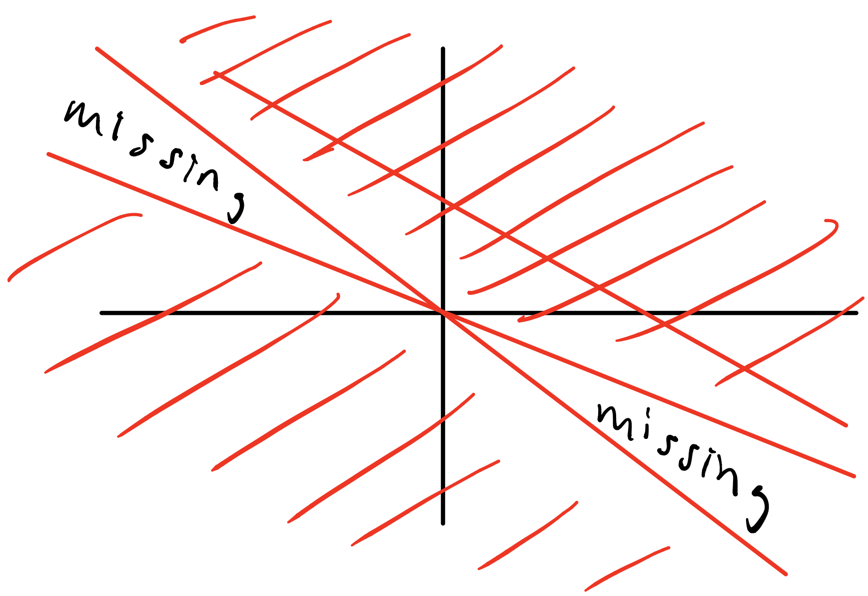
Last time:

Given variety $V \subseteq \mathbb{F}^n$, let

$$\mathrm{Cone}(V) = \{ \lambda \vec{p} : \vec{p} \in V, \lambda \in \mathbb{F} \}.$$

This is not necessarily a variety.

Example: Cone over a line that does not contain $\vec{0}$:



Claim: The plane minus a line is not Zariski closed. More generally, I claim that any Zariski-open set $\mathcal{U} \subseteq \mathbb{F}^n$ is dense, i.e.,

$$VI(\mathcal{U}) = \mathbb{F}^n.$$

Proof: Given $f \in \mathbb{F}[\vec{x}]$, consider the open set $U_f = \mathbb{F}^n \setminus V_f$, the complement of the hypersurface V_f.

I claim that $VI(U_f) = \mathbb{F}^n$.

Indeed, suppose $g \in I(U_f)$ so that

$\vec{p} \in U_f \Rightarrow g(\vec{p}) = 0$, i.e.,

$f(\vec{p}) \neq 0 \Rightarrow g(\vec{p}) = 0$.

We will show that $g(\vec{x})$ is the zero polynomial, hence $I(U_f) = 0$ and $VI(U_f) = V(0) = \mathbb{F}^n$.

Since $\forall \vec{p} \in \mathbb{F}^n$, $f(\vec{p}) \neq 0 \Rightarrow g(\vec{p}) = 0$ we have $f(\vec{p})g(\vec{p}) = 0$. Let $h(\vec{x}) := f(\vec{x})g(\vec{x})$. Since $h(\vec{p}) = 0$ for all $\vec{p}$ & $\mathbb{F}$ is infinite we have $h(\vec{x}) = 0$. Finally, since $\mathbb{F}[\vec{x}]$ is a domain and $f(\vec{x}) \neq 0$, we conclude that $g(\vec{x}) = 0$. ///

Now let $U = \mathbb{F}^n \setminus V$ be any open set, i.e., complement of a closed set. Let $f \in I(V)$ so that $V \subseteq V_f$, hence $U_f \subseteq U$. It follows that

$$\mathbb{F}^n = VI(U_f) \subseteq VI(U) \subseteq \mathbb{F}^n;$$

hence $VI(U) = \mathbb{F}^n$. ///

The projective completion:

Let $V \subseteq \mathbb{F}^n \subseteq \mathbb{F}\mathbb{P}^n$ be affine variety in chart $\mathbb{F}^n$. Think of $\mathbb{F}^n$ as $(p_1, p_2, \ldots, p_n, 1) \subseteq \mathbb{F}^{n+1}$

Then we define:

$$\overline{V} = VI(\mathrm{Cone}(V)) \subseteq \mathbb{F}^{n+1}$$

Theorem: If $V = V(I)$ for some $I \subseteq \mathbb{F}[x_1, \dots, x_n]$, then

$$\overline{V} = V(I^*), \text{ where}$$

$$I^* = \langle \underbrace{\{f^* : f \in I\}}_{\substack{\text{ideal generated by} \\ \text{homogenizations of} \\ \text{elements of } I}} \rangle \subseteq \mathbb{F}[x_1, \dots, x_{n+1}].$$

///

Examples:

- Projective completion of hypersurface.

Let $\vec{x}' = (x_1, \dots, x_n)$, $\vec{x} = (x_1, \dots, x_{n+1})$.

