Representation Theory

8/21/17

Nullstellensatz:

- Galois Correspondences:

P, Q posets. A Galois Connection is

$* : P \rightleftarrows Q : *$

s.t. $\forall\ p \in P,\ q \in Q,\quad p \le q^* \Longleftrightarrow q \le p^*$

Thm: Every Galois Connection restricts to an isomorphism on the subposets of "Closed" elements.

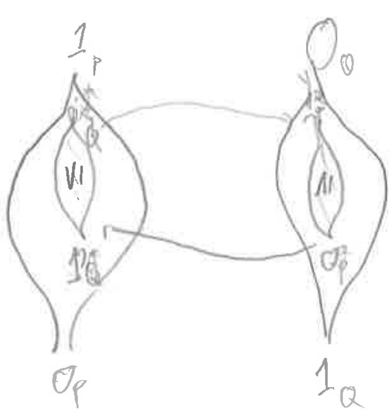
Lemma: ① $\forall\ p \in P,\ q \in Q,\quad p \le p^{**},\ q \le q^{**}$

② $\forall\ p_1, p_2 \in P,\ q_1, q_2 \in Q,\quad p_1 \le p_2 \Rightarrow p_1^* \ge p_2^*$
$\qquad q_1 \le q_2 \Rightarrow q_1^* \ge q_2^*$

③ $\forall\ p \in P,\ q \in Q,\quad p^{***} = p^*,\ q^{***} = q^*$

$* : P \rightleftarrows Q : *$

$* : Q^* \xrightarrow{\cong} P^* : *$



Let K be alg. closed field, K^n affine n-space

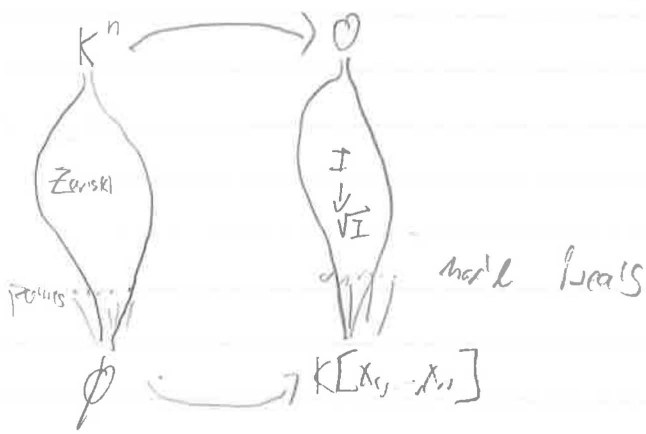
$$K[x_1,\dots,x_n] = (\varphi : K^n \longrightarrow K)$$

$$\begin{array}{ccc} K^n & & K[x_1,\dots,x_n] \\ \cup\!\shortmid & & \cup\!\shortmid \\ X & \longmapsto & I(X) \\ V(Y) & \longleftarrow\!\shortmid & Y \end{array}$$

$I(X)$ ← Vanishing ideal (ideal of fnctns simultaneously vanishing on X)

$V(Y)$ ← Common vanishing set of Y

$$X \subseteq V(Y) \implies I(X) \supseteq Y$$



$X \subseteq K^n$

$$I(X) \subseteq K[x_1,\dots,x_n], \quad K[X] := \frac{K[x_1,\dots,x_n]}{I(X)}$$ ← has no nilpotents (i.e. it's reduced)

Aff. Varieties $\longrightarrow$ Reduced Alg.

$$
\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "\varphi"'] & K[X] \\
Y \arrow[r, mapsto] & K[Y] \arrow[u, "\varphi^*"']
\end{tikzcd}
$$

We have an equivalence of categories

$Aff_K \simeq Red\ Alg_K^{op}$ $\qquad$ $F : Aff^{op} \rightleftarrows Red.\ Alg : G$

full, faithful, essentially surjective.

$FG \cong id$
$GF \cong id$

Geometry $\cong Alg^{op}$

8/23/17

$$
\begin{tikzcd}
 & \text{Category} & \\
\text{Posets} \arrow[ur] & & \text{Vector Spaces} \arrow[ul]
\end{tikzcd}
$$

Central Concepts:

Adjoint Functors
Limits / Colimits
RAPL / LAPC (right-adjoints preserve limits, left-adjoints preserve colimits)

ex: $a(b+c) = ab + ac$

$$A \otimes (B+C) = (A \otimes B) \oplus (A \otimes C)$$

Def: A Category $\mathcal{C}$ Consists of ∘) a Collection $Obj(\mathcal{C})$ of objects
"$X \in \mathcal{C}$" $\Leftrightarrow$ "$X \in Obj(\mathcal{C})$"

∘) $\forall\, X, Y \in \mathcal{C}$; a set of arrows $Hom_{\mathcal{C}}(X,Y)$

"$\alpha : X \to Y$" $\Leftrightarrow$ "$\alpha \in Hom_{\mathcal{C}}(X,Y)$"

∘) $\forall\, X, Y, Z \in \mathcal{C}$, a function

$$\circ : Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \longrightarrow Hom_{\mathcal{C}}(X,Z)$$

if $\alpha : X \to Y$, $\beta : Y \to Z$, then $\beta \circ \alpha : X \to Z$

with two axioms:

Pointed Sm

1) Identity: $\forall\, X \in \mathcal{C}$, $\exists\ id_X \in Hom_{\mathcal{C}}(X,X)$ s.t.

$\forall\ \alpha : X \to Y$, the following diagram commutes

$$\begin{tikzcd} X \arrow[loop left, "id_X"] \arrow[r, "\alpha"] & Y \arrow[loop right, "id_Y"] \end{tikzcd}$$

2) Associativity: $\forall\ X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} W$, TFDC

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[rr, bend left, "\beta \circ \alpha"] & Y \arrow[r, "\beta"] \arrow[rr, bend right, "\gamma \circ \beta"'] & Z \arrow[r, "\gamma"] & W \end{tikzcd}$$

Subcategory: $\mathcal{C}' \subseteq \mathcal{C}$

- $Obj(\mathcal{C}') \subseteq Obj(\mathcal{C})$
- $\forall x,y \in \mathcal{C}'$, $Hom_{\mathcal{C}'}(x,y) \subseteq Hom_{\mathcal{C}}(x,y)$

if this is always $=$,

$\mathcal{C}'$ is a full subcategory

ex: $Ab \subseteq Grp$ is full

ex: Let $\mathcal{P}$ be a category satisfying

$\forall x,y \in \mathcal{P}$, $|Hom_{\mathcal{P}}(x,y)| \in \{0,1\}$

What is this?

"$x \longrightarrow y$" $\Longleftrightarrow$ "$x \leq y$"

Axioms: 1) $x \leq x \quad \forall x \in \mathcal{P}$

2) $x \leq y,\ y \leq z \Rightarrow x \leq z$

This is a Preorder

Antisymmetry?

$x \leq y$ + $y \leq x \Rightarrow x \overset{\text{not defined...}}{=} y$

Solution: We can define "=" to have this property.
Now it's a poset.

Def: For $X, Y \in \mathcal{C}$, we say $X \cong_{\mathcal{C}} Y$ if $\exists\ \alpha : X \to Y,\ \beta : Y \to X$

s.t. $\beta \circ \alpha = id_X$ and $\alpha \circ \beta = id_Y$.

"Subobject of X" = equivalence class of monic arrows into X.

A Skeleton of a Category $\mathcal{C}$ is a full subcategory obtained by throwing away all but one object from each isom. class.

Sk(Preorder) = Poset
Sk(Set) = Cardinalities

"Operations" in a Category
(Limits + Colimits)

Def: Category $\mathcal{C}$ is Small if $Obj(\mathcal{C})$ is a Set.

Dichotomy:

Categories as Structures	Categories Of Structures
(Small)	Set, Ab, Top, Top$_0$
Poset	
Groupoid (every arrow is an iso)	

Def: A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ consists of

- a function $F: Obj(\mathcal{C}) \longrightarrow Obj(\mathcal{D})$
- $\forall\, c_1, c_2 \in \mathcal{C}$ a function

$$F: Hom_{\mathcal{C}}(c_1, c_2) \longrightarrow Hom_{\mathcal{D}}(F(c_1), F(c_2))$$

$$\begin{array}{ccc} c_2 & & F(c_2) \\ \alpha \uparrow & \overset{F}{\rightsquigarrow} & \uparrow F(\alpha) \\ c_1 & & F(c_1) \end{array}$$

Satisfying two axioms:

- $\forall\, c \in \mathcal{C},\ F(id_c) = id_{F(c)}$
- $\forall\ x \xrightarrow{\alpha} y \xrightarrow{\beta} z \quad \rightsquigarrow \quad F(x) \xrightarrow{F(\alpha)} F(y) \xrightarrow{F(\beta)} F(z)$

$$F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$$

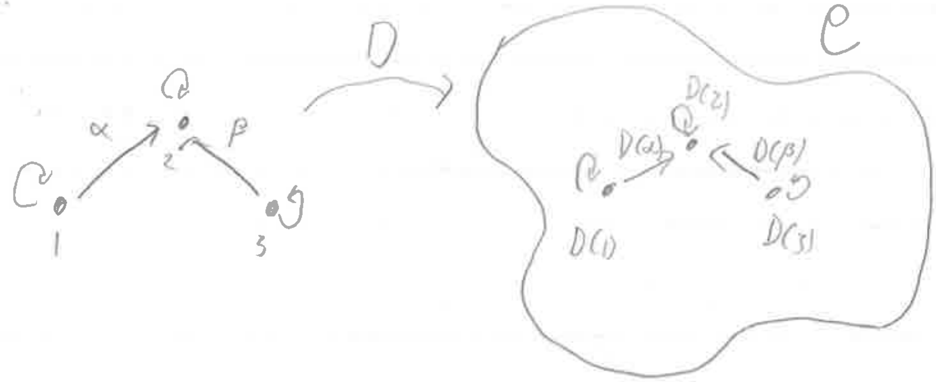
Contravariant functor: $F: \mathcal{C}^{op} \longrightarrow \mathcal{D}$
$F: \mathcal{C} \longrightarrow \mathcal{D}^{op}$

Claim: Categories + Functors form a Category, called Cat.

