The twisted cubic:

$$C = \{(t, t^2, t^3) : t \in \mathbb{F}\} \subseteq \mathbb{F}^3$$

Last time we proved:

$$C = V(x^2 - y, x^3 - z).$$

$$= V(x^2-y) \cap V(x^3-z).$$

So it's an affine variety.

Moreover, we proved that the ideal $I = (x^2-y, x^3-z)$ is prime, hence C is an irreducible variety.

Today we study the projective closure of C.

Let $\mathbb{F}^3 = \{(x,y,z)\}$

$$\subseteq \{(w:x:y:z)\} \subseteq \mathbb{F}\mathbb{P}^3$$

Claim: The closure $\overline{C} \subseteq \mathbb{F}\mathbb{P}^3$ is equal to the set

$$T = \{(s^3 : s^2t : st^2 : t^3)\} \subseteq \mathbb{F}\mathbb{P}^3$$

Proof: Equivalent to show that

$T = \{(s^3, s^2t, st^2, t^3) : s, t \in \mathbb{F}\} \subseteq \mathbb{F}^4$

is the smallest conical variety containing the set

$$C = \{(1, t, t^2, t^3) : t \in \mathbb{F}\}$$

First we observe that T is conical:

For any λ we can write $\lambda = \omega^3$, so

$$\lambda(s^3, s^2t, st^2, t^3)$$

$$= ((\omega s)^3, (\omega s)^2(\omega t), (\omega s)(\omega t)^2, (\omega t)^3)$$

$$\in T. \quad \checkmark$$

Recall: $\overline{C}$ is the Zariski closure of the Cone $(C) = \{(\lambda, \lambda t, \lambda t^2, \lambda t^3) : \lambda, t \in \mathbb{F}\}$. We will show that $T \subseteq \overline{C}$. To see this, consider any point $(s^3, s^2t, st^2, t^3) \in T$ with $s \neq 0$, so that

$$(s^3, s^2t, st^2, t^3)$$

$$= s^3\left(1, \frac{t}{s}, \left(\frac{t}{s}\right)^2, \left(\frac{t}{s}\right)^3\right) \in \mathrm{Cone}(C).$$

If we can show that T is a variety, then by minimality of $\overline{C}$ we will get $T = \overline{C}$.

Indeed, I claim that

$$T = V(\overbrace{x^2 - wy,\ xy - wz,\ xz - y^2}^{J}).$$

$$= V(x^2 - wy) \cap V(xy - wz) \cap V(xz - y^2)$$

To see this let $(w, x, y, z) = (s^3, s^2t, st^2, t^3)$ be in T. Then we have

$$x^2 - wy = (s^2t)^2 - (s^3)(st^2) = 0$$
$$xy - wz = (s^2t)(st^2) - (s^3)(t^3) = 0$$
$$xz - y^2 = (s^2t)(t^3) - (st^2)^2 = 0$$

Conversely, let $(a, b, c, d) \neq (0, 0, 0, 0)$ be in the intersection of the surfaces, so $b^2 = ac$, $bc = ad$, $bd = c^2$.

We must have $a \neq 0$ or $d \neq 0$.

By symmetry ($a \leftrightarrow d$, $b \leftrightarrow c$) we may assume $d \neq 0$. Now two cases:

(1) $a = b = c = 0$

(2) $a, b, c \neq 0$.

(1): Let $d = t^3$. Then $s = 0$ gives

$$(a,b,c,d) = (0,0,0,d)$$
$$= (s^3, s^2t, st^2, t^3) \in T \quad \checkmark$$

(2): We have $a/b = b/c = c/d$.

Let $d = t^3$ & $s = (a/b)t$, so

$$(a,b,c,d) = (t^3, s^2t, st^2, t^3) \in T \quad \checkmark$$

Corollary: This ideal satisfies

$$J \subseteq I(\overline{C}).$$

Proof: We showed that

$V(J) = \overline{C}$, hence we have

$$J \subseteq IV(J) = I(\overline{C}). \quad ///$$

Question: $J = IV(J) = \sqrt{J}$?

<u>Yes</u>, but you might not like the proof . . .

An elementary proof uses the machinery of Gröbner bases. (see Cox-Little-O'Shea, pg. 389)

A conceptual proof uses the theory of Hilbert functions, which we don't have. So our proof is a little bit ad hoc.

Claim. $I(\overline{C}) \subseteq J$.

Proof: Consider any $f \in I(\overline{C})$, so

$$f(s^3, s^2t, st^2, t^3) = 0 \quad \forall\, s, t \in \mathbb{F}.$$

We want to show $f \in J$, i.e.,

$$f = (x^2 - wy)? + (xy - wz)? + (xz - y^2)?$$

First divide by $x^2 - wy$ with respect to x to get

$$f = (x^2 - wy) f' + x \overset{?}{p(w,y,z)} + \overset{?}{q(w,y,z)}$$

✓

Now divide p & q by some polynomial in $\mathbb{F}[w,y,z] \cap J$. We use the auxiliary polynomial

$$y^3 - wz = z(xy - wz) - y(xz - y^2) \in J.$$

Divide p & q to get

$$p = (y^3 - wz) p' + y^2 p_1(w,z) + y p_2(w,z) + p_3(w,z)$$
$$q = (y^3 - wz) q' + y^2 q_1(w,z) + y q_2(w,z) + q_3(w,z).$$

Substitute $(w,x,y,z) = (s^3, s^2t, st^2, t^3)$ to get ...

Look at typed notes $\square$

Summary:

$$I\{(t,t^2,t^3)\} = I(C) = (x^2-y,\ x^3-z)\ (= I)$$

$$I\{(s^3,s^2t,st^2,t^3)\} = I(\overline{C}) = (x^2-wy,\ xy-wz,\ xz-y^2)\ (= J)$$

Corollaries:

- Let $I^* \subseteq \mathbb{F}[w,x,y,z]$ be homogenization of I. We know $I^* = I(\overline{C})$, hence $I^* = J$.
- Since proj closure of irreducible varieties are irreducible we conclude that J is in fact <u>prime</u>.

The Rational Normal Curve:

$$C = \{(t,t^2,\dots,t^n) : t \in \mathbb{F}\} \subseteq \mathbb{F}^n$$

This is an irreducible variety with

$$I(C) = (x_2^2 - x_1,\ x_3^3 - x_1,\ \dots,\ x_n^n - x_1).$$

The projective closure is equal to

$$\overline{C} = \{(s^n, s^{n-1}t, \ldots, st^{n-1}, t^n) : s,t \in \mathbb{F}\} \subseteq \mathbb{F}^{n+1},$$

which is also an irreducible variety.

The ideal of $\overline{C}$ is

$$I(\overline{C}) = \left\langle 2\times 2 \text{ minors} \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \right\rangle$$

this is the only part that doesn't follow from the above proofs.