Today: Finish the lecture from Oct 2, which got interrupted by bad internet.

Topic: "Zariski tangent spaces"

The ideal of a point:

Consider $\mathbb{F}[\vec{x}] = \mathbb{F}[x_1, \dots, x_n]$. For any point $\vec{p} \in \mathbb{F}^n$ define ideal

$$M_{\vec{p}} := \sum_{i=1}^{n} (x_i - p_i)\, \mathbb{F}[\vec{x}].$$

(a) $M_{\vec{p}}$ is kernel of $\mathbb{F}[\vec{x}] \to \mathbb{F}$ ($f(\vec{x}) \mapsto f(\vec{p})$), hence is <u>maximal</u>.

(b) Given ideals $A, B \subseteq \mathbb{F}[\vec{x}]$, let AB be smallest ideal containing the set $\{fg : f \in A, g \in B\}$. Then

$$M_{\vec{p}}^{k} = \sum_{i_1, \dots, i_k} (x_{i_1} - p_{i_1}) \cdots (x_{i_k} - p_{i_k})\, \mathbb{F}[\vec{x}]$$

(c) The taylor expansion at $\vec{x} = \vec{p}$ defines an isomorphism of vector spaces:

$$\mathbb{F}[\vec{x}] \xrightarrow{\sim} \bigoplus_{k \geq 0} \frac{M_{\vec{p}}^{k}}{M_{\vec{p}}^{k+1}}$$

where $M_{\vec{p}}^{0} := \mathbb{F}[\vec{x}]$, and

$$\dim_{\mathbb{F}}\left(M_{\vec{p}}^{k} / M_{\vec{p}}^{k+1} \right) = \binom{n+k-1}{k}.$$

(d) Any generating set of ideal $M_{\vec{p}}$ maps to a spanning set of $M_{\vec{p}} / M_{\vec{p}}^{2}$. Since $\dim(M_p / M_p^2) = n$, then M_p cannot be generated by <u>fewer</u> than n elements. [In particular, if $n \geq 2$ then $\mathbb{F}[\vec{x}]$ is not a PID.]

(a) Claim: $M_p = \{ f : f(\vec{p}) = 0 \}$.

One direction: $f \in M_p = \sum (x_i - p_i) \mathbb{F}[\vec{x}]$ then we have

$$f = (x_1 - p_1) f_1 + \cdots + (x_n - p_n) f_n$$

$$\Rightarrow f(\vec{p}) = (p_1 - p_1) f_1(\vec{p}) + \cdots + (p_n - p_n) f_n(\vec{p})$$
$$= 0 + \cdots + 0$$
$$= 0.$$

Conversely, let $f(\vec{p}) = 0$, consider Taylor expansion:

$$f(\vec{x}) = \underbrace{\cancel{f(\vec{p})}}_{0} + \sum_{\Sigma I \geq 1} \frac{1}{I!} D^I_{\vec{x}}(f)_{\vec{p}} (\vec{x} - \vec{p})^I$$

Since $\Sigma I \geq 1 \Rightarrow i_k \geq 1$ for some k, hence $f(\vec{x}) \in M_p$.

(b) $A, B \leq \mathbb{F}[\vec{x}]$ finitely generated:

$$A = f_1 \mathbb{F}[\vec{x}] + \cdots + f_l \mathbb{F}[\vec{x}]$$
$$B = g_1 \mathbb{F}[\vec{x}] + \cdots + g_m \mathbb{F}[\vec{x}].$$

Then, claim $AB = \sum f_i g_j \mathbb{F}[\vec{x}]$.

Indeed, each $f_i g_j \in AB$, hence $\mathbb{F}[\vec{x}]$-linear combinations $\in AB$.

