Last time: Various characterizations of the genus g of $f(x,y) \in \mathbb{Z}[x,y]$.

All of the theorems included "points at infinity" so we need to define those.

Equivalence of curves.

Given $f(x,y), g(x,y) \in \mathbb{Z}[x,y]$ when do we say that $C_f(\mathbb{R})$ & $C_g(\mathbb{R})$ are equivalent?

Most basic: $f = (x+y)^2 = 0$

$g = (x+y) = 0$

should be equivalent, to the line $x = -y$.

We should also allow translations

$$f(x,y) = g(x+r, y+t)$$

for some $(r,t) \in \mathbb{R}^2$ we will say $C_f(\mathbb{R})$ & $C_g(\mathbb{R})$ are equivalent.

Also allow rotations & reflections:

$$f(x,y) = g\big(x\cos\theta \mp y\sin\theta + r,\ x\sin\theta \pm y\cos\theta + t\big)$$

Rigid motions = symmetries of the Euclidean plane.

Example / Application:

General Quadric

$$f(x,y) = \alpha x^2 + \beta xy + \gamma y^2 + \delta x + \varepsilon y + \lambda = 0$$

If nondegenerate $(\beta^2 - 4\alpha\gamma \neq 0)$ then equivalent under rotation and translation to

$$g(u,v) = au^2 \pm bv^2 \pm 1 = 0.$$

(i.e., ellipse, hyperbola, $\emptyset$)

Example: $uv = 1$ equivalent under rotation by 45°

$$(u,v) = \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$$

to $$\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) = 1$$

$$x^2 - y^2 = 2. \quad ///$$

Issue: Rotation does not preserve rational points $\mathbb{Q}^2 \subseteq \mathbb{R}^2$.

For this reason we allow slightly more general transformations.

Define an <u>affine transformation</u>:

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} r \\ t \end{pmatrix}.$$

$$F(\vec{x}) = A\vec{x} + \vec{r}.$$

Invertible $\iff$ A invertible $(ad - bc \neq 0)$

in which case,

$$F^{-1}(\vec{x}) = A^{-1}\vec{x} - A^{-1}\vec{r}$$

Check: $A(A^{-1}\vec{x} - A^{-1}\vec{r}) + \vec{r}$

$$= \vec{x} - \vec{r} + \vec{r} = \vec{x} \checkmark$$

We say $f(x,y) = 0$ & $g(x,y) = 0$ are "affinely equivalent" if

$$f(x,y) = g(ax+by+r,\ cx+dy+t)$$
$$= g(u,v)$$

where $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} r \\ t \end{pmatrix}$

$$ad - bc \neq 0.$$

Remark: This enough to "diagonalize qudrics" over $\mathbb{Q}$.
[Still not over $\mathbb{Z}$ ☹]

Beyond this, what kind of equivalence should we allow?

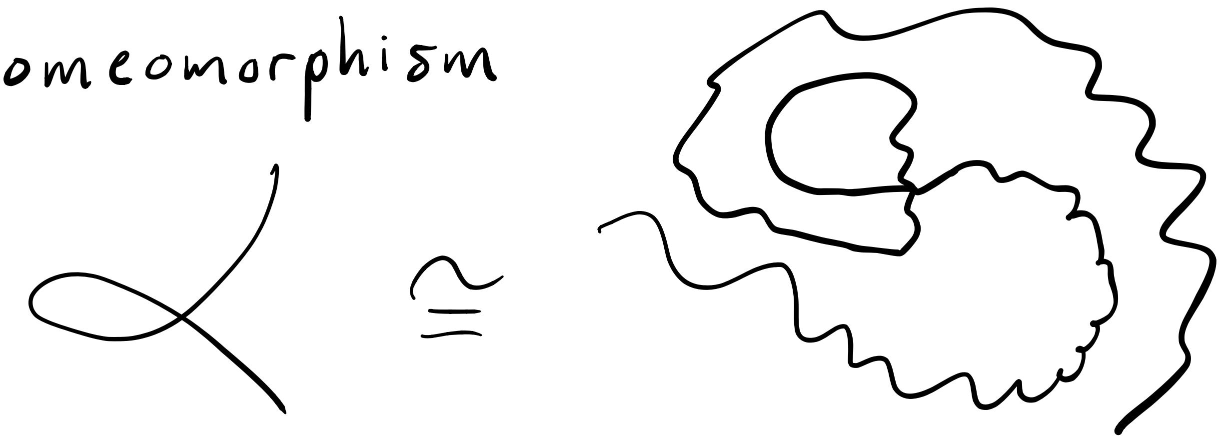
Depends what you want to do ...