Def: Let $\mathcal{I}$ be a small category (thought of as "an index set with arrows"). A diagram of shape $\mathcal{I}$ in $\mathcal{C}$ is a functor

$$D: \mathcal{I} \longrightarrow \mathcal{C}.$$

Picture:

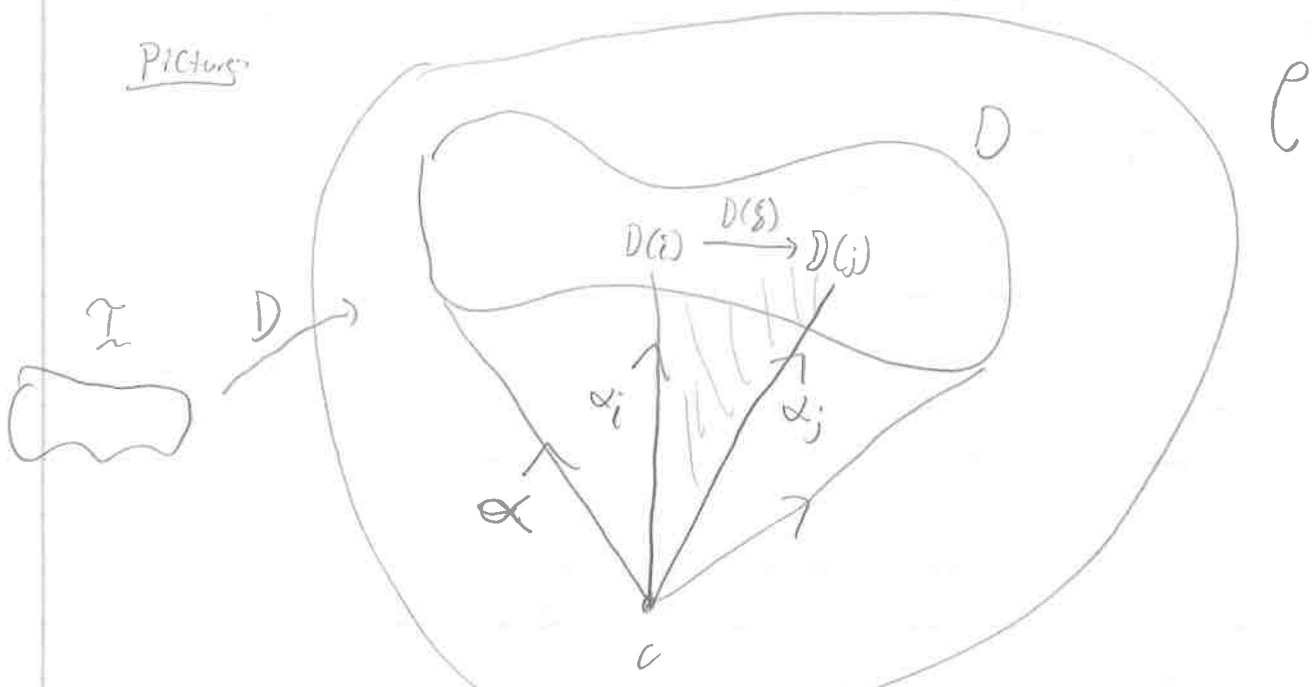


Def: A <u>Cone under the diagram D</u> consists of

- an object $c \in \mathcal{C}$ (the <u>apex</u>)
- set of arrows $\alpha_i : c \longrightarrow D(i)$ $\forall i \in \mathcal{I}$

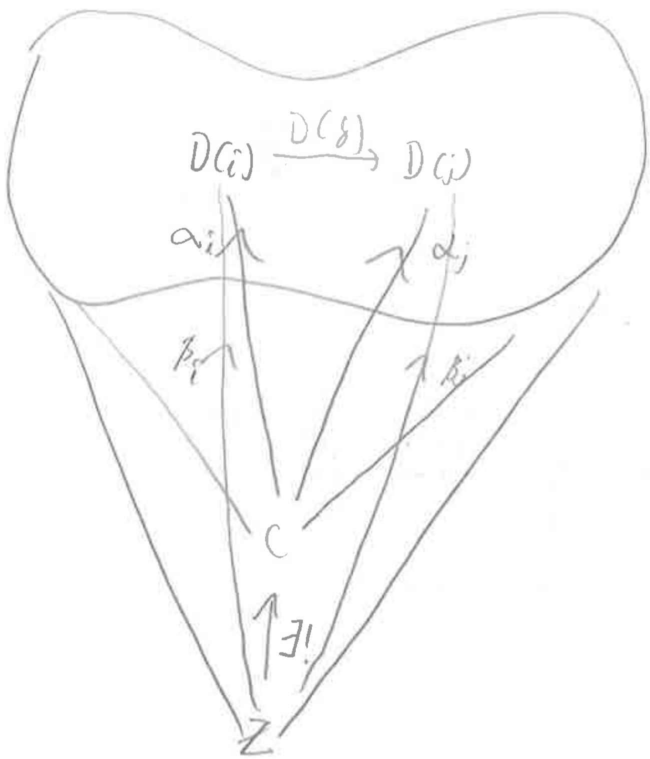
s.t. $\forall\, \delta : i \to j \in \mathcal{I}$, we have $\alpha_j = D(\delta) \circ \alpha_i$

Picture:



Denote the cone by $(C, (\alpha_i)_{i \in I})$.

Def.: We say this cone is a <u>limit</u> of the diagram D if for any other cone $(Z, (\beta_i))$ under D, there exists a <u>unique</u> arrow $u: Z \to C$ s.t.



Commutes $\forall i$

Limit of $D \equiv$ highest cone under D
Colimit of $D \equiv$ lowest cone over D

8/28/17

Current Goal:

- Limits/Colimits
- Adjoint Functors
- RAPL/LAPC

Recall: Let $\mathcal{I}$ be a small category.
A <u>diagram of shape $\mathcal{I}$</u> is a functor $D: \mathcal{I} \to \mathcal{C}$

A <u>cone under D</u> consists of

- An <u>apex</u> $c \in \mathcal{C}$
- arrows $\alpha_i : C \to D(i)$, $\forall i \in \mathcal{I}$
  s.t. $\forall \delta : i \to j$, $\alpha_j = D(\delta) \circ \alpha_i$

We say a cone (C, α) is a <u>limit of D</u> if for any other cone (Z, β) <u>under D</u>,

$\exists!\ v : Z \to C$ s.t. $\forall \delta : i \to j \in \mathcal{I}$,

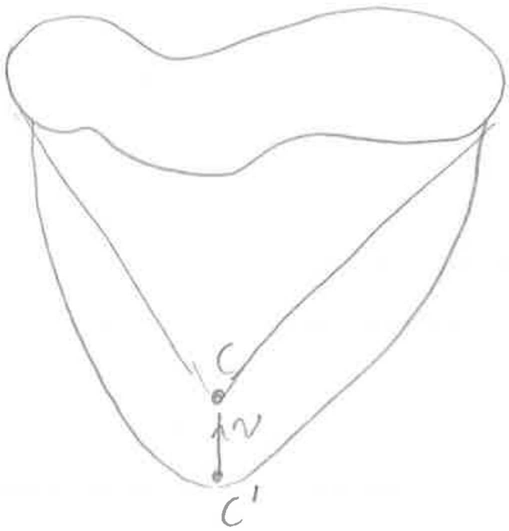
$$\begin{tikzcd}
D(i) \arrow[rr] & & D(j) \\
& C \arrow[ul, "\alpha_i"] \arrow[ur, "\alpha_j"'] & \\
& Z \arrow[u, dashed, "\exists!"'] \arrow[uul, bend left, "\beta_i"] \arrow[uur, bend right, "\beta_j"'] &
\end{tikzcd}$$

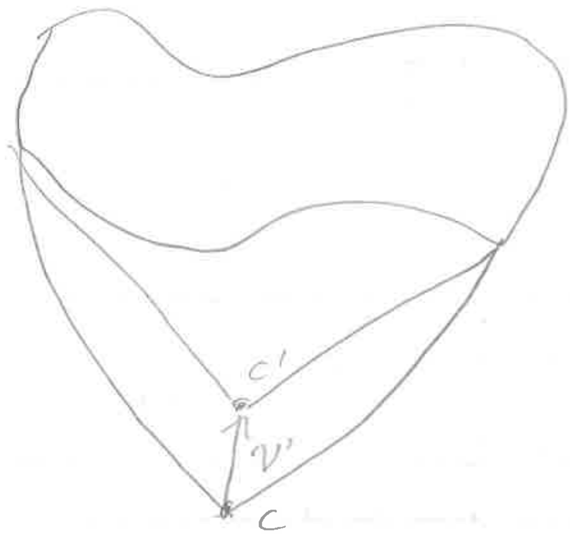
commutes.