Conversely, any element of AB is a sum of terms fgh, $f \in A$, $g \in B$,

$h \in \mathbb{F}[\vec{x}]$. Suppose

$$f = f_1\varphi_1 + \cdots + f_l\varphi_l$$

$$g = g_1\gamma_1 + \cdots + g_m\gamma_m.$$

Then $fgh = \sum_{i,j} f_i g_j (\varphi_i \gamma_j h)$. ✓

Apply to $M_p = \sum (x_i - p_i)\mathbb{F}[\vec{x}]$

to get $M_p^k = \sum (x_{i_1} - p_{i_1})\cdots(x_{i_k} - p_{i_k})\mathbb{F}[\vec{x}]$.

(c) Recall, each $f \in \mathbb{F}[\vec{x}]$ has unique expansion $f = \sum a_I (\vec{x} - \vec{p})^I$.

Indeed: $a_I = \frac{1}{I!} D_{\vec{x}}^I (f)_{\vec{p}}$.

Now for any f define the homogeneous filtration at $\vec{x} = \vec{p}$:

$$f = f_{\vec{p}}^{(0)} + f_{\vec{p}}^{(1)} + \cdots$$

$$f_{\vec{p}}^{(k)} = \sum_{\Sigma I = k} a_I (\vec{x} - \vec{p})^I$$

Send f to the formal sequence

$$\left(f_{\vec{p}}^{(0)} + M_p,\ f_p^{(1)} + M_p^2,\ f_p^{(2)} + M_p^3, \dots \right) \in \bigoplus_{k \geq 0} M_p^k / M_p^{k+1}$$

To see this is a vector space isom,
I claim $\{ (\vec{x} - \vec{p})^I + M_p^{k+1} : \Sigma I = k \}$
is a basis for M_p^k / M_p^{k+1}.

Spanning ✓

Independence? Suppose $\exists$ nontriv. linear relation: $\exists\ \Sigma I = k$ where

$$(\vec{x} - \vec{p})^I + \sum_{\substack{J \neq I \\ \Sigma J = k}} a_I (\vec{x} - \vec{p})^J \in M_p^{k+1} \qquad (*)$$

Recall: For any $I, J \in \mathbb{N}^n$,

$$D_{\vec{x}}^I (\vec{x} - \vec{p})^J = \begin{cases} \text{non-constant} & I < J \\ \text{non-zero constant} & I = J \\ 0 & I \not\leq J \end{cases}$$

So apply $D_{\vec{x}}^I$ to $(*)$:

If $\Sigma I = \Sigma J$ & $I = J$ then $I \nleq J$.

$\Rightarrow$ polynomial $\rightsquigarrow$ non-zero const.

However, $D^{I}_{\vec{x}}$ applied to any element of M_p^{k+1} gives a non-constant or zero, since $k = \Sigma I < \Sigma J$ implies that $I < J$ or $I \nleq J$. ///

$$\dim\left(M_p^k / M_p^{k+1}\right) = \#\{ I \in \mathbb{N}^n : \Sigma I = k \}$$

"Stars & Bars": $I \longleftrightarrow$ binary strings of 0s & 1s.

$$(i_1, \dots, i_n) \longmapsto \underbrace{0 \cdots 0}_{i_1} 1 \underbrace{0 \cdots 0}_{i_2} 1 \cdots 1 \underbrace{0 \cdots 0}_{i_n}$$

k copies of 0
$n-1$ copies of 1.

$\#$ of such is $\binom{n-1+k}{k} = \binom{n-1+k}{n-1}$.

Remark: $\displaystyle\sum_{k \geq 0} \dim\left(\frac{M_p^k}{M_p^{k+1}}\right) \lambda^k = \frac{1}{(1-\lambda)^n}$.

(d) Have surjective map $M_p \to M_p/M_p^2$ given by $f \mapsto (\nabla f)_{\vec{p}}(\vec{x}-\vec{p}) + M_p^2$.

Now suppose $M_p = f_1\mathbb{F}[\vec{x}] + \cdots + f_m\mathbb{F}[\vec{x}]$.