Ideas: Say $C_f(\mathbb{R}) \cong C_g(\mathbb{R})$ if

$F(C_f(\mathbb{R})) = C_g(\mathbb{R})$ for some function

$F: \mathbb{R}^2 \to \mathbb{R}^2$ that is

- homeomorphism



Doesn't preserve algebra!

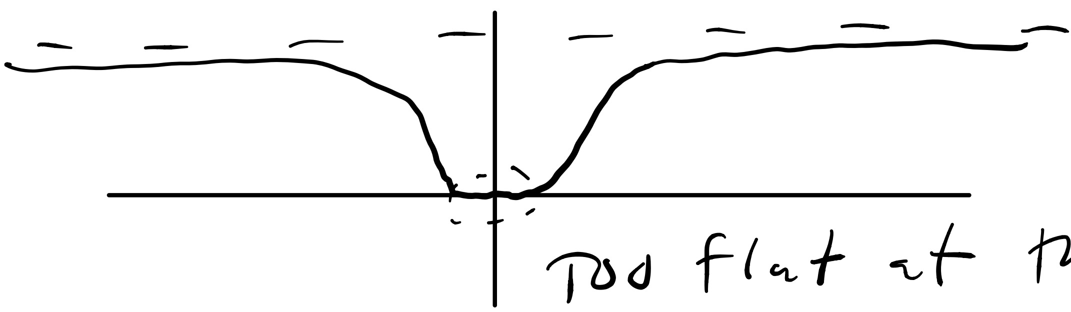
- diffeomorphism of class C^k
  (all kth partial derivatives are continuous)

- smooth diffeomorhism C^∞

- real analytic isomorphism
  (power series converge locally).

Eg: $y = e^{-1/x^2}$ is smooth but not real analytic because Taylor series at $x = 0$ is

$$y = 0 + 0x + 0x^2 + 0x^3 + \cdots$$



Too flat at the origin.

Each of these is more restrictive (more "rigid") than the previous, but still not rigid enough for us!

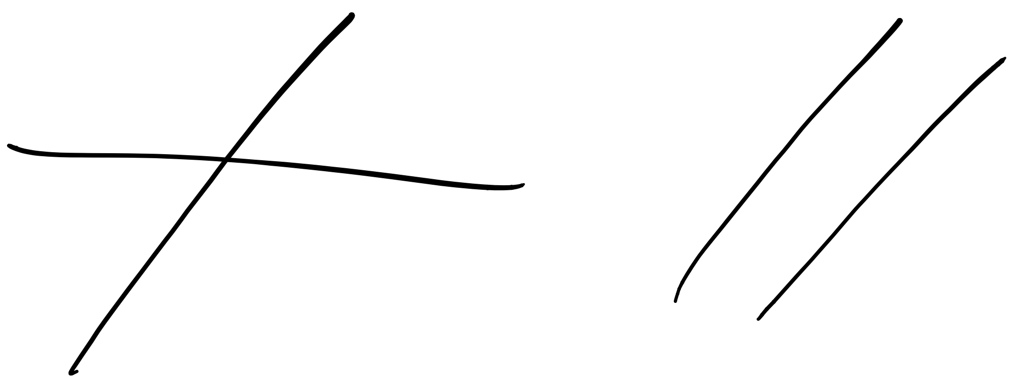
We need transformations that preserve the property of "being defined by a polynomial."

This is harder than it might seem, so I can't define it today.

First step, historically, is adding points at infinity.

[Early 1800s : Poncelet, Plücker, etc..]

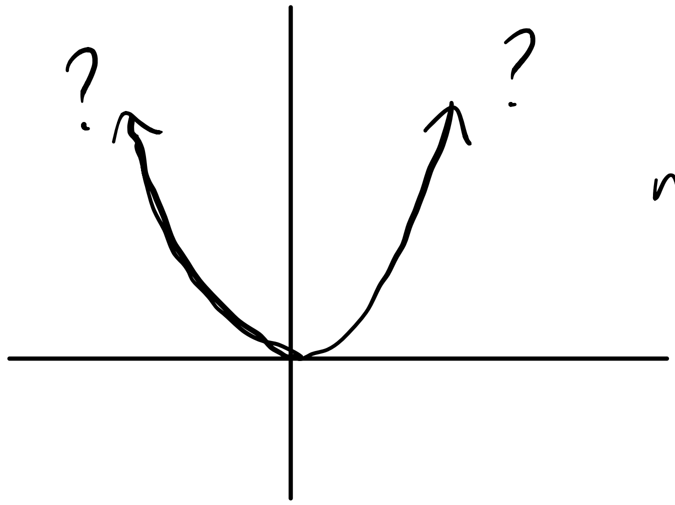
Idea: Points at ∞ correspond to slopes. Two lines of slope m meet at the point "∞_m".



any two lines meet at a <u>unique</u> point.

