Some new HW problems posted.

Homework = Algebraic Details.

Last time: PIDs & Homogeneous Polys.

This time: $R[x]$ where R is PID (more generally, R is UFD) and "Study's Lemma" (Nullstellensatz for curves & hypersurfaces.)

Motivation: To study a curve $f(x,y) = 0$ it is convenient to consider the ideal $I_f = \{ g \in \mathbb{F}[x,y] : f(a,b) = 0 \Rightarrow g(a,b) = 0 \}$ = polynomials that vanish on the curve. The "coordinate ring" of the curve is defined as $\mathbb{F}[x,y]/I_f$ = classes of polynomials that agree on the curve.

Question: Can we go backwards?

$$\mathbb{F}[x,y]/I_f \cong \mathbb{F}[x,y]/I_g$$

$\underset{?}{\Longrightarrow}$ $f(x,y) = g(x,y)$.

Yes, <u>but</u> we have to work over an algebraically closed field, and the polynomials have to be squarefree (no repeated prime factors).

Classical Statement:
"Study's Lemma"
Let $f, g \in \mathbb{F}[x,y]$, $\mathbb{F}$ alg. closed. (e.g. $\mathbb{F} = \mathbb{C}$). Let $C_f, C_g \subseteq \mathbb{F}^2$ be the corresponding curves.
If f <u>irreducible</u> then

$$C_f \subseteq C_g \iff f \mid g$$

Idea: A curve in $\mathbb{C}^2$ has a "minimal polynomial," i.e., it is defined by a <u>unique</u> polynomial.

This generalizes Descartes' Theorem:

A point in $a \in \mathbb{C}$ has a minimal polynomial:

$$f(a) = 0 \iff (x-a) \mid f(x).$$

Algebraic Background:

If R is PID, then the prime ideals of $R[x]$ are

- zero,
- $f(x)R[x]$ where $f(x)$ is irreducible. ($f(x) = p$ allowed)
- $pR[x] + f(x)R[x]$ where $p \in R$ is prime & $f(x)$ is irreducible in the ring $(R/pR)[x]$.

That's All.

The third kind of ideals are <u>not</u> principal, but they are generated by 2 elements.

Krull Dimension of a Ring:
Length of longest chain of prime ideals.

- $\dim(\text{field}) = 0$.
- $\dim(\text{PID}) = 1$
- $\dim(\text{PID}[x]) = 2$.

The proof is quite involved!

That was just a preview of HW 2.

Back to Taylor series.

Last Time: for any $\vec{a} \in \mathbb{R}^n$ &

any polynomial $f(\vec{x}) \in \mathbb{R}[x_1, \dots, x_n]$ we have the Taylor expansion

$$f(\vec{x}) = \sum_{I \in \mathbb{N}^n} \frac{(D^I_{\vec{x}} f)(\vec{a})}{I!} (\vec{x} - \vec{a})$$

$$= f(\vec{a}) + (\nabla f)_{\vec{a}} (\vec{x} - \vec{a})$$

$$+ \frac{1}{2} (\vec{x} - \vec{a})^T (Hf)_{\vec{a}} (\vec{x} - \vec{a})$$

+ higher terms.

Now we apply this to the study of tangent spaces.

A line $L \subseteq \mathbb{R}^n$ containing a point $\vec{a} \in \mathbb{R}^n$ can be parametrized as

$$L : \vec{a} + t\vec{v}$$

where $t \in \mathbb{R}$ is a parameter

$\vec{v} \in \mathbb{R}^n$ is a direction vector.

For any poly $f(\vec{x}) \in \mathbb{R}[x_1, -, x_n]$ we consider the hypersurface

$$V_f : f(\vec{x}) = 0 .$$

To compute the intersection $L \cap V_f$, we substitute $\vec{x} = \vec{a} + t\vec{v}$ into $f(\vec{x}) = 0$, to get

$$\varphi(t) := f(\vec{a} + t\vec{v}) \in \mathbb{R}[t]$$

Note $\vec{a} \in V_f \iff \varphi(0) = 0$.

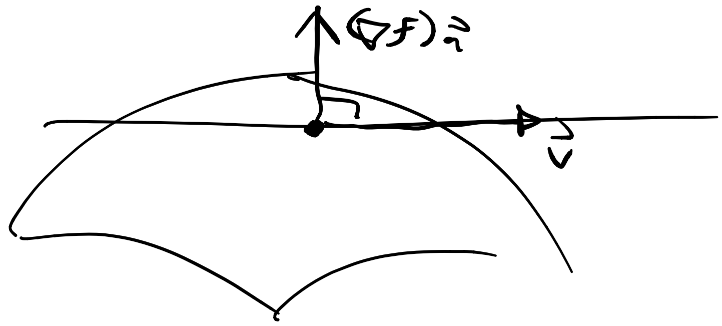
To be more explicit, expand $f(\vec{x})$ near $\vec{x} = \vec{a}$ to get

$$f(\vec{x}) = f(\vec{a}) + (\nabla f)_{\vec{a}} (\vec{x} - \vec{a})$$
$$+ \frac{1}{2} (\vec{x} - \vec{a})^T (Hf)_{\vec{a}} (\vec{x} - \vec{a}) + \cdots$$

$$\varphi(t) = f(\vec{a}) + t \, (\nabla f)_{\vec{a}} \vec{v}$$
$$+ \frac{t^2}{2} \vec{v}^T (Hf)_{\vec{a}} \vec{v} + \cdots$$

Indeed, $\varphi(0) = f(\vec{a})$ ✓

Definition: We say line L is *tangent* to V_f at $\vec{a}$ if $t=0$ is a root of multiplicity at least 2.



" double contact "

From the Taylor series:

$\vec{v}$ is a tangent vector at point $\vec{a}$

$\iff$ first 2 coefficients of $\varphi(t)$ are zero

$$\iff \begin{cases} f(\vec{a}) = 0 \\ (\nabla f)_{\vec{a}} \vec{v} = 0. \end{cases}$$

i.e. $\vec{v}$ is $\perp$ to the gradient vector at $\vec{a}$.

If $(\nabla f)_{\vec{a}} \neq \vec{0}$ then the set of tangent lines at $\vec{a}$ forms a $(n-1)$ dim hyperplane called the tangent space:

$$(\nabla f)_{\vec{a}}(\vec{x}-\vec{a}) = 0.$$

Shorthand: Write $D_{x_i} f = f_{x_i}$.

$$f_{x_1}(\vec{a})(x_1-a_1) + \cdots + f_{x_n}(\vec{a})(x_n-a_n) = 0.$$

If $(\nabla f)_{\vec{a}} = \vec{0}$ & $f(\vec{a}) = 0$, then say $\vec{a}$ is a singular point of V_f.

In this case we have $(\nabla f)_{\vec{a}} \vec{v} = 0$ for <u>all</u> $\vec{v}$ hence we could say the tangent space at $\vec{a}$ is <u>all</u> $\mathbb{R}^n$.

Emphasis: Hypersurface V_f,

- $\vec{a} \in V_f$ smooth $\Longleftrightarrow$ $\dim T_{\vec{a}} V_f = (n-1)$.
- $\vec{a} \in V_f$ singular $\Longleftrightarrow$ $\dim T_{\vec{a}} V_f = n$.

Where next?

- apply to curves
- projective space
- intrinsic version of tangent space.

Question: How does the tangent space behave under change of coordinates?