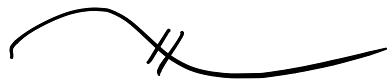
The Zariski Topology:

varieties $\longleftrightarrow$ radical ideals

irr. varieties $\longleftrightarrow$ prime ideals



Last Time: R UFD. Then

minimal primes $\leftrightarrow$ principal primes (irr. hypersurfaces) ✓

Now: Let $\mathbb{F}$ be alg closed.

maximal (prime) ideals $\subseteq \mathbb{F}[x_1, \dots, x_n]$ $\leftrightarrow$ points $\vec{p} \in \mathbb{F}^n$

$$M_{\vec{p}} \longleftrightarrow \vec{p}$$

The Hard Part (Weak NSS).

TFAE:

① $V(I) = \emptyset \Rightarrow I = \mathbb{F}[\vec{x}]$

② Every maximal $M \subseteq \mathbb{F}[\vec{x}]$ has the form $M_{\vec{p}}$ for some point $\vec{p}$.

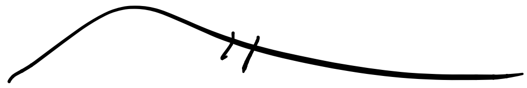
Proof: (1) $\Rightarrow$ (2): Let $M \subsetneq \mathbb{F}[\vec{x}]$ be maximal. Then we know from weak NSS that $V(M) \neq \emptyset$, say $\vec{p} \in V(M)$. But then have

$$M \subseteq M_{\vec{p}}$$

Since $V(M) = \{\vec{p} : f(\vec{p}) = 0 \ \forall f \in M\}$.
By maximality of M, $M = M_{\vec{p}}$. ✓

(2) $\Rightarrow$ (1): Consider any ideal $I \neq \mathbb{F}[\vec{x}]$. We will show $V(I) \neq \emptyset$. Indeed, if I is not maximal $\exists\ I = I_1 \subsetneq I_2$. If I_2 not maximal $\exists\ I = I_1 \subsetneq I_2 \subsetneq I_3$. This must stop because $\mathbb{F}[\vec{x}]$ is Noetherian (by HBT). Thus $\exists$ maximal ideal $M_{\vec{p}} \supseteq I$, which implies $\vec{p} \in V(I)$, i.e., $V(I) \neq \emptyset$.

✓



What is in between points & hypersurfaces in $\mathbb{F}^n$?

Let's deal with the formal structure first. This has nothing to do with polynomials.

Abstract Galois Connection:

Let $(P, \leq)$, $(Q, \leq)$ be partially ordered sets. Pair of maps $* : P \to Q$ & $* : Q \to P$ called "Galois Connection" if

$$\forall p \in P,\ \forall q \in Q,\ p \leq q^* \Leftrightarrow q \leq p^*.$$

(a) $p \leq p^{**}$, $q \leq q^{**}$

(b) $p_1 \leq p_2 \Rightarrow p_2^* \leq p_1^*$

$q_1 \leq q_2 \Rightarrow q_2^* \leq q_1^*$

(c) $p^* = p^{***}$ & $q^* = q^{***}$

(d) $P^* = \{p^* : p \in P\}$, $Q^* = \{q^* : q \in Q\}$.

Then have an anti-isomorphism of posets:

$$*: Q^* \xleftrightarrow{\sim} P^* : *$$

[Bijection reversing order.]

(e) If $(P, \leq, \vee, \wedge)$, $(Q, \leq, \vee, \wedge)$ are <u>lattices</u>, meaning $\{p_i\} \subseteq P$ has

least upper bound $\vee_i p_i \in P$

greatest lower bound $\wedge_i p_i \in P$.

Then we have

$$(\vee_i p_i)^* = \wedge_i p_i^*$$

and similarly for Q. ///

<u>Proof</u>: (a): Let $q = p^*$ so that

$$p^* \leq p^* \Rightarrow q \leq p^*$$

$$\Rightarrow p \leq q^* \Rightarrow p \leq p^{**} \checkmark$$

(b): Let $p_1 \leq p_2$. From (a) we have

$$p_1 \leq p_2 \leq p_2^{**} \Rightarrow (p_1) \leq (p_2^*)^*$$

Hence by definition, $(p_2^*) \le (p_1)^*$ ✓

(c): On one hand since $p^{**} \le p^{**}$ i.e. $(p^{**}) \le (p^*)^*$, then by definition

$(p^*) \le (p^{**})^*$ i.e. $p^* \le p^{***}$.

On the other hand, from (a) we have $p \le p^{**}$ then from (b) we have

$(p^{**})^* \le (p)^*$, i.e., $p^{***} \le p^*$. ✓

(d): First we show $* : Q^* \to P^*$ is bijective. Surjective: Every element of P^* looks like p^* for some $p \in P$.

By (c) have $p^* = p^{***} = (p^{**})^*$ so that $*$ sends $p^{**} = (p^*)^* \in Q^*$ to the element $p^* \in P^*$. ///

Injective: Suppose $\exists\, q_1^*, q_2^* \in Q^*$ that get sent to the same element $q_1^{**} = q_2^{**}$ of P^*.

Then applying (c) again gives

$$(q_1^{**})^* = (q_2^{**})^*$$

$$q_1^* = q_2^* \quad \checkmark$$

Thus we have a bijection

$$* : Q^* \longleftrightarrow P^* : *$$

which reverses order by part (b).

(e): I.O.U. $\square$

Remarks:

- We have already seen an example. Let $\varphi : R \to S$ be a ring hom. Then the pair:

$$\varphi : (2^R, \subseteq) \rightleftarrows (2^S, \supseteq) : \varphi^{-1}$$

is an abstract Galois connection. Indeed for all $X \subseteq R$, $Y \subseteq S$,

$$\varphi[X] \subseteq Y \iff X \subseteq \varphi^{-1}[Y].$$

Check:

$$\varphi[X] \subseteq Y \iff \forall x \in X,\ \varphi(x) \in Y$$
$$\iff X \subseteq \varphi^{-1}[Y].$$

Get an order-preserving bijection of posets:

$$\varphi : \underbrace{\varphi^{-1}[2^S]} \longleftrightarrow \underbrace{\varphi[2^R]} : \varphi^{-1}$$

what are these?

$$\varphi[2^R] = \{\underbrace{\varphi[\varphi^{-1}(J)]}_{= J} : \underset{\text{ideals}}{J \subseteq R}\}$$
$$= \{\text{all ideals}\}$$

$$\varphi^{-1}[2^S] = \{\underbrace{\varphi^{-1}[\varphi(J)]}_{= J + \ker\varphi} : J \subseteq S \text{ ideal}\}$$
$$= \{\text{all ideals} \supseteq \text{kernel}\}.$$

Key Point: For a Galois connection we get a bijection between <u>closed</u>

elements on each side:

$$p = p^{**} \quad \& \quad q = q^{**}$$

The hard part in any specific example is to determine exactly which elements are "closed".