Example: A hyperbola is connected!

What about a parabola?



Need to be more precise...

Define $\mathbb{RP}^2 := \mathbb{R}^3 / \text{nonzero scalars}$.

$(x, y, z) \sim (x', y', z')$ if and only if

$$\begin{aligned} x' &= \lambda x \\ y' &= \lambda y \\ z' &= \lambda z \end{aligned} \quad \text{for some } \lambda \neq 0.$$

Denote equivalence class by $(x : y : z)$.

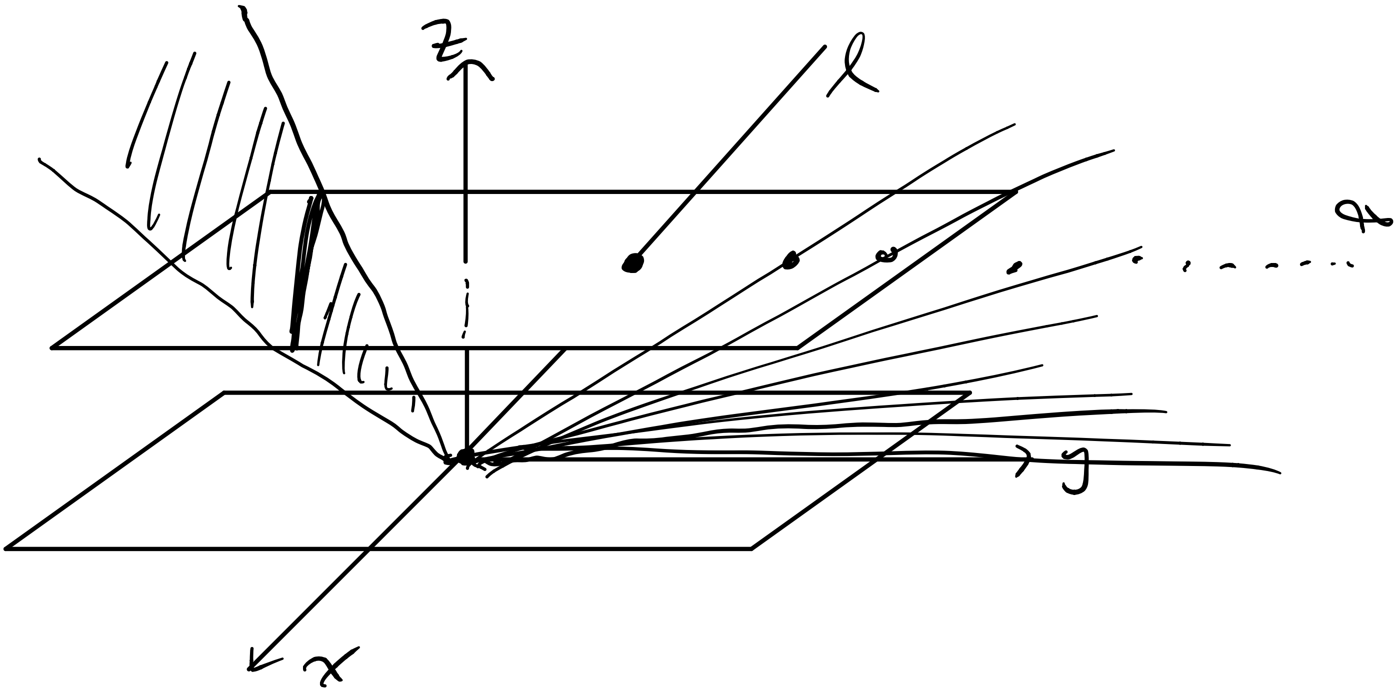
Why is this called a "plane"?

Bijection $(x : y : 1) \leftrightarrow (x, y)$

Finite points of $\mathbb{RP}^2 \longleftrightarrow \mathbb{R}^2$

Points at ∞ are $(x:y:0)$,
correspond to slopes $x/y = \text{const}$.

Picture: Points of $\mathbb{RP}^2$ are lines in $\mathbb{R}^3$ intersecting plane $z=1$.



Lines not parallel xy plane are the finite points.

Lines in $\mathbb{RP}^2$ are intersections of $z=1$ with planes in $\mathbb{R}^3$ through $\vec{0}$.