What is a "projective variety"?

Based on affine varieties we expect

proj. variety = finite intersection of proj. hypersurfaces.

Based on Study's Lemma, we expect a tight relationship between affine & projective varieties, i.e., there should be homogenization & de-homogenization functions.

To warm up: Try to find the ideal of a projective point $\vec{p} \in \mathbb{F}\mathbb{P}^n$.

Let $\vec{p} = (p_1 : p_2 : \cdots : p_{n+1}) \in \mathbb{F}\mathbb{P}^n$

This corresponds to the line

$$L = t(p_1, p_2, \ldots, p_{n+1}) \subseteq \mathbb{F}^{n+1},$$

which can be expressed (non-uniquely) as an intersection of n linear hyperplanes:

$$L = H_{p_1x_2 - p_2x_1} \cap \cdots \cap H_{p_nx_{n+1} - p_{n+1}x_n}.$$

Observe

$$H_{p_ix_{i+1} - p_{i+1}x_i} = \{ \vec{x} : p_i x_{i+1} = p_{i+1} x_i \}$$
$$= \{ \vec{x} : x_i / x_{i+1} = p_i / p_{i+1} \}.$$

Therefore, the ideal $\subseteq \mathbb{F}[x_1, \ldots, x_{n+1}]$ of the line is

$$I(L) = \sum_{i=1}^{n} (p_i x_{i+1} - p_{i+1} x_i) \mathbb{F}[\vec{x}]$$

This is <u>not</u> a maximal ideal.

Important observation: The ideal $I(L)$ is generated by homogeneous polynomials.

Homogeneous Ideals:

Let $I \leq \mathbb{F}[\vec{x}]$ be an ideal. TFAE:

1) If $f \in I$ then $f^{(k)} \in I \;\forall k$.

2) I is generated by (finitely many) homogeneous polynomials.

When these hold call I "homogeneous ideal."

Proof: (1) $\Rightarrow$ (2). From HBT, know

$$I = f_1 \mathbb{F}[\vec{x}] + \cdots + f_m \mathbb{F}[\vec{x}].$$

From (1) we know that $f_i^{(k)} \in I \ \forall i, k$.

I claim that

$$I = \sum_{i,k} f_i^{(k)} \mathbb{F}[\vec{x}].$$

$\supseteq$: Since $f_i^{(k)} \in I$, the ideal they generate is contained in I

$\subseteq$: Let $f \in I$, so $f = \sum_i f_i g_i$. Then

$$f = \sum_i \Big(\sum_k f_i^{(k)} \Big) g_i$$

$$= \sum_{i,k} f_i^{(k)} g_i.$$

(2) $\Rightarrow$ (1): Suppose $I = F_1 \mathbb{F}[\vec{x}] + \cdots + F_m \mathbb{F}[\vec{x}]$ where F_i is hogeneous of degree d_i.

Consider any $f \in I$. We want to show that $f^{(k)} \in I \ \forall k$.

By hypothesis, $f = \sum F_i g_i$ for some $g_i \in \mathbb{F}[\vec{x}]$, so that

$$\begin{aligned} f &= \sum F_i g_i \\ &= \sum_i F_i \Big(\sum_l g_i^{(l)} \Big) \\ &= \sum_{i,l} F_i g_i^{(l)} . \end{aligned}$$

Note: $F_i g_i^{(l)}$ is homogeneous of degree $d_i + l$, so $f^{(k)}$ satisfies

$$f^{(k)} = \sum_{\substack{i,l \\ d_i + l = k}} F_i g_i^{(l)} \in I . \quad \checkmark$$

[The sum might be empty: $f^{(k)} = 0 \in I$.]

Corollary: The homogeneous ideals of $\mathbb{F}[\vec{x}]$ are closed under sums (by 2) and intersections (by 1), hence form a lattice under inclusion.

Projective Zariski Topology.

Let $\mathbb{F}$ be algebraically closed.

Already know

$$\begin{array}{ccc} \text{proj subspaces in } \mathbb{FP}^n & \longleftrightarrow & \text{lin. subspaces in } \mathbb{F}^{n+1} \\ \cap & & \cap \\ \text{subsets of } \mathbb{FP}^n & \longleftrightarrow & \text{non-empty conical sets } \mathbb{F}^{n+1} \end{array}$$

"conical sets" are closed under scalar multiplication. By convention,

$\emptyset \subseteq \mathbb{FP}^n \rightsquigarrow \{\vec{0}\} \subseteq \mathbb{F}^{n+1}$.

