Why are affine & projective equivalence natural ideas? What is the geometric content?

The Fundamental Theorem of Projective Geometry:

1. If $\Phi : \mathbb{RP}^2 \to \mathbb{RP}^2$ is bijective & takes lines to lines, then

$$\Phi\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for some matrix $A \in GL_3(\mathbb{R})$, unique up to scalar multiplication.

2. If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective & takes lines to lines, then

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = A' \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ t \end{pmatrix}$$

for some unique $A' \in GL_2(\mathbb{R})$, $(s, t) \in \mathbb{R}^2$.

Remarks:

- This is AMAZING!

- We will prove ① $\Rightarrow$ ②.
- The proof of ① would take us too far afield, so I'll just sketch it. For details: Rey Casse, Projective Geometry an Introduction, Theorem 4.7.

Proof ① $\Rightarrow$ ②:

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a collineation. Since bijective on points, send parallel lines to parallel lines, hence it permutes the points at ∞ and extends to collineation $\Phi : \mathbb{RP}^2 \to \mathbb{RP}^2$.

By ① have $\Phi \vec{x} = A\vec{x}$ for some $A \in PGL_3(\mathbb{R})$. Since A preserves the "line at ∞" $z = 0$, we have

$$A = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & c \end{array}\right)$$

some $c \neq 0$. Up to scaling:

$$A = \left(\begin{array}{c|c} A' & \begin{matrix} s \\ t \end{matrix} \\ \hline 0\ 0 & 1 \end{array}\right).$$

This acts on finite points as

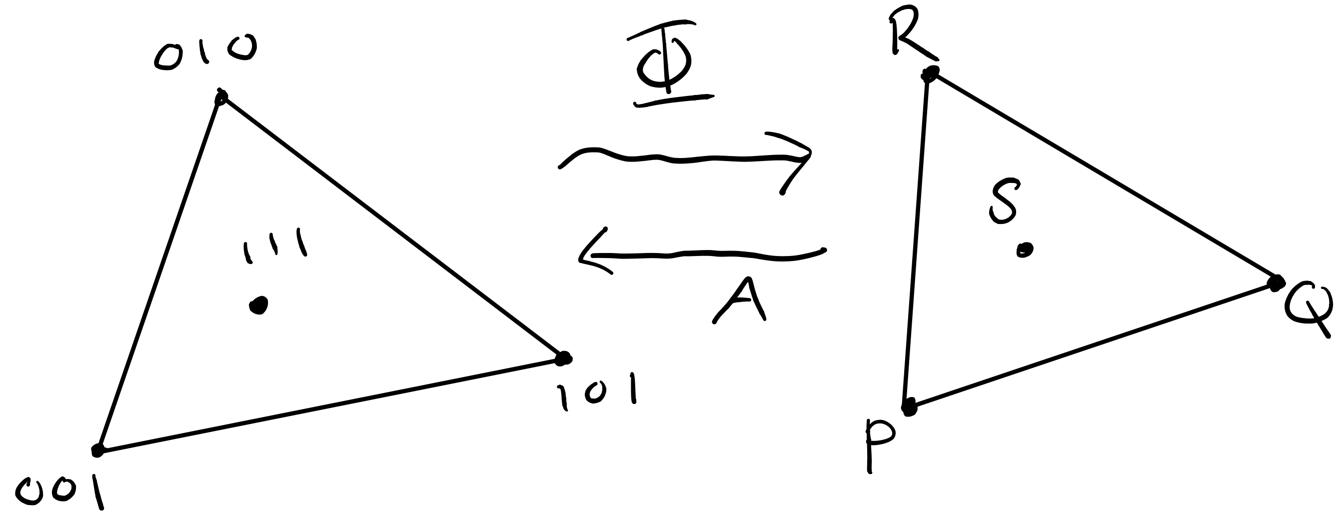
$$A\begin{pmatrix} x \\ y \\ \hline 1 \end{pmatrix} = \left(\begin{array}{c|c} A' & \begin{matrix} s \\ t \end{matrix} \\ \hline 0\ 0 & 1 \end{array}\right)\begin{pmatrix} x \\ y \\ \hline 1 \end{pmatrix}$$

$$= \begin{pmatrix} A'\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ t \end{pmatrix} \\ \hline 1 \end{pmatrix}$$

Remark: Affine maps are precisely projective maps that preserve the line at ∞.

Sketch Proof of 1:

- Let $\Phi : \mathbb{RP}^2 \to \mathbb{RP}^2$ be collineation.
- Suppose Φ sends the fundamental quadrangle to $PQRS$.



- Check, $\exists!$ projective map $A \in PGL_3$ sending PQRS back to the fundamental quadrangle.

- Hence $A \circ \Phi : \mathbb{RP}^2 \to \mathbb{RP}^2$ is a collineation <u>fixing</u> the fund. quadrangle.

- <u>The Hard Part</u>: Hence $A \circ \Phi$ acts on homogeneous coodinates by a field automorphism $\mathbb{R} \xrightarrow{\alpha} \mathbb{R}$.

$$(x : y : z) \longmapsto (x^\alpha : y^\alpha : z^\alpha).$$

[Remark: Involves von Staudt's construction of coordinates from incidence axioms & the fact that

Pappus' Theorem holds when coordinates come from a field.]

- Finally, any field automorphism of $\mathbb{R}$ is the identity, hence

$$A \circ \Phi = \mathrm{id}$$

$$\Phi = A^{-1} \in PGL_3(\mathbb{R}).$$

$\square$

Next Topic: Diagonalization of Quadratic forms.

Idea: "Conic sections" are vastly simplified in the projective plane.

Theorem: Any curve $f(x,y) = 0$ in $\mathbb{R}^2$ of degree 2 is projectively equivalent to one of:

- $x^2 = 0$
- $x^2 \pm y^2 = 0$
- $x^2 + y^2 \pm 1 = 0$.

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Actually we will prove something more general.

Theorem: Let $F(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ be homogeneous of degree 2. Then there exists linear change of variables $\Phi : \mathbb{F}^n \to \mathbb{F}^n$ such that

$$G(x_1, \ldots, x_n) = F(\Phi(x_1), \ldots, \Phi(x_n))$$

$$= s_1 x_1^2 + s_2 x_2^2 + \cdots + s_n x_n^2.$$

Here, $\mathbb{F}$ is any field where $2 \neq 0$.

In other words, any quadratic form over a field (with $2 \neq 0$) can be diagonalized.