Goal: Given a line L & hypersurface V in projective space, we want to define the "intersection multiplicity" at a point $\vec{p}$:

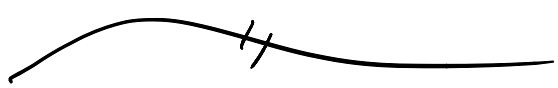
$$[L \cdot V]_{\vec{p}} \in \mathbb{N}.$$

Jargon:

$[L \cdot V]_p = 0$ don't intersect at $\vec{p}$

$[L \cdot V]_p = 1$ intersect "transversely"

$[L \cdot V]_p = 2$ tangent



Recall: A line $L \subseteq \mathbb{F}\mathbb{P}^n$ has the form $L: t_1 \vec{p}_1 + t_2 \vec{p}_2$ where $\vec{p}_1, \vec{p}_2 \in \mathbb{F}\mathbb{P}^n$ are distinct points.

Get an isomorphism (projective equivalence):

$$L \longleftrightarrow \mathbb{FP}^1 \subseteq \mathbb{FP}^n$$

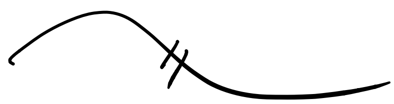
$$t_1\vec{p}_1 + t_2\vec{p}_2 \qquad (t_1 : t_2) \qquad (t_1 : t_2 : 0 \cdots : 0)$$

A reparametrization of L is induced by projective equiv $A : \mathbb{FP}^1 \to \mathbb{FP}^1$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL_2(\mathbb{F})$$

$$L \overset{A}{\rightsquigarrow} L$$

$$t_1\vec{p}_1 + t_2\vec{p}_2 \qquad (at_1 + bt_2)\vec{p}_1 + (ct_1 + dt_2)\vec{p}_2$$



Now let $V \subseteq \mathbb{FP}^n$ be a hypersurface.

This means $V = V_F : F(\vec{x}) = 0$ for some homogeneous $F(\vec{x}) \in \mathbb{F}[x_1, \dots, x_{n+1}]$.

Let $L = t_1\vec{p}_1 + t_2\vec{p}_2$

$$= \begin{pmatrix} | & | \\ \vec{p}_1 & \vec{p}_2 \\ | & | \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = P\vec{t}$$

be a parametrized line. If F is homogeneous of degree d, then by substitution obtain homogeneous polynom. of degree in t_1, t_2:

$$\Phi(t_1, t_2) := F(P\vec{t}\,) \in \mathbb{F}[t_1, t_2].$$

Then for any point $\vec{p} \in L$, say $\vec{p} = P\vec{a} = a_1\vec{p}_1 + a_2\vec{p}_2$ we want to define the multiplicity:

$[L \cdot V]_{\vec{p}}$ = "the multiplicity of $(a_1 : a_2)$ as a root of polynomial $\Phi(t_1, t_2)$."

Does this make sense?

Projective Version of Descartes' Theorem:
Let $\Phi(t_1, t_2)$ be homogeneous. Then

$(a_1 : a_2) \in \mathbb{F}\mathbb{P}^1$ is a root of Φ $\iff$ $(a_2 t_1 - a_1 t_2) \mid \Phi$

"roots correspond to linear factors"

<u>Proof</u>: If $(a_2 t_1 - a_1 t_2) \mid \Phi(t_1, t_2)$ then $\Phi(a_1, a_2) = 0$. ✓

Conversely, suppose $\Phi(a_1, a_2) = 0$, where a_1, a_2 not both zero. Two Cases:

- $a_2 \neq 0$: Suppose $\deg \Phi = n$ and let $\Phi(t_1, t_2) = t_2^m \Phi'(t_1, t_2)$ where $\Phi'(t_1, t_2)$ is homogeneous of degree $n-m$ & $t_2 \nmid \Phi'$. Note that $0 = \Phi(a_1, a_2) = a_2^m \Phi'(a_1, a_2)$ & $a_2 \neq 0 \Rightarrow \Phi'(a_1, a_2) = 0$.