<u>Uniqueness of limits</u>: A limit of D (if it exists) is unique in some sense.

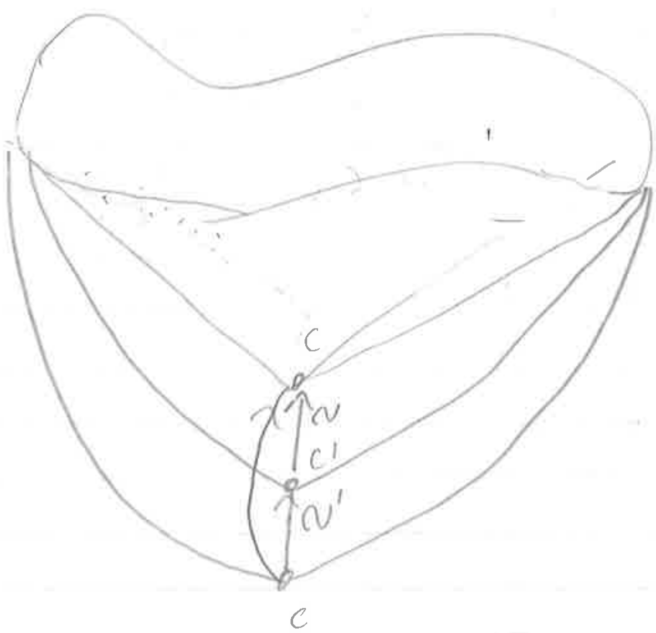
Suppose (C, α), (C', α') are limits

Canonical arrows

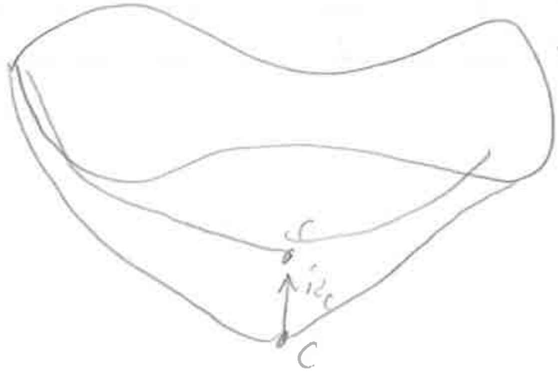




$\Rightarrow$



But we also have



Universalness $\Rightarrow \psi \circ \psi' = id_{C}, \quad \psi' \circ \psi = id_{C}$

$\Rightarrow C \cong_{\mathcal{C}} C'$, and isomorphisms ψ & ψ' are unique

wrt commuting with α & α'.

Colimits (if they exist) are unique in the same way.

ex: Let P be a poset (i.e. a small category where $|Hom_P(x,y)| \in \{0,1\}$)

Consider $\mathcal{I} = \emptyset$, the empty category

$$\emptyset : \emptyset \longrightarrow P \quad \text{empty diagram}$$

$$C \text{ is a limit of } \emptyset \iff \forall z \in P,\ z \leq C$$
$$\iff C = \hat{1}_P.$$

Let $1, 1'$ be top elts of P. Then $1 \leq 1'$, $1' \leq 1 \Rightarrow 1 \cong_P 1'$.

Similarly, colim of $\emptyset$, if it exists, is a bottom elt $0 \in P$, unique up to isomorphism.

Next, let $\mathcal{I} = \{ \overset{\bullet}{1} \quad \overset{\bullet}{2} \} = \bar{2}$

Diagram $D : \bar{2} \longrightarrow P$ is just two elements

Picture:

Cone under D = lower bound
limit of D = glb = meet

Cone over D = upper bound
Colimit of D = lub = join

General diagram in P is just a subset $S \subseteq P$

Limit + Colimit are meet + join / resp.

Example: Boolean lattice $P = 2^u$

$\vee = \cup$ $\quad 1 = u$ $\qquad$ Complete (all limits exist)
$\wedge = \cap$ $\quad 0 = \emptyset$ $\qquad$ CoComplete (all colimits exist)

$\vee \emptyset = \emptyset$ $\qquad$ mnemonic: $\sum_{\emptyset} = 0$
$\wedge \emptyset = u$ $\qquad\qquad\qquad\quad \prod_{\emptyset} = 1$

limits are "multiplicative"
Colimits are "additive"

In a general category, consider $D: \vec{n} \to \mathcal{C}$.
The limit (if it exists) is the categorical product $\prod_{i=1}^{n} D(i)$

The colimit (if it exists) is categorical sum/coproduct $\coprod_{i=1}^{n} D(i)$

empty product/coproduct is called terminal/initial object.

Examples:

In Set: $\prod$ = Cartesian product $\coprod$ = Disjoint Union
initial $= \emptyset$ terminal $= \{\bullet\}$
"0" "1"

In Grp: $\prod$ = direct product $\coprod$ = free product
initial $= \{e\}$ = final (zero object)

In CRing: $\prod$ = direct product $\coprod = \otimes_{\mathbb{Z}}$
initial $= \mathbb{Z}$ Terminal $= 0$

In Fld: $\prod$ doesn't exist, $\coprod$ doesn't exist ???

In Vec_k: $\prod, \coprod = \oplus$
initial = final $= 0$

In Ab: finite products + coproducts coincide $= \oplus$
initial = terminal $= 0$.

Given $A, B \in Ab$, define $A \oplus B = \{ (a, b) \mid a \in A, b \in B \}$

$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

Projections

$$\begin{tikzcd}
 & & A \ni c_A \\
c \in C \arrow[r, dashed, "\exists!?\ \varphi"] \arrow[urr, bend left, "\varphi_A"] \arrow[drr, bend right, "\varphi_B"'] & A \oplus B \ni (c_A, c_B) \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
 & & B \ni c_B
\end{tikzcd}$$

$c_A = \varphi_A(c)$

$c_B = \varphi_B(c)$

(gives uniqueness)

Only possible choice:

$$\varphi(c) := (\varphi_A(c), \varphi_B(c)).$$

Is this a group hom?

$$\begin{aligned}\varphi(c_1+c_2) &= (\varphi_A(c_1+c_2), \varphi_B(c_1+c_2)) \\ &= (\varphi_A(c_1)+\varphi_A(c_2), \varphi_B(c_1)+\varphi_B(c_2)) \\ &= (\varphi_A(c_1), \varphi_B(c_1)) + (\varphi_A(c_2), \varphi_B(c_2)) \\ &= \varphi(c_1) + \varphi(c_2)\end{aligned}$$

✓

Why is this object also the coproduct?

Inclusions?

$$\begin{tikzcd} \overset{a}{A} \arrow[dr, "\iota_A"] \arrow[rrd, bend left, "\varphi_A"] & & \\ & A \oplus B \arrow[r, dashed, "\exists!?"', "\varphi"] & C \\ \underset{b}{B} \arrow[ur, "\iota_B"'] \arrow[rru, bend right, "\varphi_B"'] & & \end{tikzcd}$$

$(a, 0_B)$ $(0_A, b)$

Suppose $\varphi(a,b) = c$

<u>Trick:</u> $(a,b) = (a,0) + (0,b)$, so $\varphi(a,b) = \varphi(a,0) + \varphi(0,b)$
$= \varphi(\iota_A(a)) + \varphi(\iota_B(b))$
$= \varphi_A(a) + \varphi_B(b)$ ✓

Is this a homom?

Define $\varphi(a,b) = \varphi_A(a) + \varphi_B(b)$. This is a group homom, b/c "+" is a commutative operation:

$$\varphi(a_1 + a_2, b_1 + b_2) = \varphi_A(a_1 + a_2) + \varphi_B(b_1 + b_2) = \varphi_A(a_1) + \varphi_A(a_2) + \varphi_B(b_1) + \varphi_B(b_2)$$

$$= \varphi_A(a_1) + \varphi_B(b_1) + \varphi_A(a_2) + \varphi_B(b_2)$$

$$= \varphi(a_1, b_1) + \varphi(a_2, b_2). \checkmark$$

$$\begin{tikzcd}
A \arrow[rr, "id_A"] \arrow[dr, "\iota_A"] \arrow[ddrr, bend right=60, "0_B"'] & & A \\
& A \oplus B \arrow[ur, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
B \arrow[rr, "id_B"'] \arrow[ur, "\iota_B"] \arrow[uurr, bend left, "0_A"] & & B
\end{tikzcd}$$

0_A :
$$\begin{tikzcd}
B \arrow[r, "\exists!"'] \arrow[rr, bend right, "0"'] & 0 \arrow[r, "\exists!"'] & A
\end{tikzcd}$$

In a category with a 0-object (and hence 0-arrows), a biproduct satisfies $\star$

What is a matrix?

