Need to add a hint for the following HW Problem

$$\sum x_i D_i(F) = d F$$

$$\iff F \text{ is homogeneous of degree } d.$$

Projective equivalence of conics.

Homogeneous polynomial $F(\vec{x}) = \mathbb{F}[\vec{x}]$ of degree is called a "quadratic form." If $2 \neq 0$ in $\mathbb{F}$ then we have a bijection

$$\text{Qudratic Forms} \longleftrightarrow \text{Symmetric Bilinear forms.}$$

If $\langle -,- \rangle : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is bilinear then $Q(-) = \langle -,- \rangle$ is quadratic.

Conversely, if $F(\vec{x})$ is quadratic form then

$$\langle \vec{x}, \vec{y} \rangle_F := \frac{1}{2}\left[ F(\vec{x}+\vec{y}) - F(\vec{x}) - F(\vec{y}) \right]$$

is symmetric and bilinear.

To see this, let $F(\vec{x}) = \sum_{i \leq j} c_{ij} x_i x_j$

for some $\binom{n}{2}$ coeffs $c_{ij} \in \mathbb{F}$. Then

$$F(\vec{x}+\vec{y}) = \sum_{i \leq j} c_{ij} (x_i + y_i)(x_j + y_j)$$

$$= F(\vec{x}) + F(\vec{y}) + 2 \sum c_{ii} x_i y_i + 2 \sum_{i<j} c_{ij} x_i y_j$$

Define $c_{ji} = c_{ij}$ to get

$$\langle \vec{x}, \vec{y} \rangle_F = \sum_i c_{ii} x_i y_i + \sum_{i \neq j} (c_{ij}/2) x_i y_j$$

$$= \vec{x}^T C \vec{y}$$

for symmetric matrix $C$.

Summary: Given quad form $F(\vec{x})$ there is a unique <u>symmetric</u> matrix $C^T = C$ such that

$$F(\vec{x}) = \vec{x}^T C \vec{x}.$$

The associated bilinear form is

$$\langle \vec{x}, \vec{y} \rangle_F = \vec{x}^T C \vec{y}.$$

Example:

$$F(x,y) = ax^2 + bxy + cy^2$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This is why we need $2 \neq 0$.

Theorem (Diagonalization):
Given hom. poly $F(\vec{x}) \in \mathbb{F}[\vec{x}]$ of degree 2 ($2 \neq 0$ in $\mathbb{F}$) then $\exists$ invertible matrix $A \in GL_n(\mathbb{F})$

such that

$$F(A\vec{x}) = d_1 x_1^2 + d_2 x_2^2 + \cdots + d_n x_n^2$$

for some $d_1, d_2, \ldots, d_n \in \mathbb{F}$, not all zero. In matrix terms:

Say $F(\vec{x}) = \vec{x}^T C \vec{x}$. Then

$$F(A\vec{x}) = (A\vec{x})^T C (A\vec{x})$$

$$= \vec{x}^T (A^T C A) \vec{x}.$$

i.e. we will find $A$ such that

$$A^T C A = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{pmatrix}.$$

[ Remark: This is NOT conjugation.

$$A^T C A \neq A^{-1} C A. \quad ]$$

Proof: Given $A \in GL_n(\mathbb{F})$, define

$$G(\vec{x}) = F(A\vec{x}) = \vec{x}^T (A^T C A) \vec{x}.$$

$$\langle \vec{x}, \vec{y} \rangle_G = \langle A\vec{x}, A\vec{y} \rangle_F = \vec{x}^T (A^T C A) \vec{y}.$$

Let $\vec{e}_i \in \mathbb{F}^n$ be standard basis

Let $\vec{a}_i = A\vec{e}_i$ be ith column of $A$.

Then

$$\langle \vec{e}_i, \vec{e}_j \rangle_G = \langle \vec{a}_i, \vec{a}_j \rangle_F = \vec{e}_i (A^T C A) \vec{e}_j$$

$$= ij \text{ entry of } A^T C A.$$

Goal: Find a basis $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{F}^n$ such that $\langle \vec{a}_i, \vec{a}_j \rangle_F = 0$ for $i \neq j$.

Then $A = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix}$ is the desired invertible matrix.

