HW2 is up, due next Fri.

Near Future:

- Tangent hyperplanes to hypersurfaces.
- Projective tangent spaces.
- Zariski Tangent spaces.
- Curves: tangents, inflections, Bézout's Theorem.

Medium Future:

- Curves as compact complex manifolds.
- Curves as field extensions of $\mathbb{C}$ of transcendence degree 1.

Projective space in general.

Let $\mathbb{F}$ be a field.

$$\mathbb{F}\mathbb{P}^n := \left(\mathbb{F}^{n+1} \setminus 0\right) \Big/ (\text{nonzero scalars})$$

$(x_1, \dots, x_{n+1}) \sim (x'_1, \dots, x'_{n+1})$

$$\iff \exists \lambda \in \mathbb{F} \setminus 0, \forall i, x'_i = \lambda x_i$$

Let $(x_1 : x_2 : \cdots : x_{n+1})$ denote equivalence classes.

$\mathbb{F}\mathbb{P}^n$ is covered by $n+1$ affine charts

$$U_i = \{ \vec{x} \in \mathbb{F}\mathbb{P}^n : x_i \neq 0 \}$$

$$= \{ (x_1 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_{n+1}) \}$$

$$\leftrightarrow \mathbb{F}^n = \{ (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \}$$

Polynomials do <u>not</u> define functions on proj space. However, if $F[\vec{x}] \in \mathbb{F}[x_1, \dots, x_{n+1}]$ is homog. then for all $\lambda \in \mathbb{F} \setminus 0$ we have

$$F(\vec{x}) = 0 \iff F(\lambda \vec{x}) = 0,$$

hence we obtain a projective hypersurface $V_F : F(\vec{x}) = 0 \subseteq \mathbb{F}\mathbb{P}^n$.

If $F(\vec{x}) = \vec{a} \circ \vec{x}$

$$= a_1 x_1 + \cdots + a_{n+1} x_{n+1}$$

is a linear form, then corresponding hypersurface is <u>projective hyperplane</u>

$$H_{\vec{a}} : a_1 x_1 + \cdots + a_{n+1} x_{n+1} = 0.$$

In particular, $H_i : x_i = 0$ is the complement of ith affine chart

$$H_i = \mathbb{FP}^n \setminus U_i$$

"ith coordinate hyperplane at infinity."

Note that we have a bijection

$$H_{n+1} \longleftrightarrow \mathbb{RP}^{n-1}$$

$$(x_1 : \cdots : x_n : 0) \longleftrightarrow (x_1 : \cdots : x_n)$$

Call this the standard embedding

$$\mathbb{RP}^{n-1} \subseteq \mathbb{RP}^n.$$

More generally, I claim any proj. hyperplane is "projectively equivalent to $\mathbb{RP}^{n-1}$.

Def: The projective linear group $PGL_{n+1}(\mathbb{F})$ consists of $GL_{n+1}(\mathbb{F})$ up to scalar multiplication.

If $F(\vec{x}) \in \mathbb{F}[\vec{x}]$ is homogeneous of degree d, and $A \in GL_{n+1}(\mathbb{F})$ then $G(\vec{x}) = F(A\vec{x})$ is hom. of degree d, and we say

$$V_F \ \& \ V_G$$

are projectively equivalent.

[Remark: Degree is a projective invariant.]

Theorem: Any two hyperplanes (i.e., proj. hypersurfaces of degree 1) are projectively equivalent.

<u>Proof</u>: For any $\vec{a} \in \mathbb{F}^{n+1}$ we will show that $H_{\vec{a}} \approx H_{n+1} = \mathbb{R}\mathbb{P}^{n-1}$.

Indeed, choose $A \in GL_{n+1}(\mathbb{F})$ with $(n+1)$st column $\vec{a}$, so that

$$A\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \vec{a}$$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = A^{-1}\vec{a}$$

Then $H_{A^{-1}\vec{a}} = H_{n+1}$ ✓

In fact, we can choose A to be orthogonal $(A^T A = I)$ so that

$H_{\vec{a}} \approx H_{n+1}$ via "generalized rotation."

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Punchline:

$$\mathbb{F}\mathbb{P}^n = \mathbb{F}^n \cup \text{hyperplane at } \infty.$$

<u>Any</u> hyperplane can be viewed as the hyperplane at ∞,

up to projective equivalence.

How strong is projective equivalence?

Fundamental Theorem of Projective Geometry: Let $\mathbb{R}$ = <u>real</u> numbers.

If $\Phi : \mathbb{RP}^n \to \mathbb{RP}^n$ is bijective on points and "preserves incidence" among subspaces, then

$$\Phi = A \in PGL_{n+1}(\mathbb{R}).$$

So $PGL_{n+1}(\mathbb{R})$ is as general as you could possibly hope for.

[Used the fact that $\mathbb{R}$ has no nontrivial field automorphisms.]

As with curves, tight relationship between affine & projective hyp. surfaces.

Let $F(\vec{x}) \in \mathbb{F}[x_1, \dots, x_{n+1}]$ be hom. of degree d. Define ith dehomogenization:

$$F_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = F(\vec{x})\Big|_{x_i = 1}.$$

non-homogeneous.

If $x_i^m \mid F$ & $x_i^{m+1} \nmid F$ then

F_i has degree $d-m$.

Conversely, if $f \in \mathbb{F}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}]$ of degree d, define ith homogenization:

$$f^i(x_1, \ldots, x_{n+1}) = x_i^d \cdot f\left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right)$$

homog. of degree d.

Observe:

- $(f^i)_i = f$
- F is homog with $x_i^m \mid F$, $x_i^{m+1} \nmid F$ then $(F_i)^i = F/x_i^m$

Exercise: Check.

Geometric Meaning:

Any affine hypersurface $V_f \in U_i$ of degree d in ith affine chart has a unique projective completion $V_f \subseteq V_F \subseteq \mathbb{RP}^n$ of degree d that does not contain the hyperplane at infinity H_i.

[If $V_F \subseteq \mathbb{RP}^n$ does contain H_i the dehomogenization

$$V_{F_i} = V_F \cap U_i$$

forgets this information.]