Consider a group hom

$$\varphi : A \oplus B \longrightarrow A \oplus B$$

φ defines four "component" homoms:

$$\varphi_{ij} := \pi_i \circ \varphi \circ \iota_j \qquad \text{for } i,j \in \{A, B\}$$

By universal properties, φ is uniquely determined by its components $\varphi_{AA}, \varphi_{AB}, \varphi_{BA}, \varphi_{BB}$.

$$\varphi \text{ "="} \begin{pmatrix} \varphi_{AA} & \varphi_{AB} \\ \varphi_{BA} & \varphi_{BB} \end{pmatrix}$$

Composition = matrix multiplication

$$\begin{pmatrix} \varphi_{AA} & \varphi_{AB} \\ \varphi_{BA} & \varphi_{BB} \end{pmatrix} \circ \begin{pmatrix} \varphi'_{AA} & \varphi'_{AB} \\ \varphi'_{BA} & \varphi'_{BB} \end{pmatrix} = \begin{pmatrix} \varphi_{AA} \circ \varphi'_{AA} + \varphi_{AB} \circ \varphi'_{BA} & \_\_\_ \\ \_\_\_ & \_\_\_ \end{pmatrix}$$

8/30/17 Why are limits/colimits interesting?

- They're ubiquitous
- Because they are preserved by right/left adjoint functors, which are also ubiquitous

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A Galois connection b/w posets is

$$*: P \rightleftarrows Q : *$$

with $p \le q^* \iff q \le p^* \quad \forall p \in P, q \in Q$

Theorem (RAPL): Galois connections send joins to meets, i.e.: Suppose we have $S \subseteq P$ where $\bigvee_P S$ exists; then the meet of the image exists, and is given by $\bigwedge_Q S^* = (\bigvee_P S)^*$.

Dually: $(\bigvee_Q T)^* = \bigwedge_P (T^*)$.

Pf: We know that $\bigvee_P S \in P$ exists.
By definition, we have, $\forall q \in Q$, that $q \le (\bigvee_P S)^* \iff \bigvee_P S \le q^*$

$$\iff s \le q^* \quad \forall s \in S$$
$$\iff q \le s^* \quad \forall s \in S$$

$q \le (\bigvee_P S)^* \iff q \le s^* \quad \forall s \in S.$

To show $(\bigvee_P S)^* = \bigwedge_Q S^*$:

- $q = (\bigvee_P S)^* \Rightarrow (\bigvee_P S^*) \le s^* \quad \forall s \in S$
  $\Rightarrow (\bigvee_P S)^*$ is a lower bound for S^*.

- For any other lower bound z,

$$z \le s^* \ \forall s \in S \Rightarrow z \le (\vee_P S)^*$$

✓

$\square$

Corollary: Meet of closed elts is closed. (almost a topology...)

Special case: If $0_P = \vee_P \emptyset$ exists, then

$$0_P^* = (\vee_P \emptyset)^* = \wedge_Q \emptyset^* = \wedge_Q \emptyset = 1_Q$$

Change language:

Let P, Q be posets. An adjunction is a pair of fnctns

$$L : P \rightleftarrows Q^{op} : R$$

Satisfying

$$p \le R(q) \iff L(p) \le q \qquad \forall p \in P, q \in Q$$

Compare to $\langle p, R(q) \rangle = \langle L(p), q \rangle$ vector spaces

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Notation: "$L \dashv V$"

left adjoint, right adjoint

Thm (RAPL): For $\forall S \subseteq P$, $T \subseteq Q$, we have:

- if $\bigvee_P S$ exists then
  $\bigvee_Q L(S) = L(\bigvee_P S)$ exists "left adjoints preserve colimits"

- if $\bigwedge_Q T$ exists, then
  $\bigwedge_P R(T) = R(\bigwedge_Q T)$ exists "right adjoints preserve limits"

Thm (Uniqueness of Adjoints):

Consider a function $L : P \longrightarrow Q$.
If we have two adjunctions

$$L : P \rightleftarrows Q : R \qquad L : P \rightleftarrows Q : R'$$

then $R = R'$.

Pf: $\forall p \in P$, $q \in Q$, we have

$$p \leq R(q) \Leftrightarrow L(p) \leq q \Leftrightarrow p \leq R'(q)$$

$$p \leq R(q) \Leftrightarrow p \leq R'(q)$$

Put • $p = R(q)$. Then $R(q) \leq R(q) \Rightarrow R(q) \leq R'(q)$

• $p = R'(q)$. Then $R'(q) \leq R'(q) \Rightarrow R'(q) \leq R(q)$.

$\Rightarrow R(q) \cong_P R'(q) \quad \forall q$

(So if P is really a poset, $R(q) = R'(q) \quad \forall q$). $\square$

- L, R Order-preserving maps

"monad"
- $R \circ L$ is a Closure Operator $2^P \to 2^P$

"Comonad"
- $L \circ R$ is an "interior operator" $2^Q \to 2^Q$

$LRL \cong L$ + $RLR \cong R$

Example: Consider a set function $f: U \to V$.

This induces a function on subsets

$$f: 2^U \longrightarrow 2^V$$

Note: f is monotone + preserves arbitrary unions:

- $U_1 \subseteq U_2 \Rightarrow f(U_1) \subseteq f(U_2)$
- $f(U_1 \cup U_2) = f(U_1) \cup f(U_2)$

Maybe $f: 2^U \longrightarrow 2^V$ is left-adjoint to something..

Indeed, $f \dashv f^{-1}$, i.e.

$$f : 2^U \rightleftarrows 2^V : f^{-1}$$

However f doesn't preserve intersections, so the string of adjunctions cannot be extended on the left

$$\cancel{X} \dashv f \dashv f^{-1} \dashv f! \dashv \cancel{X},$$

where $f!(S) := \{ v \in V \mid f^{-1}(v) \subseteq S \}$

Compare to $f(S) := \{ v \in V \mid f^{-1}(v) \cap S \neq \emptyset \}$

<u>Note:</u> $f_!(S) \subseteq f(S)$.

<u>Rephrase:</u> $f(S) = \{ v \in V \mid \exists\, u \in f^{-1}(v),\ u \in S \}$
$f_!(S) = \{ v \in V \mid \forall\, u \in f^{-1}(v),\ u \in S \}$

"Sheaf-theoretic logic"

$f \dashv f^{-1} \dashv f_!$

Sets
Sheaves

$$\begin{tikzcd} \mathrm{Qcoh}(\mathrm{Spec}\,A) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Qcoh}(\mathrm{Spec}\,B) \arrow[d, phantom, "\cong" rotate=90] \\ A\text{-mod} \arrow[r, bend left] \arrow[r] \arrow[r, bend right] & B\text{-mod} \end{tikzcd}$$

<u>Categories $\approx$ Vector Spaces</u>

vector space V over field K $=$ category $\mathcal{C}$

The field K $=$ Category of <u>Sets</u>

K-linear fnctn $=$ functor

dual space $V^* = \mathrm{Set}^{\mathcal{C}}$

bilinear pairing $V \times V \to K$ $=$ $\mathrm{Hom}(-,-)$ bifunctor
$(X,Y) \in \mathcal{C} \longmapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$

adjoint operators = adjoint functors

RAPL + Uniqueness of adjoints = $\mathrm{Hom}_e(-,-)$ is "continuous" + "non-degenerate" (Yoneda)

$\mathbb{C}G$-modules: U, V

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(U,V) = \langle U, V \rangle_G$$

Irreducible $\mathbb{C}G$-modules are an "orthonormal basis"

Let U be an irreducible $\mathbb{K}$-module, ($\mathbb{K}$ alg. closed)

$\varphi : U \to U$ homomorphism

$\ker(\varphi) \subseteq U$ submodule $\Rightarrow$ $\ker(\varphi) = 0$ or U
$\mathrm{im}(\varphi) \subseteq U$ submodule, so $\mathrm{im}(\varphi) = 0$ or U

$\ker(\varphi)$	$\mathrm{im}(\varphi)$	
0	U	φ iso
U	0	$\varphi = 0$

in this case, isomorphism $U \to U$
$\qquad\qquad\qquad\qquad \|$
$\qquad\qquad\quad$ scalar mult.

$(\varphi - \lambda I) : U \to U$ is the 0 map exactly when $\varphi = \lambda 1$.

$$\Rightarrow \mathrm{Hom}_{\mathbb{C}G}(\mathrm{irr}, \mathrm{irr}) \cong \mathbb{C} \Rightarrow \langle \mathrm{irr}, \mathrm{irr}' \rangle_G = \begin{cases} 1, & \mathrm{irr} \cong \mathrm{irr}' \\ 0, & \mathrm{irr} \not\cong \mathrm{irr}' \end{cases}$$

9/25/17 <u>Application of RAPL to posets:</u>

Let P, Q be posets in which all joins/colimits exist, and let $*: P \rightleftarrows Q : *$ be a Galois Connection.

Then the meet of closed elts exists and is also closed.

<u>Pf</u>: $\forall$ $S \subseteq P$, $T \subseteq Q$, we have $\bigwedge_Q S^* = (\bigvee_P S)^*$, which exists, and

$\bigwedge_P T^* = (\bigvee_Q T)^*$, which exists.

$\square$

(<u>Note</u>: In general, the join of closed elts needn't be closed).