To begin: Since $F$ has degree 2, we know $C \neq 0$, hence $\exists$ some $\vec{x}, \vec{y} \in \mathbb{F}^n$ with

$$\langle \vec{x}, \vec{y} \rangle_F = \vec{x}^T C \vec{y} \neq 0.$$

We want to choose $\vec{a}_1 \in \mathbb{F}^n$ with $\langle \vec{a}_1, \vec{a}_1 \rangle_F \neq 0$. If $\langle \vec{x}, \vec{x} \rangle_F \neq 0$ or $\langle \vec{y}, \vec{y} \rangle_F \neq 0$ then take $\vec{a}_1 = \vec{x}$ or $\vec{a}_1 = \vec{y}$. Otherwise, suppose $\langle \vec{x}, \vec{x} \rangle_F = \langle \vec{y}, \vec{y} \rangle_F = 0$. Then

$$\langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle_F = \cancel{\langle \vec{x}, \vec{x} \rangle_F} + \cancel{\langle \vec{y}, \vec{y} \rangle_F} + 2\langle \vec{x}, \vec{y} \rangle_F \neq 0.$$

$(2 \neq 0)$.

Then take $\vec{a}_1 = \vec{x} + \vec{y}$. ✓

Now we have $\langle \vec{a}_1, \vec{a}_1 \rangle_F \neq 0$. $(\vec{a}_1 \neq \vec{0})$

Define subspace

$$V_1 = \{ \vec{x} \in \mathbb{F}^n : \langle \vec{a}_1, \vec{x} \rangle_F = 0 \}$$

Observations:

- $\vec{a}_1 \notin V_1$.

- $\vec{a}_1^T C \neq \vec{0}^T$ because
  $\vec{a}_1^T C \vec{a}_1 = \langle \vec{a}_1, \vec{a}_1 \rangle_F \neq 0$.

- $\langle \vec{a}_1, \vec{x} \rangle_F = 0$
  $\underbrace{\vec{a}_1^T C}\, \vec{x} = 0$ hyperplane.

- $V_1$ is $(n-1)$-dimensional.

If $\langle \vec{x}, \vec{y} \rangle_F = 0 \ \forall\, \vec{x}, \vec{y} \in V_1$
then let $\vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n \in V_1$
be <u>any basis</u>. Done.

Otherwise, repeat argument to get
$\langle \vec{a}_2, \vec{a}_2 \rangle_F \neq 0$ for some $\vec{a}_2 \in V_1$.

Let
$$V_2 = \{ \vec{x} \in \mathbb{F}^n : \langle \vec{a}_1, \vec{x} \rangle = \langle \vec{a}_2, \vec{x} \rangle = 0 \}.$$

Observe
- $\vec{a}_1, \vec{a}_2 \notin V_2$

- $\vec{a}_1^T C, \vec{a}_2^T C \neq \vec{0}^T$

  are not parallel because

  $\vec{a}_2 = t\vec{a}_1 \Rightarrow \langle \vec{a}_1, \vec{a}_2 \rangle_F = t\langle \vec{a}_1, \vec{a}_1 \rangle_F \neq 0.$

- $V_2$ is $(n-2)$-dimensional.

Repeat the argument. $\square$

Apply to quadratic curves.

Corollary: $f(x,y) = 0 \subseteq \mathbb{R}^2$ has degree 2. Then proj. equivalent to

- $x^2 = 0$
- $x^2 \pm y^2 = 0$
- $x^2 + y^2 \pm 1 = 0$

Proof: $F(A\vec{x}) = d_1 x^2 + d_2 y^2 + d_3 z^2$

for some $A \in GL_3(\mathbb{R})$

$d_1, d_2, d_3 \in \mathbb{R}$ not all zero.

Let $S = (s_{ij})$ be diagonal with

$$s_{ii} = \begin{cases} 1/\sqrt{d_i} & d_i > 0 \\ 1/\sqrt{-d_i} & d_i < 0 \\ 1 & d_i = 0. \end{cases}$$

Then $F(SA\vec{x}) = \delta_1 x^2 + \delta_2 y^2 + \delta_3 z^2$

where $\delta_1, \delta_2, \delta_3 \in \{\pm 1, 0\}$.

Finally let $P \in GL_3(\mathbb{R})$ be a permutation so that

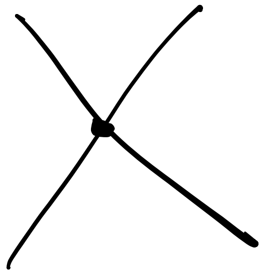
$$F(\pm PSA\vec{x})$$

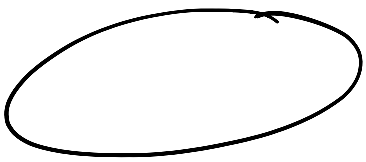
has the desired form. $\square$

Remark: If we allow complex projective change of variables, then

$$f \approx x^2 = 0,\ x^2 + y^2 = 0,\ x^2 + y^2 + 1 = 0.$$

What is the geometric meaning of this?

 $\approx$ one point

 $\approx$ empty set.