Given square-free $f \in \mathbb{F}[\vec{x}']$,

$$I = I(V_f) = f\,\mathbb{F}[\vec{x}'].$$

Hence

$$\overline{V_f} = V(I^*).$$

$$= \langle (fg)^* : g \in \mathbb{F}[\vec{x}'] \rangle.$$

$$= \langle f^* g^* : g \in \mathbb{F}[\vec{x}'] \rangle$$

$$= f^* \mathbb{F}[\vec{x}].$$

Hence $\overline{V_f}$ is the projective hypersurface V_{f^*}.

- Projective completion of point.

$$\vec{p} = (p_1, \dots, p_n) \in \mathbb{F}^n.$$

$$I = I(\vec{p}) = M_{\vec{p}} = \sum_i (x_i - p_i)\mathbb{F}[\vec{x}']$$

For geometric reasons:

$$\overline{\{\vec{p}\}} = \{(\lambda p_1 : \lambda p_2 : \cdots : \lambda p_n : \lambda)\}$$

We saw that the homogeneous ideal of this line is

$$I^* = \sum_{i<j\le n} (x_i p_j - x_j p_i)\, \mathbb{F}[\vec{x}]$$

$$+ \sum_i (x_i - x_{n+1} p_i)\, \mathbb{F}[\vec{x}]$$

$$= \sum_i (x_i - x_{n+1} p_i)\, \mathbb{F}[\vec{x}].$$

Note that we obtained I^* by homogenizing the generators of I:

$$x_i - p_i \longmapsto x_i - p_i x_{n+1}.$$

Warning: Usually I^* is not generated by homogenizations of generators of I.

Example: The twisted cubic.

$$C := \{(t, t^2, t^3) : t \in \mathbb{F}\} \subseteq \mathbb{F}^3.$$

Intuitively, this is a one-dimensional curve in three-dimensional space. But the algebra behind this is not so clear.

Claim: C is a variety.

Proof: Define the ideal

$$I = (x^2 - y)\mathbb{F}[x,y,z] + (x^3 - z)\mathbb{F}[x,y,z].$$

Claim that $C = V(I)$. To see this let $(t, t^2, t^3) \in C$. Then for any $f = (x^2-y)g + (x^3-z)h \in I$ we have

$$f(t,t^2,t^3) = (t^2-t^2)g(t,t^2,t^3) + (t^3-t^3)h(t,t^2,t^3) = 0.$$

Hence $C \subseteq V(I)$. Conversely, if $(a,b,c) \in V(I)$ then since x^2-y, $x^3-z \in I$, have $a^2-b=0$, $a^3-c=0$.

Hence $(a,b,c) = (a, a^2, a^3) \in C$. ///

Question: $I(C) = I$?

Claim: I is prime, hence $C = V(I)$ is irreducible.

Proof: Consider $f, g \in \mathbb{F}[x,y,z]$ with $fg \in I$. We will show that

$$f \in I \text{ or } g \in I.$$

Divide f by $x^3 - z$ in $\mathbb{F}[x,y][z]$:

$$f = (x^3 - z) f_1(x,y,z) + r(x,y,z),$$

where $\deg_z(r) < \deg_z(x^3 - z) = 1$

$$\Longrightarrow \deg_z(r) = 0 \Longrightarrow r(x,y,z) = r(x,y).$$

Divide $r(x,y)$ by $x^2 - y$ in $\mathbb{F}[x][y]$:

$$r = (x^2 - y) f_2(x,y) + f_3(x,y).$$

with $\deg_y(f_3) < \deg_y(x^2 - y) = 1$.

Hence $f_3(x,y) = f_3(x) \in \mathbb{F}[x]$.

Apply same argument to g:

$$f = (x^3 - z) f_1(x,y,z) + (x^2 - y) f_2(x,y) + f_3(x)$$
$$g = (x^3 - z) g_1(x,y,z) + (x^2 - y) g_2(x,y) + g_3(x).$$

Substitute (t, t^2, t^3):

$$f(t, t^2, t^3) = f_3(t)$$
$$g(t, t^2, t^3) = g_3(t)$$

Define $h(x) = f_3(x) g_3(x) \in \mathbb{F}[x]$.

Since $fg \in I$, have

$$\begin{aligned} h(t) &= f_3(t) g_3(t) \\ &= f(t, t^2, t^3) g(t, t^2, t^3) \\ &= (fg)(t, t^2, t^3) = 0 \end{aligned}$$

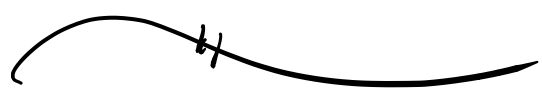
Since $h(x)$ has infinitely many roots $t \in \mathbb{F}$, have $h(x) = 0$

$$\Longrightarrow f_3(x) = 0 \text{ or } g_3(x) = 0.$$

Corollary: The ideal I is radical, hence $I(C) = IV(I) = I$.

So we understand the ring theory of the curve C.

C is an intersection of 2 surfaces, which is what you would expect for a 1D curve. Moreover, the ideal $I(C)$ is generated by minimal polynomials for these surfaces.



Next time: The projective closure

$$\overline{C} = \{(s^3 : s^2 t : s t^2 : t^3)\}$$

is more complicated!