{ii} = g_{jj} = g_{kk}$. By repeating the argument for all $i < j < k$ we find that $g = \lambda I$ for some λ. Finally, we have $\lambda^n = \det(\lambda I) = \det(g) = 1$. □

In particular, we find that the center of each group $\mathrm{SL}_n(\mathbb{C}), \mathrm{U}(n), \mathrm{SU}(n)$ lies in the standard maximal torus.[7] Thus we may consider the preimage of the center under the exponential homomorphisms:

$$\begin{aligned} \exp : \mathbb{R}^n &\twoheadrightarrow T \subseteq \mathrm{U}(n) \subseteq \mathrm{GL}_n(\mathbb{C}) \\ \exp : \mathbb{R}^n_0 &\twoheadrightarrow T_0 \subseteq \mathrm{SU}(n) \subseteq \mathrm{SL}_n(\mathbb{C}). \end{aligned}$$

The preimage of $Z(\mathrm{U}(n))) = \{\lambda I : |\lambda| = 1\}$ is only partly discrete. If $\mathbf{1} \in \mathbb{R}^n$ denotes the vector of all ones then we have $\exp(\mathbf{x}) \in Z(\mathrm{U}(n))$ if and only if $\mathbf{x} - \alpha\mathbf{1} \in \mathbb{Z}^n$ for some $\alpha \in \mathbb{R}$. We conclude that

$$\exp^{-1}(\{\lambda I : |\lambda| = 1\}) = \mathbb{Z}^n + \mathbb{R}\mathbf{1},$$

which is the same as

$$\mathbb{Z}^{n-1} \oplus \mathbb{R}\mathbf{1} = \{(x_1 + \alpha, \ldots, x_{n-1} + \alpha, \alpha) : x_i \in \mathbb{Z}, \alpha \in \mathbb{R}\}.$$

Then since $\exp : \mathbb{R}^n \to T$ is surjective, the First Isomorphism Theorem confirms that

$$Z(\mathrm{U}(n)) \cong \frac{\exp^{-1}(Z(\mathrm{U}(n)))}{\ker(\exp)} = \frac{\mathbb{Z}^{n-1} \oplus \mathbb{R}\mathbf{1}}{\mathbb{Z}^n} \cong \mathbb{R}/\mathbb{Z},$$

[7] For the compact groups $\mathrm{U}(n), \mathrm{SU}(n)$ this agrees with the general theorem that the center is the *intersection* of all maximal tori. For the noncompact group $\mathrm{SL}_n(\mathbb{C})$ this is a bit more surprising.

as expected. The case of $Z(\mathrm{SU}(n))$ is more interesting.

Theorem/Definition (Root and Weight Lattices in General). If K is a *simply connected* and *compact* Lie group then we always have a surjective exponential homomorphism onto a maximal torus:

$$\exp : \mathfrak{t} \twoheadrightarrow T \subseteq K.$$

The kernel Λ (being discrete) is called the *root lattice*. Recall that the center of K is always contained in the maximal torus: $Z \subseteq T$. We define the weight lattice Ω (also discrete) as the preimage of the center:

$$\ker(\exp) = \Lambda \subseteq \Omega = \exp^{-1}(Z) \subseteq \mathfrak{t}.$$

The root lattice and the weight lattice are both isomorphic to $\mathbb{Z}^r$ where $r = \dim_{\mathbb{R}} \mathfrak{t} = \dim_{\mathbb{R}} T$ is called the *rank* of K. Since the homomorphism $\exp : \mathfrak{t} \to T$ is surjective, it follows from the First Isomorphism Theorem that

$$\frac{\Omega}{\Lambda} = \frac{\exp^{-1}(Z)}{\ker(\exp)} \cong Z.$$

We saw above that discrete normal subgroups of K are contained in the center Z. Therefore the exponential map gives us an explicit bijection

$$\{\text{groups between } \Lambda \text{ and } \Omega\} \leftrightarrow \{\text{discrete normal subgroups of } K\}.$$

Since K is simply connected, the Galois connection on covering spaces gives bijections

$$\begin{array}{ccccc} \{\text{groups between } \Lambda \text{ and } \Omega\} & \leftrightarrow & \{\text{subgroups of } Z\} & \leftrightarrow & \{\text{groups covered by } K\} \\ \exp^{-1}(\Gamma) & \leftrightarrow & \Gamma & \leftrightarrow & K/\Gamma \end{array}$$

with fundamental groups given by

$$\pi_1(K/\Gamma) \cong \Gamma \cong \frac{\exp^{-1}(\Gamma)}{\Lambda}.$$

In particular, the *adjoint group* K/Z has no discrete normal subgroups its fundamental group is isomorphic to the center:

$$\pi_1(K/Z) \cong Z \cong \frac{\Omega}{\Lambda}.$$

///

Theorem/Definition (Root and Weight Lattices in Type A). We will see below that the compact group $\mathrm{SU}(n)$ is simply connected. Recall the exponential homomorphism:

$$\begin{aligned} \exp : \mathbb{R}^n_0 &\twoheadrightarrow T_0 \\ (x_1, \ldots, x_n) &\mapsto \begin{pmatrix} e^{2\pi x_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i x_n} \end{pmatrix}. \end{aligned}$$