<u>Ex</u>: Zariski Topology

$I : K^n \rightleftarrows k[X_1, \ldots, X_n] : V$

<u>Automatic</u>: $I(\bigcup_i S_i) = \bigcap_i I(S_i)$

$V(\sum_i J_i) = \bigcap_i V(J_i)$

$V(J_1 \cap J_2) \overset{?}{=} V(J_1) \cup V(J_2)$

By the general nonsense, we have $J_1 \cap J_2 \subseteq J_1$, $J_1 \cap J_2 \subseteq J_2$

$\Rightarrow V(J_1) \subseteq V(J_1 \cap J_2)$ $\quad V(J_2) \subseteq V(J_1 \cap J_2)$

$\Rightarrow V(J_1) \cup V(J_2) \subseteq V(J_1 \cap J_2)$ ✓

How about $\supseteq$?

We will show that $(V(J_1) \cup V(J_2))^c \subseteq (V(J_1 \cap J_2))^c$

Consider $x \notin V(J_1) \cup V(J_2) \Rightarrow x \notin V(J_1)$, & $x \notin V(J_2)$.

$\therefore \exists\ f_1 \in J_1,\ f_2 \in J_2,$ s.t. $f_1(x) \neq 0,\ f_2(x) \neq 0.$

Now, observe that $(f_1 f_2)(x) = (f_1(x))(f_2(x)) \neq 0$, but $f_1 f_2 \in J_1 \cap J_2$, so $x \notin V(J_1 \cap J_2)$.

$\square$

We conclude that Zariski-closed sets are closed under arbitrary intersections and under finite unions. Hence, the Zariski topology is actually a topology.

How can we extend this idea of adjunctions to general categories?

$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$

We have two functors

$$L : \mathcal{C} \longrightarrow \mathcal{D}$$
$$R : \mathcal{D} \longrightarrow \mathcal{C}.$$

What conditions make this an adjunction?

Recall that for posets, we say $x \leq y \iff |\mathrm{Hom}(x,y)| = 1$

$\therefore$ "$x \leq R(y) \iff L(x) \leq y$" $\iff$ "$\mathrm{Hom}(x, R(y)) = 1 \iff \mathrm{Hom}(L(x), y) = 1$"

$\iff$ "$\mathrm{Hom}(x, R(y)) \overset{\exists}{\longleftrightarrow} \mathrm{Hom}(L(x), y)$"

Idea: Maybe we call this an adjunction if we have bijections b/w the hom sets

$$\mathrm{Hom}(x, R(y)) \longleftrightarrow \mathrm{Hom}(L(x), y) \qquad \forall\, x \in \mathcal{C},\ y \in \mathcal{D}$$

Right idea, but it doesn't go far enough:
This family of bijections has to fit together in some kind of "natural" way.

We need, for example, that L determines R uniquely. How would we prove it?

$\forall\, x \in \mathcal{C},\ y \in \mathcal{D}$, we have bijections

$$\mathrm{Hom}(x, R(y)) \longleftrightarrow \mathrm{Hom}(L(x), y) \longleftrightarrow \mathrm{Hom}(x, R'(y))$$

$$\Rightarrow \mathrm{Hom}(x, R(y)) \longleftrightarrow \mathrm{Hom}(x, R'(y)).$$

Fix $y \in \mathcal{D}$. This holds for all x, so we want to conclude that $R(y) \cong_{\mathcal{C}} R'(y)$.

We need a property that forces this:

$$\mathrm{Hom}(x, y) = \mathrm{Hom}(x, y')\ \forall x \Rightarrow y = y'$$

e.g.: $\langle x, y\rangle = \langle x, y'\rangle \quad \forall x$

If $\langle\,,\,\rangle$ is bilinear + nondegenerate, $\langle x, y-y'\rangle = 0 \quad \forall x$

$$\Rightarrow y - y' = 0 \Rightarrow y = y'.$$

Let $\langle -, - \rangle$ be a nondegenerate billinear form on a k-vector space V.

For each $x \in V$, we get two linear maps $V \longrightarrow V^*$

$$\begin{aligned} V &\hookrightarrow V^* \\ x &\longmapsto \langle x, - \rangle \\ x &\longmapsto \langle -, x \rangle \end{aligned}$$

Grothendieck generalizes this to categories:

<u>The hom bifunctor:</u>

Given a category $\mathcal{C}$, we have some kind of "function"

$$\begin{aligned} Hom : \mathcal{C}^2 &\longrightarrow Set \\ (x, y) &\longmapsto Hom_{\mathcal{C}}(x, y) \end{aligned}$$

We want to think of this like an "inner product" on the category.

- It's a <u>functor</u>

$$Hom: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow Set$$

An arrow in $\mathcal{C}^{op} \times \mathcal{C}$ is a <u>pair</u> of arrows

$$\begin{array}{c} (x_2, y_2) \\ \uparrow \\ (x_1, y_1) \end{array} \quad \overset{?}{=} \quad \begin{array}{cc} x_2 & y_2 \\ \uparrow & \uparrow \\ x_1 & y_1 \end{array}$$

To say that $\mathrm{Hom}(-,-)$ is a "bifunctor" means that it has two component functors;

For each $X \in \mathcal{C}$, we set

$$H^X = \mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow \mathrm{Set}$$

and

$$H_X = \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$

Why are they functors?

$$\alpha : Y_1 \longrightarrow Y_2$$

$$\begin{aligned} H^X(\alpha) : \mathrm{Hom}(X, Y_1) &\longrightarrow \mathrm{Hom}(X, Y_2) \\ \varphi &\longmapsto \alpha \circ \varphi \end{aligned}$$

$$\begin{tikzcd} X \arrow[d, "\varphi"'] \arrow[dr, "\alpha \circ \varphi"] & \\ Y_1 \arrow[r, "\alpha"'] & Y_2 \end{tikzcd}$$

$$H^X(\alpha_1 \circ \alpha_2)(\varphi) = \alpha_1 \circ \alpha_2 \circ \varphi = \alpha_1 \circ (\alpha_2 \circ \varphi) = H^X(\alpha_1)(\alpha_2 \circ \varphi)$$

$$= H^X(\alpha_1)\big(H^X(\alpha_2)(\varphi)\big)$$

$$\Rightarrow \quad H^X(\alpha_1 \circ \alpha_2) = H^X(\alpha_1) \circ H^X(\alpha_2)$$

This is a "representation" of the "abstract $\circ$ operation" as some kind of composition.

$$\begin{aligned} H_X(\alpha) : \mathrm{Hom}(Y_1, X) &\longrightarrow \mathrm{Hom}(Y_2, X) \\ \varphi &\longmapsto \varphi \circ \alpha \end{aligned}$$

$$\begin{tikzcd} Y_1 \arrow[dr, "\varphi"'] & Y_2 \arrow[l, "\alpha"'] \arrow[d, dashed, "\varphi \circ \alpha"] \\ & X \end{tikzcd}$$

$$H_X(\alpha_1 \circ \alpha_2)(\varphi) = H_X(\alpha_2)\big(H_X(\alpha_1)(\varphi)\big)$$

b/c $\varphi \circ (\alpha_1 \circ \alpha_2) = (\varphi \circ \alpha_1) \circ \alpha_2$

What kind of functors are H^X & H_X?

Answer: Yoneda's Lemma

Statement: Rough idea: $\mathrm{Hom}(x, y) \cong \mathrm{Hom}(x, y') \ \forall x \Rightarrow y \cong y'$

By analogy with $V \hookrightarrow V^*$, Yoneda gives

$$\mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{C}^{op}}$$
$$\mathcal{C}^{op} \hookrightarrow \mathrm{Set}^{\mathcal{C}}$$

More precisely,

$$\mathrm{Nat}_{\mathrm{Set}^{\mathcal{C}^{op}}}(H^X, F) \leftrightarrow \mathrm{Hom}(X, F(-))$$

9/27/17

Flashback to RAPL for posets

Q: If $L : P \rightleftarrows Q : R$, $L \dashv R$, and

$\bigvee L(S)$ exists in Q, does $\bigvee S$ necessarily exist?

"Do left adjoints create colimits?"

ex: If $L \dashv R$ then
① L preserves order
② L preserves any colimits that exist in P.

$L(x) \leq y \iff x \leq R(y)$

$$L(S) \leq \bigvee L(S) \quad \forall S$$
$$\Rightarrow S \leq R\left(\bigvee L(S)\right) \quad \forall S$$

Conversely, let $L : P \longrightarrow Q$ be any function of posets and suppose that P has all colimits.

If (1) + (2) hold, then that L has a necessarily unique right adjoint.

"Freyd's Adjoint Functor theorem"

ex: $? : Ab \rightleftarrows Ab : Hom(A, -)$

$\uparrow$

$A \otimes (-)$

$\langle , \rangle : V \times V \longrightarrow K$

Bilinear + Nondegenerate Pairing

For each $x \in V$, get two linear functions

$H^x : V \longrightarrow K$, $H_x : V \longrightarrow K$
$y \longmapsto \langle x, y \rangle$ $\qquad y \longmapsto \langle y, x \rangle$

Moreover, we have that $H^x, H_x \in V^* = K^V$

So, we really have two functions

$H^{(-)} : V \hookrightarrow K^V$ $\qquad H_{(-)} : V \hookrightarrow K^V$
$x \longmapsto H^x$ $\qquad x \longmapsto H_x$

Two embeddings $V \hookrightarrow K^V$

Categories: $\mathrm{Hom}(-,-): \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Set}$

This is a bifunctor.

Bilinear?
Nondegenerate?

