From Last Time:

Let $f, g \in \mathbb{F}[x,y]$, f irreducible & $f \nmid g$.

Then $C_f \cap C_g$ is a finite set.

Key Step: f, g coprime in $\mathbb{F}[x,y]$

$\Rightarrow$ f, g coprime in $\mathbb{F}(x)[y]$ (PID)

$\Rightarrow$ $fF + gG = 1$, $F, G \in \mathbb{F}(x)[y]$.

$\Rightarrow$ $f\tilde{f} + g\tilde{g} = h$, $\tilde{f}, \tilde{g} \in \mathbb{F}[x,y]$, $h \in \mathbb{F}[x]$.

If $(a,b) \in C_f \cap C_g$, evaluate to get $h(a) = 0$. Since $h \neq 0$ there are finitely many such a.

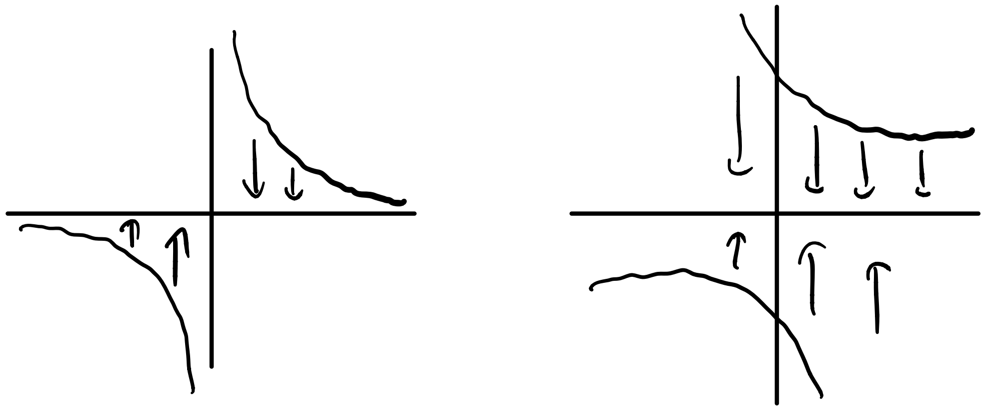
We would <u>like</u> to say that each $f(a,y) \in \mathbb{F}[y]$ has finitely many roots $y = b$. But what if $f(a,y) = 0$?

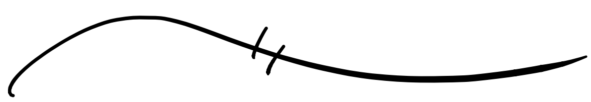
OOPS.

TRICK: Run symmetric proof in the ring $\mathbb{F}(y)[x]$.

But in higher dimensions this trick doesn't work. In that case we need a clever change of variables:

"normalization"





Study's Lemma for Hypersurfaces:

Let $\mathbb{F}$ be alg. closed.

(a) For any $f \in \mathbb{F}[x_1, \dots, x_n]$, $f \neq 0$,
$\exists \vec{p} \in \mathbb{F}^n$, $f(\vec{p}) \neq 0$.

(b) Given $f, g \in \mathbb{F}[\vec{x}]$ with f irred. & $V_f \subseteq V_g$ then $f \mid g$.

(c) Bijections:

hypersurfaces $\leftrightarrow$ square-free polys

irreducible " $\leftrightarrow$ irreducible "

<u>Proof</u>: (a): Induction on n.

$n=1$: A non-zero poly in 1 variable has $< \infty$ roots. If $\# \mathbb{F} = \infty$ $\exists\, a \in \mathbb{F}$, $f(a) \neq 0$. $n \geq 2$:

Let $f \in \mathbb{F}[x_1, \ldots, x_n]$, $f \neq 0$,

Let $\vec{x}_i' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

$\exists\, i$ such that

$$f(\vec{x}) = \sum f_k(\vec{x}_i')\, x_i^k$$

with $f_k(\vec{x}_i') \in \mathbb{F}[\vec{x}_i']$ not all zero.

Let $g \in \mathbb{F}[\vec{x}_i']$ be product of the nonzero f_k, so $g \neq 0$, $g \in \mathbb{F}[\vec{x}_i']$.

By induction, $\exists\, \vec{p}_i' \in \mathbb{F}^{n-1}$, $g(\vec{p}_i') \neq 0$.

$\Rightarrow \exists\, k$, $f_k(\vec{p}_i') \neq 0$, hence $f(\vec{p}_i', x_i) \in \mathbb{F}[x_i]$ is nonzero.