We saw above that $\Lambda = \ker(\exp)$ is the intersection of the hyperplane $\mathbb{R}^n_0 \subseteq \mathbb{R}^n$ with the standard lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$. In particular, Λ has the *standard root basis*

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \ldots \quad \alpha_{n-1} = e_{n-1} - e_n,$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. I claim that the weight lattice $\exp^{-1}(Z(\mathrm{SU}(n))$ is the *orthogonal projection* of the lattice $\mathbb{Z}^n$ onto the hyperplane $\mathbb{R}^n_0$. In particular, we will show that Ω has an integer basis given by the *fundamental weights*,

$$\begin{aligned}
\omega_1 &= L_1 \\
\omega_2 &= L_1 + L_2 \\
&\vdots \\
\omega_{n-1} &= L_1 + L_2 + \cdots + L_{n-1},
\end{aligned}$$

where $L_i = e_i - \frac{1}{n}(e_1 + \cdots + e_n)$ is the orthogonal projection of $e_i \in \mathbb{R}^n$ onto the hyperplane $\mathbb{R}^n_0$. Then from the First Isomorphism Theorem it will follow that

$$\frac{\Omega}{\Lambda} = \frac{\exp^{-1}(Z(\mathrm{SU}(n))}{\ker(\exp)} \cong Z(\mathrm{SU}(n)) = \{\lambda I : \lambda^n = 1\} \cong \frac{\mathbb{Z}}{n\mathbb{Z}}.$$

Finally, we note that the root basis α_i and the weight basis ω_i are dual in the sense of abstract root systems. That is, we have

$$(\omega_i, \alpha_j) = \delta_{ij}.$$

///

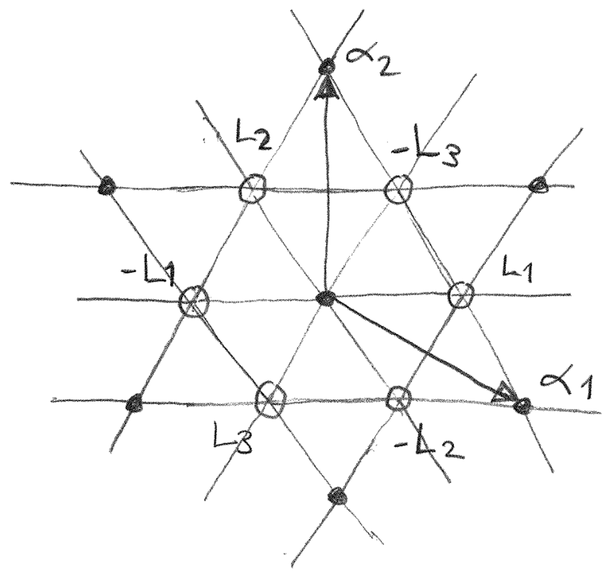
Proof. First we consider the projection of $\mathbb{Z}^n$ onto $\mathbb{R}^n_0$. Let A be any $n \times k$ real matrix whose columns are a basis for a k-dimensional subspace $V \subseteq \mathbb{R}^n$. Then the $n \times n$ matrix $A(A^TA)^{-1}A^T$ is the orthogonal projection onto this subspace and the matrix $I - A(A^TA)^{-1}A^T$ is the orthogonal projection onto the orthogonal complement $V^\perp \subseteq \mathbb{R}^n$. Let $\mathbf{1} \in \mathbb{R}^n$ be the vector of all ones so that $\mathbb{R}^n_0 = (\mathbb{R}\mathbf{1})^\perp$. Since the matrix

$$\mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T = \frac{\mathbf{1}\mathbf{1}^T}{\mathbf{1}^T\mathbf{1}} = \frac{1}{n}\mathbf{1}\mathbf{1}^T = \frac{1}{n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

is the projection onto the line $\mathbb{R}\mathbf{1}$ we conclude that $P := I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$ is the projection onto the hyperplane $\mathbb{R}^n_0$. In particular, the standard basis vector $e_i \in \mathbb{R}^n$ projects to

$$L_i := (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T)e_i = e_i - \frac{1}{n}(e_1 + e_2 + \cdots + e_n) \in \mathbb{R}^n_0.$$

Here is a picture of the situation when $n = 3$:



I claim that $P\mathbb{Z}^n$ is the weight lattice and that $L_1, L_2, \ldots, L_{n-1}$ is an integral basis:

$$\Omega = P\mathbb{Z}^n = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2 \oplus \cdots \oplus \mathbb{Z}L_{n-1}.$$

To see this, first note $L_1, L_2, \ldots, L_n$ is an integral spanning set because

$$\begin{aligned} P\mathbb{Z}^n &= P(\mathbb{Z}e_1 + \mathbb{Z}e_2 + \cdots + \mathbb{Z}e_n) \\ &= \mathbb{Z}Pe_1 + \mathbb{Z}Pe_2 + \cdots + \mathbb{Z}Pe_n \\ &= \mathbb{Z}L_1 + \mathbb{Z}L_2 + \cdots + \mathbb{Z}L_n. \end{aligned}$$

But the vector L_n is redundant because of the integral relation

$$\begin{aligned} P\mathbf{1} &= \mathbf{0} \\ P(e_1 + e_2 + \cdots + e_n) &= \mathbf{0} \\ Pe_1 + Pe_2 + \cdots + Pe_n &= \mathbf{0} \\ L_1 + L_2 + \cdots + L_n &= \mathbf{0}. \end{aligned}$$

Finally, since $\dim \mathbb{R}^n_0 = n-1$ we see that the vectors $L_1, \ldots, L_{n-1}$ are linearly independent.

Now let us show that the projection of $\mathbb{Z}^n$ is the same as the weight lattice, i.e., the preimage of $Z(\mathrm{SU}(n))$ under the exponential map $\mathbb{R}^n_0 \to T_0 \subseteq \mathrm{SU}(n)$. To do this we first compute the preimage under the map $\exp : \mathbb{R}^n \to T_0$ and then intersect with $\mathbb{R}^n_0$. So consider any $\mathbf{x} \in \mathbb{R}^n$ and observe that $\exp(\mathbf{x}) \in Z(\mathrm{SU}(n)) = \{\lambda I : \lambda^n = 1\}$ if and only if $\mathbf{x} - \alpha\mathbf{1} \in \mathbb{Z}^n$ for some $\alpha \in \frac{1}{n}\mathbb{Z}$. It follows that the preimage of $Z(\mathrm{SU}(n))$ under the map $\exp : \mathbb{R}^n \to T_0$ is

$$\exp^{-1}(Z(\mathrm{SU}(n))) = \mathbb{Z}^n + \frac{1}{n}\mathbb{Z}\mathbf{1},$$

which is the same as

$$\{(x_1+\alpha, x_2+\alpha, \ldots, x_{n-1}+\alpha, \alpha) : x_i \in \mathbb{Z}, \alpha \in \frac{1}{n}\mathbb{Z}\}.$$

Then the preimage under the map $\exp : \mathbb{R}^n_0 \to T_0$ is the intersection:

$$\begin{aligned}
&\exp^{-1}(Z(\mathrm{SU}(n))) \cap \mathbb{R}^n_0 \\
&= \{(x_1+\alpha, \ldots, x_{n-1}+\alpha, \alpha) : x_i \in \mathbb{Z}, \alpha \in \frac{1}{n}\mathbb{Z}, x_1+\cdots+x_{n-1}+n\alpha = 0\} \\
&= \{(x_1+\alpha, \ldots, x_{n-1}+\alpha, \alpha) : x_i \in \mathbb{Z}, \alpha = -\frac{1}{n}(x_1+\cdots+x_{n-1})\} \\
&= \{x_1L_1 + x_2L_2 + \cdots + x_{n-1}L_{n-1} : x_i \in \mathbb{Z}\} \\
&= \mathbb{Z}L_1 + \mathbb{Z}L_2 + \cdots + \mathbb{Z}L_n \\
&= P\mathbb{Z}^n.
\end{aligned}$$

Finally, we connect this to the abstract theory of root systems by choosing a canonical basis. Since the system of integer linear equations

$$\begin{aligned}
\omega_1 &= L_1 \\
\omega_2 &= L_1 + L_2 \\
&\vdots \\
\omega_{n-1} &= L_1 + L_2 + \cdots + L_{n-1},
\end{aligned}$$

has determinant 1 (which is an invertible element of $\mathbb{Z}$) we see that ω_i is another integer basis for the weight lattice. To show that this is the system of *fundamental weights* corresponding to the *simple roots* α_j we must show that $(\omega_i, \alpha_j) = \delta_{ij}$. For this we first note that

$$\begin{aligned}
(L_i, L_j) &= (e_i - \frac{1}{n}\mathbf{1}, e_j - \frac{1}{n}\mathbf{1}) \\
&= (e_i, e_j) - \frac{1}{n}(e_i, \mathbf{1}) - \frac{1}{n}(\mathbf{1}, e_j) + \frac{1}{n}(\mathbf{1}, \mathbf{1}) \\
&= (e_i, e_j) - \frac{1}{n}\cdot 1 - \frac{1}{n}\cdot 1 + \frac{1}{n}\cdot n \\
&= (e_i, e_j) - \frac{1}{n}.
\end{aligned}$$