For each object $x \in \mathcal{C}$, we have two functors

$$H^x : \mathcal{C} \longrightarrow \mathrm{Set} \qquad\qquad H_x : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$
$$y \longmapsto \mathrm{Hom}(x,y) \qquad\qquad y \longmapsto \mathrm{Hom}(y,x)$$
$$\alpha \longmapsto \alpha \circ (-) \qquad\qquad \alpha \longmapsto (-) \circ \alpha = \text{"}\alpha^{op} \circ_{\mathcal{C}^{op}} (-)^{op}\text{"}$$

Given categories $\mathcal{C}, \mathcal{D}$, write $\mathcal{D}^{\mathcal{C}} := \{\text{functors } \mathcal{C} \to \mathcal{D}\}$

Thus, we have some kind of "functions"

$$x \longmapsto H^x$$
$$H^{(-)} : \mathcal{C} \longrightarrow \mathrm{Set}^{\mathcal{C}}$$
$$H_{(-)} : \mathcal{C} \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}$$
$$x \longmapsto H_x$$

They should be functors

How could they act on arrows?

Consider $\gamma : c_1 \longrightarrow c_2$ in $\mathcal{C}^{op}$

Do we get an "arrow" $H^\gamma : H^{c_1} \longrightarrow H^{c_2}$?

$H^\gamma : \mathrm{Hom}(c_1, -) \longrightarrow \mathrm{Hom}(c_2, -)$?

It needs to be a family of arrows:

$\forall\, d \in \mathcal{C}$, we need an arrow

$$\begin{tikzcd} c_1 \arrow[d, "\varphi"'] & c_2 \arrow[l, "\gamma"'] \arrow[dl] \\ d & \end{tikzcd}$$

$$H^\gamma_d : \mathrm{Hom}(c_1, d) \longrightarrow \mathrm{Hom}(c_2, d)$$
$$\varphi \longmapsto \varphi \circ \gamma$$

$H^\gamma(-) = (-)\circ\gamma$ ← This is a family of arrows.

How does it work with arrows?

$d_1 \xrightarrow{\delta} d_2$ $\qquad H^\gamma_{d_1}$? $H^\gamma_{d_2}$

$$\begin{tikzcd} H^{c_1}(d_2) \arrow[r, "H^\gamma_{d_2}"] & H^{c_2}(d_2) \\ H^{c_1}(d_1) \arrow[u, "H^{c_1}(\delta)"] \arrow[r, "H^\gamma_{d_1}"'] & H^{c_2}(d_1) \arrow[u, "H^{c_2}(\delta)"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}(c_1, d_2) \arrow[r, "(-)\circ\gamma"] & \mathrm{Hom}(c_2, d_2) \\ \mathrm{Hom}(c_1, d_1) \arrow[u, "\delta\circ(-)"] \arrow[r, "(-)\circ\gamma"'] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{Hom}(c_2, d_1) \arrow[u, "\delta\circ(-)"'] \end{tikzcd}$$

$(\delta\circ\varphi)\circ\gamma = \delta\circ(\varphi\circ\gamma)$ ✓

Natural Transformations:

Def: Given two functors $F_1, F_2 : \mathcal{C} \longrightarrow \mathcal{D}$, a natural transformation

$$\Phi : F_1 \Rightarrow F_2$$

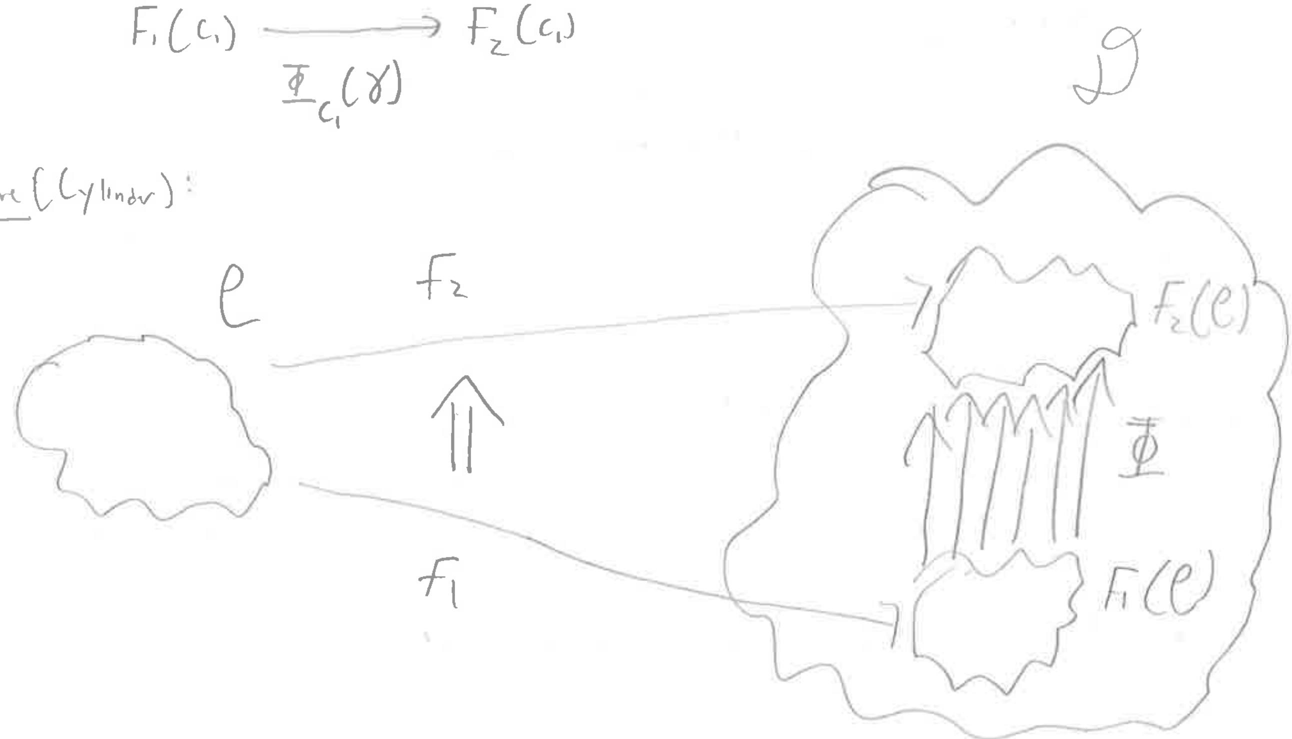
consists of a family of arrows in $\mathcal{D}$

$$\Phi_c : F_1(c) \longrightarrow F_2(c) \quad (\text{one for each object of } \mathcal{C})$$

Such that for each arrow $c_1 \xrightarrow{\gamma} c_2$ in $\mathcal{C}$, we have a commutative square

$$\begin{tikzcd}
F_1(c_2) \arrow[r, "\Phi_{c_2}(\gamma)"] & F_2(c_2) \\
F_1(c_1) \arrow[u, "F_1(\gamma)"] \arrow[r, "\Phi_{c_1}(\gamma)"'] \arrow[ur, phantom, "\circlearrowright"] & F_2(c_1) \arrow[u, "F_2(\gamma)"']
\end{tikzcd}$$

Picture (Cylinder):



Observation: functors $\mathcal{C} \longrightarrow \mathcal{D}$ are like "diagrams of shape $\mathcal{C}$ living in $\mathcal{D}$".

Natural transformations are "cylinders" b/w diagrams.

We get a category of diagrams $\mathcal{D}^{\mathcal{C}}$ where the arrows are natural transformations.

ex: If $\mathcal{I}$ is a small category then $\mathcal{C}^{\mathcal{I}}$ is the category of shape $\mathcal{I}$ in $\mathcal{C}$.

For each object $c \in \mathcal{C}$, we can define the constant diagram

$$
\begin{aligned}
c^{\mathcal{I}} : \mathcal{I} &\longrightarrow \mathcal{C} \\
i &\longmapsto c \\
\varphi &\longmapsto id_c
\end{aligned}
$$

For any diagram $D : \mathcal{I} \longrightarrow \mathcal{C}$, $\Phi : c^{\mathcal{I}} \Longrightarrow D$
cone under D

$\Phi : D \Longrightarrow c^{\mathcal{I}}$
cone over D

Limits + Colimits: For a diagram $D : \mathcal{I} \longrightarrow \mathcal{C}$, we have a category of cones under D:

$$\mathrm{Cone}(D) = \{ c^{\mathcal{I}} \Longrightarrow D \}$$

Arrows:

$$
\begin{tikzcd}
 & D & \\
c_1^{\mathcal{I}} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & c_2^{\mathcal{I}} \arrow[ul, Rightarrow]
\end{tikzcd}
$$

A limit, if it exists, is a terminal object in $\mathrm{Cone}(D)$.

Since $\mathrm{Set}^{\mathcal{C}}$ & $\mathrm{Set}^{\mathcal{C}^{op}}$ are categories, we can now state that

$$H^{(-)} : \mathcal{C}^{op} \longrightarrow \mathrm{Set}^{\mathcal{C}}$$

$$H_{(-)} : \mathcal{C} \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}$$

are functors. Bilinearity ✓

Non-degeneracy?

Def: A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is called full if all functions

$$F: \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2))$$

are surjective; faithful if all functions

$$F: \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2))$$

are injective; and an embedding (fully faithful) if all functions

$$F: \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2))$$

are bijective.