From base case $\exists\ p_i \in \mathbb{F}$,

$$f(\vec{p}) = f(\vec{p}_i', p_i) \neq 0. \qquad ///$$

(b): Normalization:

If $f \in \mathbb{F}[\vec{x}]$ is non-constant of degree $d \geq 1$, I claim we can make a linear sub $\vec{x} \mapsto A\vec{x}$ so that

$$f(A\vec{x}) = c x_i^d + \text{lower terms in } x_i$$

for some i & $c \in \mathbb{F} \setminus 0$. To see this, let

$$f = f^{(d)} + \cdots + f^{(1)} + f^{(0)}$$

be homogeneous filtration. Since $f^{(d)}(\vec{x}) \neq 0$, $\exists\ \vec{p} \in \mathbb{F}^n$, $f^{(d)}(\vec{p}) \neq 0$.

Since $f^{(d)}$ is homogeneous, $\boxed{\exists\ i,\ a_i \neq 0.}$

Define $\vec{x} \mapsto A\vec{x}$ by

$$x_i \mapsto a_i x_i \quad \& \quad x_j \mapsto x_j + a_j x_i$$

This is invertible since $p_i \neq 0$.

Finally:

$$f(A\vec{x}) = f^{(d)}(A\vec{x}) + \text{lower terms}$$

$$= f^{(d)}(\vec{?})\, x_i^d + \text{lower terms.}$$

///

Elimination: Now let $f, g \in \mathbb{F}[\vec{x}]$, f irreducible, $V_f \subseteq V_g$. We will prove $f \mid g$. So assume $f \nmid g$.

$\Longrightarrow f, g$ coprime in $\mathbb{F}[\vec{x}]$.

Since f non-constant, make a change so that

$$f(A\vec{x}) = c\, x_i^d + \text{lower terms.}$$

Change of vars preserves divisibility and containment of hyp. surfaces, so let's just say $f(\vec{x}) = c x_i^d + \cdots$.

Let $\vec{x}_i' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Then (Gauss) f, g still coprime in

$\mathbb{F}(\vec{x}_i')[x_i] \supseteq \mathbb{F}[\vec{x}_i'][x_i] = \mathbb{F}[\vec{x}]$.

Since $\mathbb{F}(\vec{x}_i')[x_i]$ is PID, have

$$fF + gG = 1$$

for some $F, G \in \mathbb{F}(\vec{x}_i')[x_i]$.

Let $h(\vec{x}_i') \in \mathbb{F}[\vec{x}_i']$ be common mult of coeffs of F & G, multiply:

$$f\tilde{f} + g\tilde{g} = h \in \mathbb{F}[\vec{x}_i']$$

Since $h \neq 0$, (a) $\Rightarrow \exists\, \vec{p}_i' \in \mathbb{F}^{n-1}$, $h(\vec{p}_i') \neq 0$. Since $f(\vec{x}) = cx_i^d + \cdots$, have $f(\vec{p}_i', x_i) \in \mathbb{F}[x_i]$ is <u>nonzero</u>.

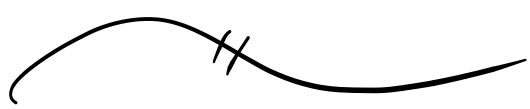
Since $\mathbb{F}$ alg closed, $\exists\, p_i \in \mathbb{F}$, $f(\vec{p}) = f(\vec{p}_i', p_i) = 0$. Finally, since $V_f \subseteq V_g$, $g(\vec{p}) = 0$, hence

$$f(\vec{p})\bar{f}(\vec{p}) + g(\vec{p})\bar{g}(\vec{p}) = h(\vec{p}_i')$$
$$0 = h(\vec{p}_i')$$

Contradiction. ///

(c) Say hypsurface is reducible if $V = V_1 \cup V_2$, $V_1, V_2 \subsetneq V$.

Proof of the bijections follows verbatim from the proof for curves. ///



Sketch: Study's Lemma extends to <u>projective</u> hypersurfaces.

(a) The factors of a homogeneous polynomial are homogeneous.

(b) F, G hom, F irr, $V_F \subseteq V_G \Rightarrow F \mid G$

(c) Bijections.

Proof: (a): Let F be hom of degree d, so $F = F^{(d)}$. Let $F = q_1 \cdots q_m$ be irreducible fact. with $\deg(q_i) = d_i \geq 1$. We will show that $q_i = q_i^{(d_i)}$. To see this note that

$$\begin{aligned} F^{(d)} &= q_1^{(d_1)} \cdots q_m^{(d_m)}. \\ \| \\ F &= q_1 \cdots q_m \end{aligned}$$

By <u>uniqueness</u> of factorization, each $q_i^{(d_i)}$ is irreducible, hence $q_i^{(d_i)} \sim q_j$ for some j. Hence q_j is homogeneous, hence $q_j = q_j^{(d_j)}$.

[See typed notes for (b) & (c).]