Want to show $(\nabla f_i)_{\vec{p}}(\vec{x}-\vec{p}) + M_p^2$ are a spanning set for vector space M_p/M_p^2.

Indeed: Every elt of M_p/M_p^2 looks like $(\nabla f)_{\vec{p}}(\vec{x}-\vec{p})$ for some $\boxed{f \in M_p.}$

By hypothesis, $f = f_1 g_1 + \cdots + f_m g_m$ for some $g_1, \ldots, g_m \in \mathbb{F}[\vec{x}]$. Product Rule:

$$(\nabla f)_{\vec{p}} = \sum_i \nabla(f_i g_i)_{\vec{p}}$$

$$= \sum \left((\nabla f_i)_{\vec{p}}\, g_i(\vec{p}) + \overset{0}{\cancel{f_i(\vec{p})}}\, (\nabla g_i)_{\vec{p}} \right)$$

$$= \sum g_i(\vec{p})\, (\nabla f_i)_{\vec{p}},$$

hence $(\nabla f)_{\vec{p}}(\vec{x}-\vec{p}) + M_p^2$

$$= \sum_i g_i(\vec{p})(\nabla f_i)_p(\vec{x}-\vec{p}) + M_p^2. \quad ///$$

Zariski Tangent Space of Hypersurface.

Let $V_f \subseteq \mathbb{F}^n$ be hypersurface,

Let $\vec{p} \in V_f$, let $T_{\vec{p}} V_f$ be the

tangent space $\{ \vec{v} : (\nabla f)_{\vec{p}} \vec{v} = 0 \}$.

Then:

$$(T_{\vec{p}} V_f)^* \approx M_p \big/ \left(M_p^2 + f(x)\mathbb{F}[x]\right)$$

<u>Proof</u>:

$$\begin{tikzcd} M_p \arrow[r, two heads] \arrow[rr, bend right, two heads, "\text{surjective}"'] & (\mathbb{F}^n)^* \arrow[r, two heads] & (T_p V_f)^* \end{tikzcd}$$

$$f \longmapsto \left[\vec{v} \mapsto (\nabla f)_{\vec{p}} \vec{v} \right]$$

Claim: $\ker = M_p^2 + f(x)\mathbb{F}[x]$.

One direction easy (see notes).

Other direction. Let g be in $\ker$.

i.e., $\vec{v} \longmapsto (\nabla g)_{\vec{p}} \vec{v} = 0 \quad \forall\, \vec{v} \in T_p V_f$.

i.e., $(\nabla f)_{\vec{p}} \vec{v} = 0 \implies (\nabla g)_{\vec{p}} \vec{v} = 0$.

i.e., $H_{(\nabla f)_p} \subseteq H_{(\nabla g)_p}$

by $\underline{\text{dimension}}$ $\Longrightarrow$ $H_{(\nabla f)_p} = H_{(\nabla g)_p}$.

$$\Longrightarrow \quad H_{(\nabla f)_p}^{\perp} = H_{(\nabla g)_p}^{\perp}.$$

$$\Longrightarrow \quad (\nabla g)_{\vec{p}} = \lambda (\nabla f)_{\vec{p}}.$$

Finally, let $h = g - \lambda f \in \mathbb{F}[\vec{x}]$.

$$h(\vec{p}) = g(\vec{p}) - \lambda f(\vec{p}) = 0 - \lambda 0 = 0.$$

$$(\nabla h)_{\vec{p}} = (\nabla g)_{\vec{p}} - \lambda (\nabla f)_{\vec{p}} = \vec{0}.$$

$$\Longrightarrow \quad h \in M_p^2.$$

$$g - \lambda f \in M_p^2$$

$$g \in M_p^2 + \{\lambda f : \lambda \in \mathbb{F}\}$$

$$\subseteq M_p^2 + f(\vec{x})\,\mathbb{F}[\vec{x}] \quad \checkmark \quad ///$$

Remark: $\mathbb{F}$ need not be algebraically closed!