MTH 785: Commutative Algebra.

Historical Framework:

Algebraic curves / Riemann Surfaces

Dedekind-Weber (1882):

Reprove Riemann's Theorems in the language of algebraic number theory.

$$f(x,y) = 0.$$

$f(x,y) \in \mathbb{Z}[x,y]$ polynomial.

Problems:

- Find all pos. integer solutions.
  Diophantus (~300 AD)
- Find all $\mathbb{Z}$ solutions.
- Find all $\mathbb{Q}$ solutions.

Diophantine Problems.

HARD!

Example: Fermat's Last Theorem

$$x^n + y^n = 1$$

has no rational solution $(x,y) \in \mathbb{Q}^2$ when $n \geq 3$. (Wiles, 1994)

Hilbert's 10th Problem:

Give an algorithm to determine when $f(x,y) = 0$ <u>has</u> any $\mathbb{Z}$ solution.

Matiyasevich (1970):

There is no such algorithm.

Issue: We have no tools.

Move to $\mathbb{R}$.

$f(x,y) = 0$ defines a "curve" in $\mathbb{R}^2$.

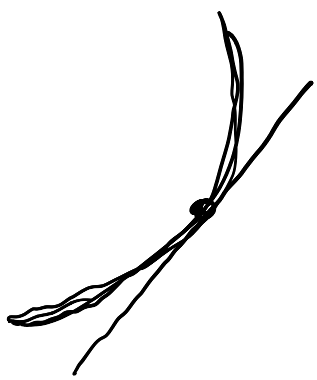
Problems:

- Classify singular points.
- Count bitangents, inflections, etc.
- Count connected components.

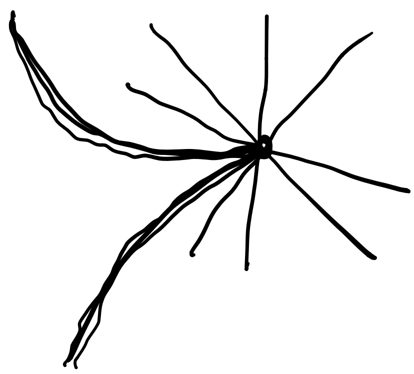
If $f(a,b) = 0$ for $(a,b) \in \mathbb{R}^2$, the tangent line is

$$\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) = 0.$$

If $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$ this is <u>not</u> a <u>line</u>! (2dim tangent space).



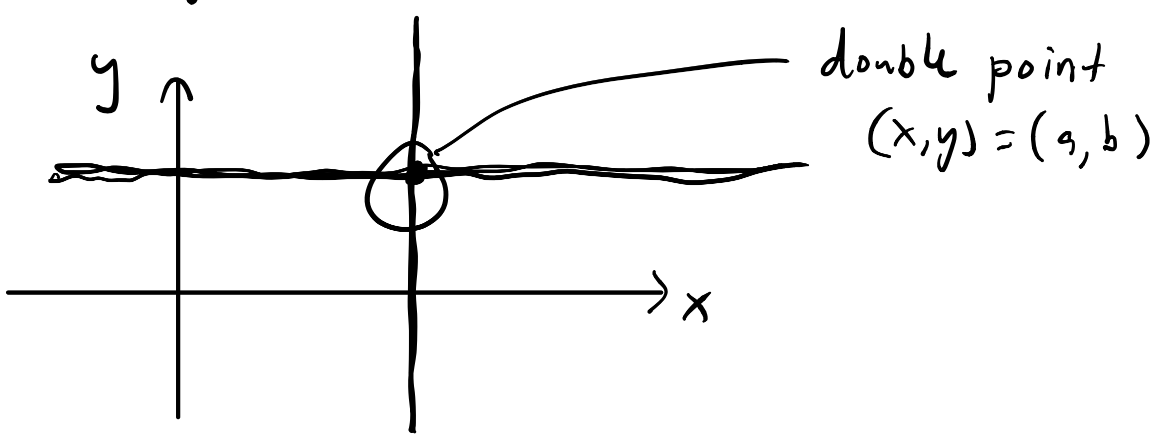
smooth
"regular point"