Lemma: Embeddings are

- essentially injective on objects:

$$c_1 \cong_{\mathcal{C}} c_2 \iff F(c_1) \cong_{\mathcal{D}} F(c_2)$$

- essentially monic: Given $\mathcal{B} \overset{G_1}{\underset{G_2}{\rightrightarrows}} \mathcal{C} \xrightarrow{F} \mathcal{D}$

$$G_1 \cong_{\mathcal{C}^{\mathcal{B}}} G_2 \iff F \circ G_1 \cong_{\mathcal{D}^{\mathcal{B}}} F \circ G_2$$

"left-cancellable up to natural iso."

$$H^X \cong H^Y \iff X \cong Y$$

10/11/17: Recall: A natural transformation is a "cylinder between diagrams".

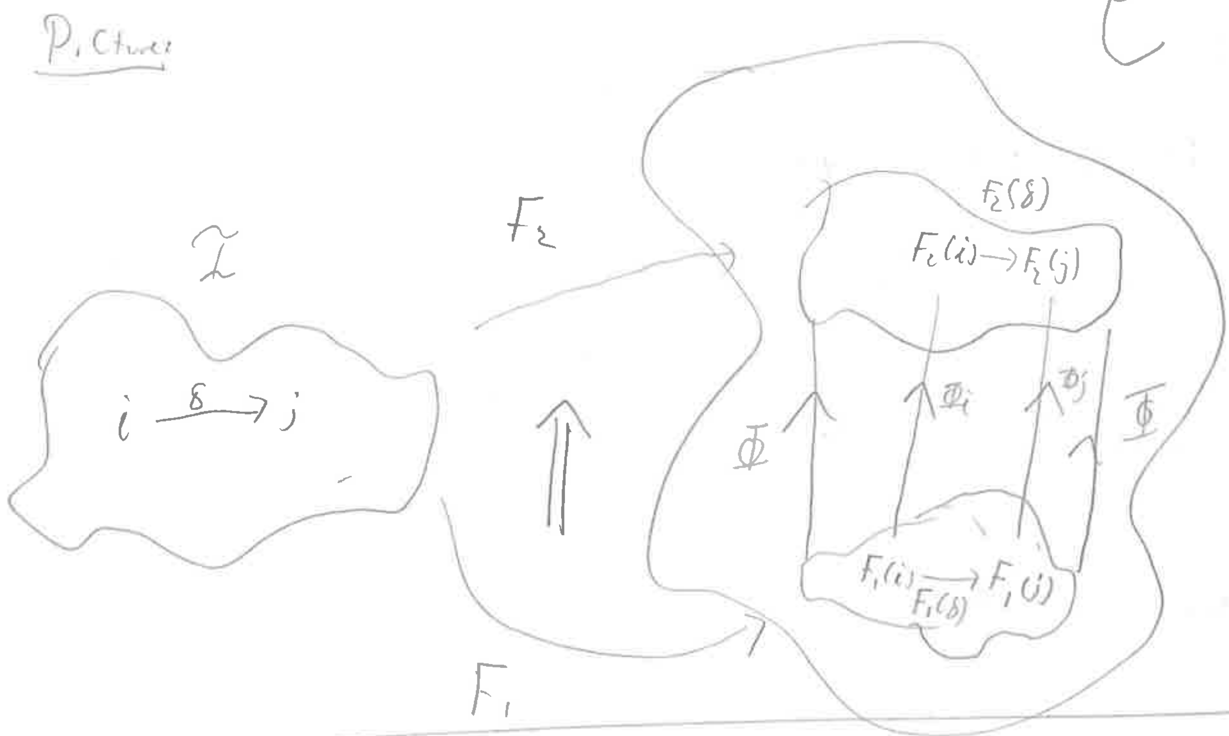
Let $F_1, F_2 : \mathcal{I} \longrightarrow \mathcal{C}$. A natural transformation consists of, $\forall i \in \mathcal{I}$, an arrow $\Phi_i : F_1(i) \longrightarrow F_2(i)$ s.t.

$\forall g : i \longrightarrow j \in \mathcal{I}$, the square

$$\begin{tikzcd} F_2(i) \arrow[r, "F_2(g)"] & F_2(j) \\ F_1(i) \arrow[u, "\Phi_i"] \arrow[r, "F_1(g)"'] \arrow[ur, phantom, "\circlearrowright"] & F_1(j) \arrow[u, "\Phi_j"'] \end{tikzcd}$$

Commutes.

Picture



Def: Consider functors $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$. We say that this is an adjunction $(L \dashv R)$ if there exists a family of bijections

$$\Phi_{c,d} : \mathrm{Hom}_{\mathcal{C}}(c, R(d)) \longleftrightarrow \mathrm{Hom}_{\mathcal{D}}(L(c), d)$$

that is natural in $(c, d) \in \mathcal{C}^{op} \times \mathcal{D}$

In other words, we have an isomorphism of functors

$$\Phi : \mathrm{Hom}_{\mathcal{C}}(-, R(-)) \cong \mathrm{Hom}_{\mathcal{D}}(L(-), -)$$

in the category $[\mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathrm{Set}]$

Analogy: $\langle -, R(-) \rangle = \langle L(-), - \rangle$

Convenient to write $\overline{\Phi}_{c,d}$ & $\overline{\Phi}^{-1}_{c,d}$ simply as

$$\varphi \longmapsto \overline{\varphi} \longmapsto \overline{\overline{\varphi}} = \varphi.$$

Then, naturality just says

$$\left.\begin{aligned} \overline{\varphi \circ L(\gamma)} &= \overline{\varphi} \circ \gamma \\ \overline{R(\delta) \circ \psi} &= \delta \circ \overline{\psi} \end{aligned}\right\} \text{ whenever composition is defined.}$$

How do we know that this is the "correct" definition?

If $L \dashv R$ and $L' \dashv R$, then we should have $L \cong L'$.

Assume we have isomorphisms

$$\mathrm{Hom}_{\mathscr{D}}(L(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, R(-)) \cong \mathrm{Hom}_{\mathscr{D}}(L'(-), -).$$

We have a family of bijections

$$\Phi_{c,d} : \mathrm{Hom}_{\mathscr{D}}(L(c), d) \longleftrightarrow \mathrm{Hom}_{\mathscr{D}}(L'(c), d), \text{ natural in}$$

$(c,d) \in \mathcal{C}^{op} \times \mathscr{D}$.

For each fixed $c \in \mathcal{C}^{op}$, naturality in $d \in \mathscr{D}$ means we have an iso:

$$\mathrm{Hom}_{\mathscr{D}}(L(c), -) \cong \mathrm{Hom}_{\mathscr{D}}(L'(c), -)$$

in the category $[\mathscr{D} \longrightarrow \mathrm{Set}]$

Equivalently, $H^{L(c)} \cong H^{L'(c)}$, or

$$(H^{(-)} \circ L)(c) \cong (H^{(-)} \circ L')(c)$$

$$\begin{tikzcd} c \in \mathcal{C}^{op} \arrow[r, "L"] \arrow[rr, bend right, "H^{(-)} \circ L"'] & \mathcal{D}^{op} \arrow[r, "H^{(-)}"] & Set^{\mathcal{D}} \ni H^{L(c)} \end{tikzcd}$$

Now, naturality in $c \in \mathcal{C}^{op}$ gives us an isomorphism

$$H^{(-)} \circ L \cong H^{(-)} \circ L'$$

in the category $[\mathcal{C}^{op} \longrightarrow Set^{\mathcal{D}}]$

Question: $H^{(-)} \circ L \cong H^{(-)} \circ L' \Longrightarrow L \cong L'$?

I.e., is $H^{(-)}$ left-cancellable, at least up to isomorphism?

Issue: "injective", "embedding", "left-cancellable"

Def: We say $c \in Ar(\mathcal{C})$ is monic if it is left-cancellable, i.e. $\forall \alpha, \beta \in Ar(\mathcal{C})$, $c \circ \alpha = c \circ \beta \Leftrightarrow \alpha = \beta$

Exercise: In Set, monic $\equiv$ injective

Jargon: We are hoping that $H^{(-)}$ is "monic up to isomorphism" a.k.a. "essentially monic"

Category Embeddings:

Every functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ comes with a family of Hom functions $\forall (c_1, c_2) \in \mathcal{C}^2$, $F: Hom_{\mathcal{C}}(c_1, c_2) \longrightarrow Hom_{\mathcal{D}}(F(c_1), F(c_2))$.

