Motivation for the nonsense/garbage in commutative algebra.

Last time: Given $f(x,y) \in \mathbb{Z}[x,y]$, $\exists$ special integer $g \geq 0$ (called the "genus") that somehow influences the A-points $C_f(A)$ of the curve C_f for any ring.

Examples:

- $g \geq 1$: $\# C_f(\mathbb{Z}) < \infty$
- $g \geq 2$: $\# C_f(\mathbb{Q}) < \infty$

- $\#$ components $C_f(\mathbb{R}) \leq g+1$
- $|\# C_f(\mathbb{F}_q) - (q+1)| \leq 2g\sqrt{q}$

What <u>is</u> the genus?
How can it be computed?

The key is to look at <u>complex points</u>.

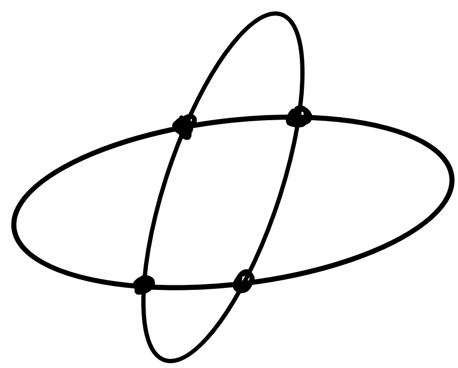
Some important theorems:

Let $f(x,y) \in \mathbb{Z}[x,y]$ have degree d.

- (Newton): Any line intersects the curve in at most d distinct points. [Exactly d, if we count complex points, with multiplicity, and at ∞.]

- (Bézout, 1870s): Any curve of degree n intersects the curve in at most nd distinct points.

Example:



- (Plücker, 1830s):

There exists an integer $g \geq 0$, such that ($n \gg 0$) for any $nd - g$ points on C_f, $\exists$ deg n curve intersecting C_f at these points, but <u>not</u> for $nd - g + 1$ points on the curve.

g = "deficiency"

Plücker's formula:
If C_f has only nodes & cusps, then

$$g = \frac{(d-1)(d-2)}{2} - \#\text{nodes} - \#\text{cusps}$$

- Modern (Serre?):

For a singular point $\alpha = (a,b) \in \mathbb{C}^2$ of C_f, let $f^\alpha(x,y) = f(x+a, y+b)$, so that f^α has singular point at $(0,0)$.

The ring $\mathbb{C}[[x,y]]/(f^\alpha_x, f^\alpha_y)$ has even finite dimension $2\delta_\alpha$ as a vector space.

[δ_α is the "delta invariant" of the singularity.]

Then

$$g = \frac{(d-1)(d-2)}{2} - \sum_\alpha \delta_\alpha$$

where we sum over singular points.

Example:

$\delta_\alpha = 1$ for node or cusp

- (Abel, 1820s):

Modern:

$g =$ dim of space of "holomorphic 1-forms" on the curve.

Old: Addition formulas for "abelian functions". Eg. $\arcsin(a) = \int^a \frac{1}{\sqrt{1-x^2}}\,dx$

Euler:

$$\arcsin(a) + \arcsin(b) = \arcsin(c)$$

where $c = a\sqrt{1-b^2} + b\sqrt{1-a^2}$.

Abel: More generally, $A(a) = \int^a r(x,y)\,dx$ where y defined implicitly by $f(x,y) = 0$.

[First non-elementary example:

$$A(a) = \int^a \frac{1}{\sqrt{1-x^4}}\,dx \quad \left(f(x,y) = y^2 + x^4 - 1,\ r(y) = \frac{1}{y}\right)$$

called an elliptic integral.]

Abel proved there exists a number $g \geq 0$ such that for any $a_1, a_2, \ldots, a_n \in \mathbb{C}$ there exist $b_1, b_2, \ldots, b_g \in \mathbb{C}$ with

$$A(a_1) + \cdots + A(a_n) = A(b_1) + \cdots + A(b_g)$$
$$+ \text{ elementary function.}$$

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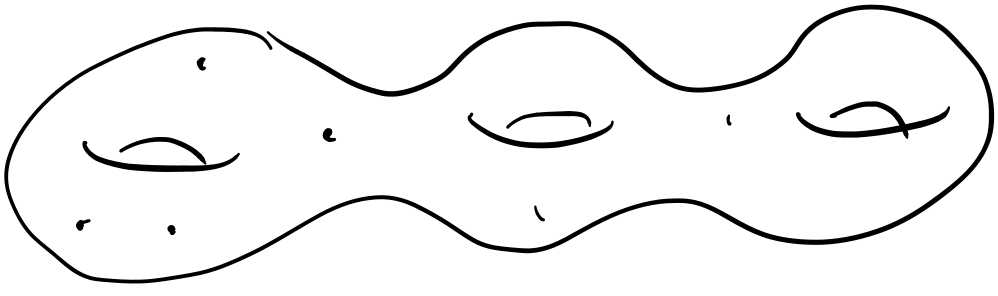
- (Riemann, 1850s):

The curve $C_f(\mathbb{C})$ is a compact orientable surface with finitely many points deleted (points at ∞)



$g = \#$ handles.