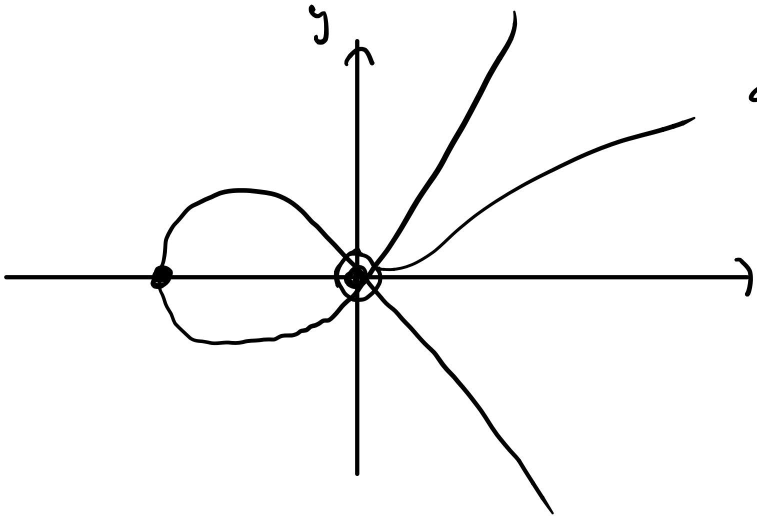
too many tangents.
"singular point"

Examples:

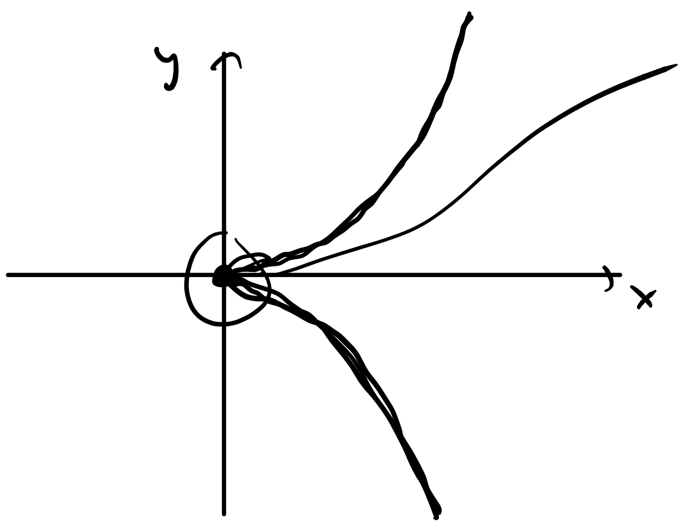
- $(x-a)(y-b) = 0$



- $y^2 = x^3 + x^2 = x^2(x+1)$.



- $y^2 = x^3$



Equivalence of singularities is subtle.

Double Points & Cusps

are the easiest examples.

Curve with no singularities is "smooth"

- $y^2 = f(x)$ $F(x,y) = y^2 - f(x) = 0$

$f(x) \in \mathbb{R}[x]$ degree 3,

has no repeated roots.

Claim: No singularities.

Proof: If $(a,b) \in \mathbb{R}^2$ is a singular point then

$$b^2 = f(a) \quad \& \quad \frac{\partial F}{\partial x}(a,b) = \frac{\partial F}{\partial y}(a,b) = 0$$

$$f'(a) = 2b = 0.$$

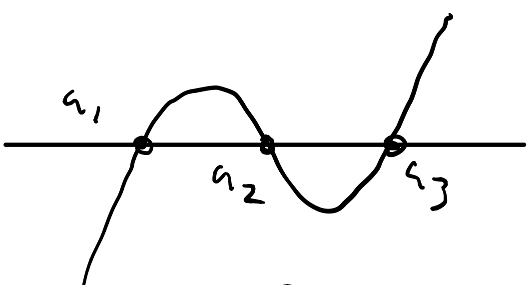
$$b = 0$$

$$f(a) = f'(a) = 0.$$

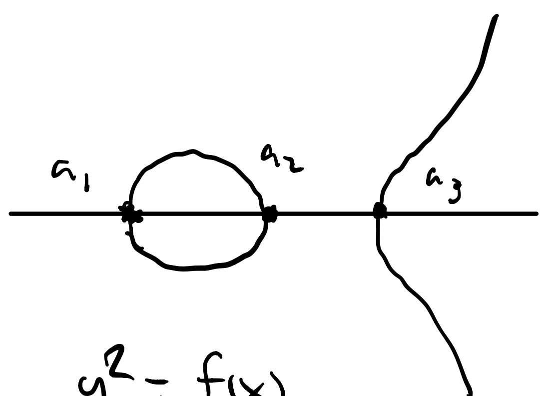
$\Longrightarrow$ a is repeated root.

Jargon: Elliptic Curve.

Picture: $y^2 = (x-a_1)(x-a_2)(x-a_3)$



$y = f(x)$



$y^2 = f(x)$

Connected Components ?

$f(x,y) = 0$, $f(x,y) \in \mathbb{R}[x,y]$

Harnack's Theorem:

If $f(x,y)$ has degree d and "no singularities" then

$$\#\text{ components of } f(x,y)=0 \leq \underbrace{\frac{(d-1)(d-2)}{2}}_{\text{"genus"}} + 1.$$

Next time!