But the empty conical set does not correspond to any subset of $\mathbb{FP}^n$.

We should look for a Galois connection:

$$I : \begin{pmatrix}\text{non-empty conical} \\ \text{sets} \subseteq \mathbb{F}^{n+1}\end{pmatrix} \rightleftarrows \begin{pmatrix}\text{some kind} \\ \text{of ideals} \subseteq \mathbb{F}[\vec{x}]\end{pmatrix} : V$$

Theorem (Weak Nullstellensatz):

- $S \subseteq \mathbb{F}^{n+1}$ conical $\Rightarrow I(S)$ homogeneous

- $I \subseteq \mathbb{F}[\vec{x}]$ homogeneous $\Rightarrow V(I)$ conical.

Proof: Let S be conical, consider $I = I(S)$.
If $f \in I$ we will show $f^{(k)} \in I \ \forall k$.
To show this, consider any point $\vec{p} \in S$
so that $\lambda \vec{p} \in S$ for all λ &

$$f(\lambda \vec{p}) = 0 \text{ for all } \lambda.$$

Then

$$0 = f(\lambda \vec{p}) = \sum f^{(k)}(\lambda \vec{p})$$

$$= \sum \lambda^k f^{(k)}(\vec{p})$$

Consider the polynomial

$$g(y) := \sum y^k f^{(k)}(\vec{p}) \in \mathbb{F}[y].$$

Since this polynomial has ∞ roots
it must be the zero polynomial

$$\Longrightarrow f^{(k)}(\vec{p}) = 0 \quad \forall k.$$

Since this holds $\forall \vec{p} \in S$,

$$f^{(k)} \in I \ \forall k.$$

✓

Conversely let $I \neq \mathbb{F}[\vec{x}]$ be homogeneous and consider $V = V(I)$. We will show that V is conical. Indeed, we have

$$I = F_1 \mathbb{F}[\vec{x}] + \cdots + F_m \mathbb{F}[\vec{x}]$$

for some homogeneous F_i.

Let $\vec{p} \in V$, so that $F_i(\vec{p}) = 0 \ \forall i$.

Then for any $\lambda \vec{p}$ we have

$$F_i(\lambda \vec{p}) = \lambda^{\deg F_i} F_i(\vec{p}) = 0.$$

And for any $f = \sum F_i g_i \in I$ we have

$$f(\lambda \vec{p}) = \sum F_i(\lambda \vec{p}) g_i(\lambda \vec{p}) = 0.$$

Hence $\lambda \vec{p} \in V$ as desired. ✓

We have a Galois connection:

$$I : \begin{pmatrix} \text{non-empty} \\ \text{conical sets} \end{pmatrix} \rightleftarrows \begin{pmatrix} \text{non-unit} \\ \text{hom. ideals} \end{pmatrix} : V$$

We get some free results:

- Zariski closure of conical set is conical.
- Conical variety is $\cap$ finitely many conical hypersurfaces.

[Proj variety = $\cap$ finitely many proj. hypersurfaces.]

- Unique union of irreducible proj. varieties.
- Radical closure of hom. ideal is hom.
- Radical hom. ideal is unique intersection of finitely many prime hom. ideals.

$$I : \begin{pmatrix}\text{non-empty} \\ \text{conical varieties}\end{pmatrix} \xleftrightarrow{\sim} \begin{pmatrix}\text{non-unit} \\ \text{hom. rad. ideals}\end{pmatrix} : V$$

Important Observation:

Unique minimal non-empty conical variety $= \{\vec{0}\}$ $\longleftrightarrow$ unique maximal radical homo. ideal $M_{\vec{0}}$.

$M_{\vec{0}} \subseteq \mathbb{F}[x_1, \ldots, x_{n+1}]$ is called the "irrelevant ideal" because it corresponds to the empty set $\emptyset \subseteq \mathbb{FP}^n$.

The whole ring $\mathbb{F}[\vec{x}]$ does not correspond to a subset of $\mathbb{FP}^n$.