We also note that $L_i - L_{i+1} = (e_i - \frac{1}{n}\mathbf{1}) - (e_{i+1} - \frac{1}{n}\mathbf{1}) = e_i - e_{i+1} = \alpha_i$. Then we have

$$\begin{aligned}
(\omega_i, \alpha_j) &= (L_1 + \cdots + L_i, L_j - L_{j+1}) \\
&= \sum_{k=1}^{i}(L_k, L_j) - \sum_{k=1}^{i}(L_k, L_{j+1}) \\
&= \sum_{k=1}^{i}\left[(e_k, e_j) - \frac{1}{n}\right] - \sum_{k=1}^{i}\left[(e_k, e_{j+1}) - \frac{1}{n}\right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{i}(e_k, e_j) - \sum_{k=1}^{i}(e_k, e_{j+1}) \\
&= (e_1 + \cdots + e_i, e_j - e_{j+1}) = \delta_{ij}.
\end{aligned}$$

$\square$

I owe you a proof that $\mathrm{SU}(n)$ is simply connected.

Sketch Proof that $\mathrm{SU}(n)$ is Simply Connected. First note that the trivial group $\mathrm{SU}(1) = \{1\}$ is simply connected, i.e.,

$$\pi_1(\mathrm{SU}(1)) = 1.$$

Now let $n > 1$ and observe that the unitary matrices $\mathrm{U}(n)$ preserve the length 1 vectors in $\mathbb{C}^n$, which form a real $2n$-dimensional sphere $S^{2n} \subseteq \mathbb{C}^n$. In fact, the subgroup $\mathrm{SU}(n)$ acts transitively on this sphere with stabilizer isomorphic to $\mathrm{SU}(n-1)$. From the fiber bundle

$$\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to \mathrm{SU}(n)/\mathrm{SU}(n-1) = S^{2n}$$

we obtain a long exact sequence of homotopy groups:

$$\cdots \to \pi_2(S^{2n}) \to \pi_1(\mathrm{SU}(n-1)) \to \pi_1(\mathrm{SU}(n)) \to \pi_1(S^{2n}) \to \cdots .$$

Then since $\pi_2(S^{2n}) = \pi_1(S^{2n}) = 1$ we conclude that

$$\pi_1(\mathrm{SU}(n-1)) \cong \pi_1(\mathrm{SU}(n))$$

and the result follows by induction. $\square$

By Cartan-Iwasawa-Malcev it follows that $\mathrm{SL}_n(\mathbb{C})$ is also simply connected. Now let me summarize the situation in type A.

- The groups $\mathrm{SU}(n)$ and $\mathrm{SL}_n(\mathbb{C})$ are simply connected and have center isomorphic to the weight lattice modulo the root lattice:

  $$Z(\mathrm{SU}(n)) = Z(\mathrm{SL}_n(\mathbb{C})) \cong \Omega/\Lambda \cong \mathbb{Z}/n.$$

- We saw that $\mathrm{SU}(n)$ is the maximal compact subgroup in $\mathrm{SL}_n(\mathbb{C})$. Later we will see that $\mathrm{SL}_n(\mathbb{C})$ is the "complexification" of $\mathrm{SU}(n)$. Every root system has a unique pair of simply connected groups $K \subseteq K_{\mathbb{C}}$ with these properties.
- The groups $\mathrm{PSU}(n) = \mathrm{SU}(n)/Z(\mathrm{SU}(n))$ and $\mathrm{PSL}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C})/Z(\mathrm{SL}_n(\mathbb{C}))$ have no discrete normal subgroups and have fundamental group isomorphic to the weight lattice modulo the root lattice

  $$\pi_1(\mathrm{PSU}(n)) = \pi_1(\mathrm{PSL}_n(\mathbb{C})) \cong \Omega/\Lambda \cong \mathbb{Z}/n.$$

- Later we will see that $\mathrm{PSU}(n)$ and $\mathrm{PSL}_n(\mathbb{C})$ also have no non-discrete normal subgroups, hence they are both simple groups. Again, every root system has a unique pair of groups $PK \subseteq PK_{\mathbb{C}}$ with these properties.

Types B and D

Maximal Tori in $\mathrm{SO}(n)$. How does this relate to type B and D root systems?

Unfortunately the group $\mathrm{SO}(n)$ is not simply connected. Cartan-Dieudonné. $\mathrm{SO}(n)$ generated by simple rotations. Quaternions and Clifford Algebras.

Weyl Groups

Abelian Groups and Fourier Analysis

Tori. Quantization of charge. Characters of tori.

Weyl Integration

Class functions determined by tori.