Commutative Algebra in Context — Fall 2020
Homework 1 — Drew Armstrong

Turn in any 3 problems by Mon, Sept 14.

Problem 1. Prove that a polynomial $f(x) \in \mathbb{F}[x]$ of degree d over a field $\mathbb{F}$ has at most d roots[1] in $\mathbb{F}$. [Hint: Given $\alpha \in \mathbb{F}$ and $f(\alpha) = 0$ we can divide $f(x)$ by $x - \alpha$ to get $f(x) = (x - \alpha)g(x)$ with $g(x) \in \mathbb{F}(x)$ of degree $d - 1$. By induction on degree we know that $g(x)$ has at most $d - 1$ roots in $\mathbb{F}$.]

Problem 2. A commutative ring A is called an *integral domain* (or just *domain*) if $a, b \in A \setminus \{0\}$ implies $ab \in A \setminus \{0\}$.

(a) Prove that A is a domain if and only if it is a *subring of a field*.

(b) If A is a domain, use (a) and Problem 1 to prove that a nonzero polynomial $f(x) \in A[x]$ has only finitely many roots in A.

Problem 3. Let A be an infinite domain and suppose that $f(x), g(x) \in A[x]$ satisfy $f(\alpha) = g(\alpha)$ for infinitely many $\alpha \in A$. In this case prove that $f(x) = g(x)$ as polynomials (i.e., they have the same coefficients).

Problem 4. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be vector of independent variables and let A be a domain. Define the degree function $A[\mathbf{x}] \setminus \{0\} \to \mathbb{N}$ and prove that it satisfies $\deg(fg) = \deg(f) + \deg(g)$. In particular, this implies that $A[\mathbf{x}]$ is also a domain.

Problem 5. Let A be a commutative ring and let $F(\mathbf{x}) \in A[\mathbf{x}]$ be a polynomial in some finite list of variables $\mathbf{x} = (x_1, \ldots, x_n)$. Consider the following conditions:

(H1) Every term of $F(\mathbf{x})$ has the form $a x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ where $i_1 + \cdots + i_n = d$ and $0 \neq a \in A$.

(H2) We have $F(\lambda \mathbf{x}) = \lambda^d F(\mathbf{x})$ for all scalars $\lambda \in A$.

Polynomials satisfying (H1) are called *homogeneous of degree d*. Prove we always have (H1)$\Rightarrow$(H2). If A is an infinite domain prove that we also have (H2)$\Rightarrow$(H1). [Hint for (H2)$\Rightarrow$(H1): For any polynomial $F(\mathbf{x}) \in A[\mathbf{x}]$ and variable y note that $F(y\mathbf{x}) = \sum_{k \geq 0} y^k F^{(k)}(\mathbf{x})$, where the sum has finitely many terms and $F^{(k)}(\mathbf{x})$ is homogeneous of degree k in the sense of (H1). Use Problems 3 and 4 to show that $F(\mathbf{x}) = F^{(d)}(\mathbf{x})$.]

6. Let A be an infinite domain and consider an invertible matrix $\Phi \in \mathrm{GL}_n(A)$. Let $F(\mathbf{x}) \in A[\mathbf{x}]$ be homogeneous of degree d. In this case prove that $G(\mathbf{x}) := F(\Phi \mathbf{x}) \in A[\mathbf{x}]$ is also homogeneous of degree d. [Hint: Use Problem 5.]

[1] Distinct, or counted with multiplicity.

7. For any ring A, the A-linear function $D_x : A[x] \to A[x]$ is defined by

$$D_x(x^k) := \begin{cases} kx^{k-1} & k > 0, \\ 0 & k = 0. \end{cases}$$

Prove that the following properties are satisfied for all $f(x), g(x) \in A[x]$:

(a) $D_x(fg) = D_x(f)g + fD_x(g)$,

(b) $D_x(g^k) = kg^{k-1}D_x(g)$,

(c) $D_x(f \circ g) = (D_x(f) \circ g)D_x(g)$.

8. Euler's Formula. Let $\mathbf{x} = (x_1, \ldots, x_n)$. We define the function $D_{x_i} : A[\mathbf{x}] \to A[\mathbf{x}]$ as in Problem 5 by thinking of $A[\mathbf{x}] = A_i[x_i]$ as the ring of polynomials in x_i with coefficients from $A_i := A[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Now fix some $F(\mathbf{x}) \in A[\mathbf{x}]$ and $d \geq 0$ and consider the following condition:

(H3) $\sum_i x_i D_i(F) = dF$

Prove that (H1)⇒(H3) for any ring. If A is a domain of characteristic zero (necessarily infinite), prove that we also have (H3)⇒(H1). [Hint for (H3)⇒(H1): Write $F(\mathbf{x}) = \sum_k F^{(k)}(\mathbf{x})$ where each $F^{(k)}(\mathbf{x})$ is a sum of monomials of degree k. Then since the operator $\sum_i x_i D_{x_i}$ is linear we have $dF = \sum_i x_i D_{x_i}(F) = \sum_k \sum_i x_i D_{x_i}(F^{(k)})$.]