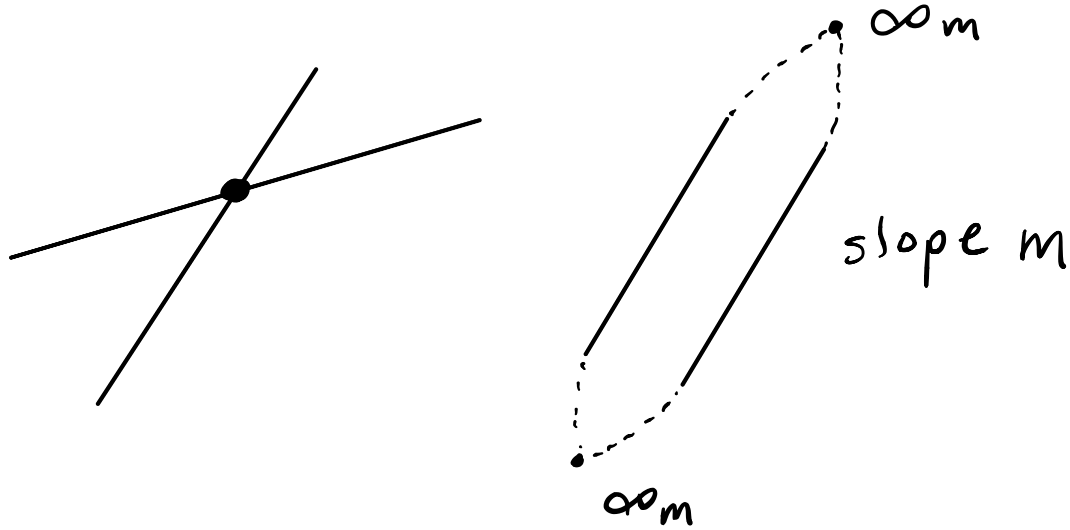
Points at infinity via "Homogeneous coordinates"

Idea: Points at infinity correspond to slopes $m \in \mathbb{R} \cup \{\infty\}$. Any two non-equal lines meet at a unique point:



To be precise:

Let $\mathbb{RP}^2 := \left(\mathbb{R}^3 \setminus \{0\}\right) \big/ \text{nonzero scalars.}$

$$(x,y,z) \sim (x',y',z') \iff \begin{cases} x' = \lambda x \\ y' = \lambda y \\ z' = \lambda z \end{cases} \quad \lambda \neq 0$$

$(x:y:z)$ = equivalence class of (x,y,z)

Finite points:

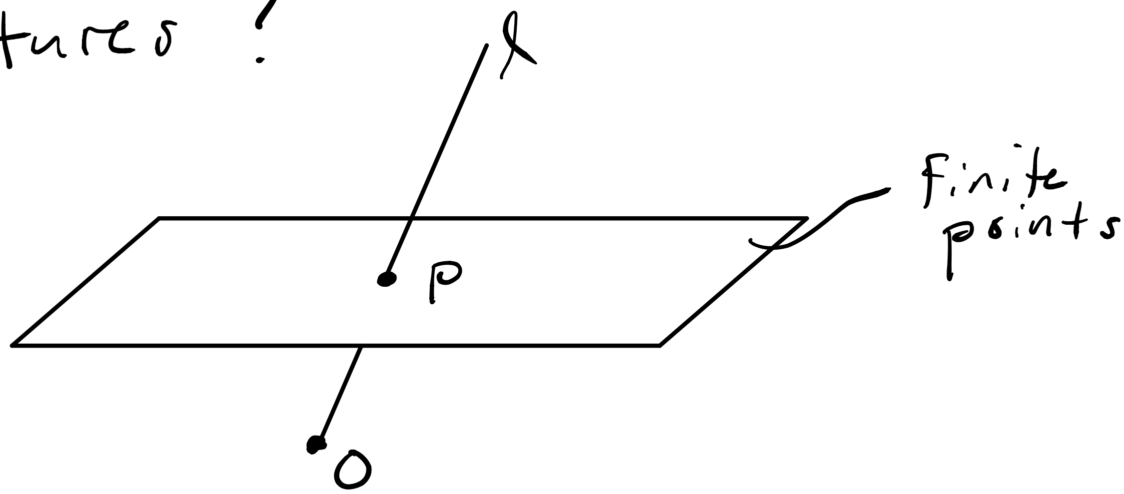
$$(x:y:1) \longleftrightarrow (x,y)$$

Points at infinity:

$$(x:y:0) \longleftrightarrow \text{slopes } y/x.$$

$$(0:1:0) \longleftrightarrow \text{infinite slope}$$

Pictures?



Finite points = non horizontal lines through O.

Infinite points = horizontal lines through O.

i.e., $\mathbb{RP}^2$ = lines through a fixed point in $\mathbb{R}^3$

Curves in projective plane:

Say polynomial $F(x,y,z) \in \mathbb{R}[x,y,z]$ is homogeneous of degree d if

$$F(\lambda x, \lambda y, \lambda z) = \lambda^d F(x,y,z)$$

for all $\lambda \neq 0$. In this case, the equation

$$F(x,y,z) = 0$$

preserves equivalence, hence defines a subset $C_F \subseteq \mathbb{RP}^2$.