Consider the dehomogenization

$$\varphi(t_1) := \Phi(t_1, 1) = \Phi'(t_1, 1)$$

Since $a_2 \neq 0$, have $(a_1 : a_2) \sim (\frac{a_1}{a_2} : 1)$
and since Φ homogeneous, have

$$\varphi(a_1/a_2) = \Phi\left(\frac{a_1}{a_2}, 1\right) = \Phi(a_1, a_2) = 0.$$

Usual Descartes' Theorem:

$$\varphi(t_1) = (t_1 - a_1/a_2)\,\psi(t_1)$$

for some $\psi(t_1)$ of degree $n-m-1$.

Re-homogenize:

$$\Phi'(t_1, t_2) = t_2^{\,n-m}\,\varphi(t_1/t_2)$$

$$= \frac{1}{a_2}(a_2 t_1 - a_1 t_2)\, t_2^{\,n-m-1}\,\psi(t_1/t_2)$$

$$= \frac{1}{a_2}(a_2 t_1 - a_1 t_2)\,\Psi(t_1, t_2)$$

for some homogeneous $\Psi(t_1, t_2)$
of degree $n-m-1$.

Done. ✓

- $a_1 \neq 0$: Symmetric Proof. ///

It follows that multiplicity of $(a_1 : a_2) \in \mathbb{F}\mathbb{P}^1$ is well-defined as the highest power of $(a_2 t_1 - a_1 t_2)$ dividing $\Phi(t_1, t_2)$.

[Remark: More generally one can show that every homogeneous poly over a field has unique factorization into irred. homog. polynomials.]

Back to $[L \cdot V]_{\vec{p}}$.

Let $L = P\vec{t}$, $V = V_F$, $\vec{p} = P\vec{a}$.

Then we define

$$[L \cdot V]_{\vec{p}} = \text{multiplicity of } \vec{a} \in \mathbb{F}\mathbb{P}^1$$

as a root of $\Phi(\vec{t}) = F(P\vec{t})$.

I claim that this number is well-defined up to projective equivalence & reparametrization of the line.

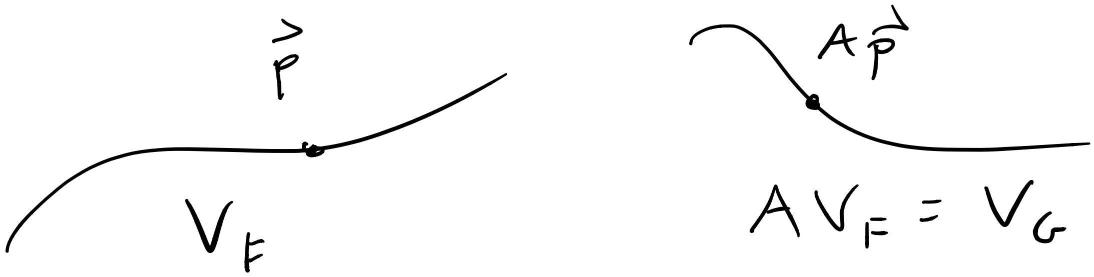
Proof: Let $A \in PGL_{n+1}(\mathbb{F})$.

Line L gets sent to $AL = AP\vec{t}$.

hypersurface $V = V_F$ gets sent to

$$AV = V_G$$

where $G(\vec{x}) = F(A^{-1}\vec{x})$.



$F(\vec{p}) = 0$

$G(A\vec{p}) = 0$

$F(A^{-1}A\vec{p}) = F(\vec{p})$ ✓

Then $\Phi(t_1, t_2) = F(P\vec{t})$

$$= F(A^{-1}AP\vec{t}) = G(AP\vec{t}).$$

$$[L \cdot V]_{\vec{p}} = [AL \cdot AV]_{A\vec{p}}.$$

Next let $\Sigma \in PGL_2(\mathbb{F})$ be a reparametrization of the line.

$$L = P\vec{t} \rightsquigarrow L = P\Sigma\vec{t}$$

Observe $\Sigma\vec{t}$ is a root of $F(P\Sigma\vec{t})$ of mult $m \iff \vec{t}$ is a root of $F(P\vec{t})$ of mult. m.

Reason: $\Phi(\vec{t}) \rightsquigarrow \Phi(\Sigma\vec{t})$ preserves the degrees of the homogeneous factors of Φ. ///

What's missing?

Dependence of $V = V_F$ on the polynomial F? (Nullstellensatz)

Idea: Intersection multiplicity $[L \cdot V]_{\vec{p}}$ should be intrinsic to the geometry, i.e., independent of of the algebraic mode of expression.

Historically, it has been very difficult to make this precise...