Setting $z=1$ gives

$$F(x,y,1) = 0,$$

which is the curve $f(x,y) = 0$ in the affine plane $\mathbb{R}^2$ where

$$f(x,y) := F(x,y,1)$$

is called the de-homogenization of F at $z=1$.

Thus we have $C_f \subseteq C_F$, and the points $C_F \setminus C_f$ are called the points at infinity of C_f.

Conversely, given $f(x,y) \in \mathbb{R}[x,y]$ of degree d, we define the "homogenization" by

$$F(x,y,z) := \underbrace{z^d f\left(\frac{x}{z}, \frac{y}{z}\right)}_{\substack{\text{actually a polynomial,} \\ \text{(and homogeneous)}}}.$$

Then $f(x,y) = 0$

$\Longleftrightarrow F(x,y,1) = 0$ finite points.

And the points at infinity $(x:y:0)$ of the curve $f(x,y) = 0$ are defined by

$$F(x,y,0) = 0.$$

Examples:

- Hyperbola $f(x,y) = x^2 - y^2 - 1 = 0$.

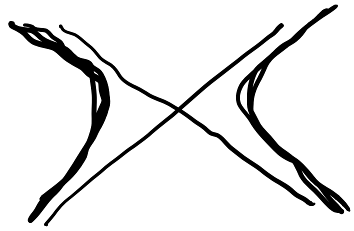
Degree is 2, so

$$F(x,y,z) = z^2 f\left(\tfrac{x}{z}, \tfrac{y}{z}\right)$$
$$= z^2\left[\left(\tfrac{x}{z}\right)^2 - \left(\tfrac{y}{z}\right)^2 - 1\right]$$
$$= x^2 - y^2 - z^2.$$

Points at ∞ are the roots of

$$F(x:y:0) = 0$$
$$x^2 - y^2 = 0$$
$$(x-y)(x+y) = 0.$$

There are two points at infinity corresponding to slopes $y/x = \pm 1$.



- Parabola $f(x,y) = x^2 - y = 0$.

$$F(x,y,z) = z^2\left(\left(\frac{x}{z}\right)^2 - \left(\frac{y}{z}\right)\right)$$

$$= x^2 - yz.$$

Points at ∞:

$$F(x:y:0) = 0$$

$$x^2 = 0$$

Hence $(0:1:0)$ (vertical slope) is a <u>double point</u> at ∞.

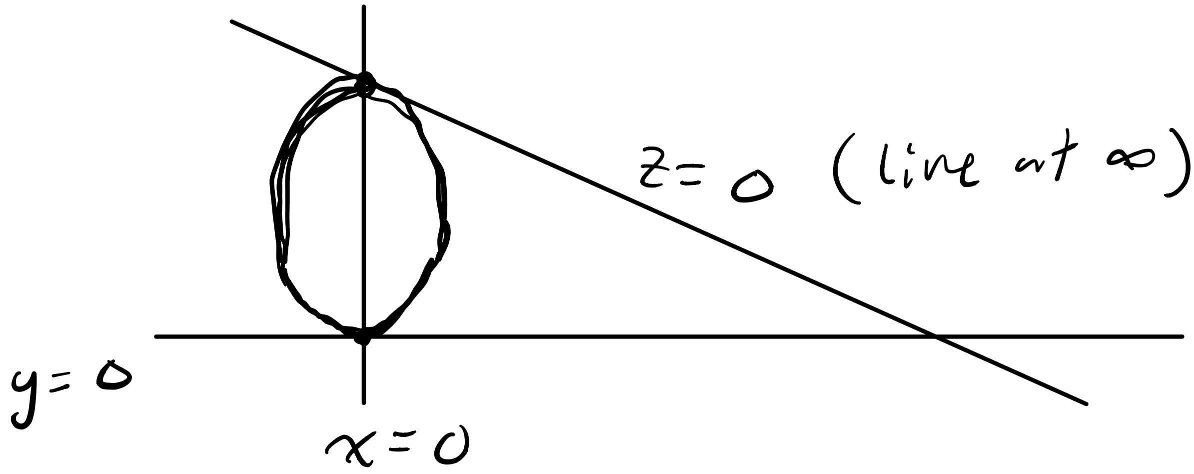
Recall: Line at ∞ is $z=0$.

Instead, de-homogenize at $y=1$ to get the curve

$$x^2 - z = 0.$$

Meaning: The curve is tangent to the line $z=0$.

Often you will see the following:



What does it mean?

$$\mathbb{RP}^2 = \mathbb{R}^3 \setminus \{0\} \Big/ \text{scalars}$$

$$= \text{points of unit sphere } S^2 \subseteq \mathbb{R}^3 \Big/ \text{antipodal map.}$$

Visualize the curve $F(x,y,z) = 0$ in $\mathbb{RP}^2$ by intersecting the surface in $\mathbb{R}^3$ (a cone through the origin) with the surface S^2.

LOOK AT MAPLE.