Call F faithful if all Hom functions are injective
full " " " surjective
embedding " " " bijective

Embedding Lemma: Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an embedding. Then F is essentially injective, i.e.,

$$C_1 \cong_{\mathcal{C}} C_2 \iff F(C_1) \cong_{\mathcal{D}} F(C_2)$$

AND

F is essentially monic, i.e., for any functors $G_1, G_2 : \mathcal{B} \to \mathcal{C}$,

$$G_1 \cong G_2 \iff F \circ G_1 \cong F \circ G_2$$

Pf: Suppose $\alpha : C_1 \longleftrightarrow C_2 : \beta$ is an isomorphism in $\mathcal{C}$.
Apply F to get arrows:

$$F(\alpha) : F(C_1) \rightleftarrows F(C_2) : F(\beta)$$

Note: $F(\alpha) \circ F(\beta) = F(\alpha \circ \beta) = F(id_{C_2}) = id_{F(C_2)}$ ✓

Similarly, $F(\beta) \circ F(\alpha) = id_{F(C_1)}$ ✓

Conversely, suppose $\alpha' : F(C_1) \longleftrightarrow F(C_2) : \beta'$

By fullness, $\exists\ \alpha : C_1 \rightleftarrows C_2 : \beta$ with $F(\alpha) = \alpha'$, $F(\beta) = \beta'$

$$F(\alpha \circ \beta) = F(\alpha) \circ F(\beta) = \alpha' \circ \beta' = id_{F(C_2)} = F(id_{C_2}).$$

By faithfulness, this means that $\alpha \circ \beta = id_{C_2}$. ✓
Similarly, $\beta \circ \alpha = id_{C_1}$ ✓

$G_1 \cong G_2 \Rightarrow F \circ G_1 \cong F \circ G_2$ always ✓

Conversely, Suppose $\exists\ \Phi' : F \circ G_1 \overset{\sim}{\Rightarrow} F \circ G_2$, i.e.

$\forall\ \beta : b_1 \to b_2 \in \mathcal{B}$

$$\begin{tikzcd} F(G_1(b_2)) \arrow[r, "\Phi'_{b_2}"] & F(G_2(b_2)) \\ F(G_1(b_1)) \arrow[u, "F(G_1(\beta))"] \arrow[r, "\Phi'_{b_1}"'] \arrow[ur, phantom, "\circlearrowright"] & F(G_2(b_1)) \arrow[u, "F(G_2(\beta))"'] \end{tikzcd}$$

By 1^{st} part, $\exists\ \Phi_b : G_1(b) \xrightarrow{\sim} G_2(b)$,

where $F(\Phi_b) = \Phi'_b$. Does the resulting square

$$\begin{tikzcd} G_1(b_2) \arrow[r, "\Phi_{b_2}", "\sim"'] & G_2(b_2) \\ G_1(b_1) \arrow[u, "G_1(\beta)"] \arrow[r, "\sim", "\Phi_{b_1}"'] & G_2(b_1) \arrow[u, "G_2(\beta)"'] \end{tikzcd}$$

Commute?

$$F(\Phi_{b_2} \circ G_1(\beta)) = F(\Phi_{b_2}) \circ F(G_1(\beta)) = \Phi'_{b_2} \circ F(G_1(\beta))$$
$$= F(G_2(\beta)) \circ \Phi'_{b_1}$$
$$= F(G_2(\beta) \circ \Phi_{b_1})$$

Since F is faithful $\Phi_{b_2} \circ G_1(\beta) = G_2(\beta) \circ \Phi_{b_1}$ $\square$

10/6/17 <u>Theorem</u> (Uniqueness of adjoints):

If $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ adjunction
$L' : \mathcal{C} \rightleftarrows \mathcal{D} : R$ adjunction

$$\Rightarrow L \cong L'$$

<u>Ex</u>: $(-)\otimes(-) : Ab \otimes Ab \longrightarrow Ab$
$\mathrm{Hom}(-,-) : Ab^{op} \times Ab \longrightarrow Ab$

<u>Pf</u>: $\mathrm{Hom}(L(-),-) \cong \mathrm{Hom}(-,R(-)) \cong \mathrm{Hom}(L'(-),-)$

$$\Rightarrow \mathrm{Hom}(L(-),-) \cong \mathrm{Hom}(L'(-),-)$$

$\Leftrightarrow H^{(-)} \circ L \cong H^{(-)} \circ L' \Longrightarrow L \cong L'$, since $H^{(-)}$ is an embedding

$\square$

The Yoneda Lemma (In "fullest generality"):

Two functors $\mathcal{C} \times Set^{\mathcal{C}} \longrightarrow Set$:

$$\mathrm{Eval} : \mathcal{C} \times Set^{\mathcal{C}} \longrightarrow Set$$
$$(c, F) \longmapsto F(c)$$

$$\mathrm{Yon} : \mathcal{C} \times Set^{\mathcal{C}} \longrightarrow Set$$
$$(c, F) \longmapsto \mathrm{Nat}(H^c, F)$$

<u>Claim</u>: Natural isomorphism

$$\Xi_{c,F} : \mathrm{Nat}(H^c, F) \xrightarrow{\sim} F(c).$$
$$(\Phi : H^c \Rightarrow F) \longmapsto \Phi_c(id_c)$$

Corollaries:

$H^{(-)} : \mathcal{C}^{op} \longrightarrow Set^{\mathcal{C}}$ is an embedding

$c \longmapsto H^c$

Pf: Let $F : \mathcal{C} \longrightarrow Set$ be H^d

Get bijections $Nat(H^c, H^d) \longleftrightarrow H^d(c)$

$$Hom_{Set^{\mathcal{C}}}(H^c, H^d) \longleftrightarrow Hom_{\mathcal{C}}(d, c) = Hom_{\mathcal{C}^{op}}(c, d)$$

$\Rightarrow H^{(-)}$ is fully faithful

$\square$

"regular representation"

Note: This implies that $c_1 \cong c_2 \iff H^{c_1} \cong H^{c_2}$

$c_1 \cong c_2 \iff H_{c_1} \cong H_{c_2}$

$H_{(-)} : \mathcal{C} \longrightarrow Set^{\mathcal{C}^{op}}$ free co-completion

$Y : \mathcal{C} \longrightarrow \hat{\mathcal{C}}$

The Yoneda functor embeds $\mathcal{C}$ into its category of presheaves

Recall: Let $Y : \mathcal{C} \longrightarrow Set^{\mathcal{C}^{op}}$ be Yoneda.

Then $\exists$

$$U \dashv V \dashv W \dashv X \dashv Y \iff \mathcal{C} \cong Set$$

RAPL: Right-adjoints preserve limits, left-adjoints preserve colimits.

Idea/Motivation: Let $\langle -, - \rangle : V \times V \longrightarrow \mathbb{C}$ be a continuous + nondegenerate bilinear function. Let $L : V \rightleftarrows V : R$ be adjoint operators, i.e.

$$\langle u, R(v) \rangle = \langle L(u), v \rangle \qquad \forall\, u, v \in V$$

Claim: If $\lim_i v_i \in V$ exists, then $\lim_i R(v_i)$ also exists, and

$$\lim_i R(v_i) = R(\lim_i v_i)$$

Pf: $\forall\, u \in V$, we have
$$\begin{aligned}\langle u, R(\lim v_i) \rangle &= \langle L(u), \lim v_i \rangle \\ &= \lim \langle L(u), v_i \rangle \\ &= \lim \langle u, R(v_i) \rangle \\ &= \langle u, \lim R(v_i) \rangle\end{aligned}$$

$$\Rightarrow R(\lim v_i) = \lim R(v_i).$$

$\square$

Question: Given $c \in \mathcal{C}$, is the functor $H^c : \mathcal{C} \longrightarrow \mathrm{Set}$ continuous?

Check: Suppose the diagram $D : \mathcal{I} \longrightarrow \mathcal{C}$ has a limit ℓ, with cone Δ, i.e.

$$\forall\, i \xrightarrow{\delta} j \qquad \begin{tikzcd} & D(i) \arrow[rr, "D(\delta)"] & & D(j) \\ & & \ell \arrow[ul, "\Delta_i"] \arrow[ur, "\Delta_j"'] & \\ & & ? \arrow[u, "\exists!"] & \end{tikzcd} \qquad \mathcal{C}$$

Then, applying H^c, we have

$$
\begin{tikzcd}
\overset{\varphi_i(s)}{H^c(D(i))} \arrow[rr, "H^c(D(s))"] & & \overset{\varphi_j(s)}{H^c(D(j))} \\
& \overset{\bar{\varphi}(s)}{H^c(\ell)} \arrow[ul, "H^c\Lambda_i"'] \arrow[ur, "H^c\Lambda_j"] & \\
& S \arrow[u, dashed, "\exists!\bar{\varphi}"'] \arrow[uul, bend left, "\varphi_i"] \arrow[uur, bend right, "\varphi_j"'] &
\end{tikzcd}
\qquad \text{SET}
$$

$$s \in S$$

Go back to the first diagram! $\varphi_i(s), \varphi_j(s) \in \mathrm{Hom}(C, D(i)),$
$\mathrm{Hom}(C, D(j)),$

$$
\begin{tikzcd}
D(i) \arrow[rr, "D(s)"] & & D(j) \\
& \ell \arrow[ul, "\Lambda_j"'] \arrow[ur, "\Lambda_i"] & \\
& C \arrow[u, dashed, "\exists!\bar{\varphi}(s)"'] \arrow[uul, bend left, "\varphi_i(s)"] \arrow[uur, bend right, "\varphi_j(s)"'] &
\end{tikzcd}
$$

$$\Rightarrow \varphi_i(s) = \Lambda_i \circ \bar{\varphi}(s), \quad \varphi_j(s) = \Lambda_j \circ \bar{\varphi}(s)$$

$\Rightarrow H^C$ is continuous!

∎

$\mathrm{Hom}(-,-)$ is continuous in 2^{nd} coordinate. What about 1^{st} coord?

Claim: $\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{C}}\, \Lambda, -) = \lim \mathrm{Hom}_{\mathcal{C}}(\Lambda, -)$

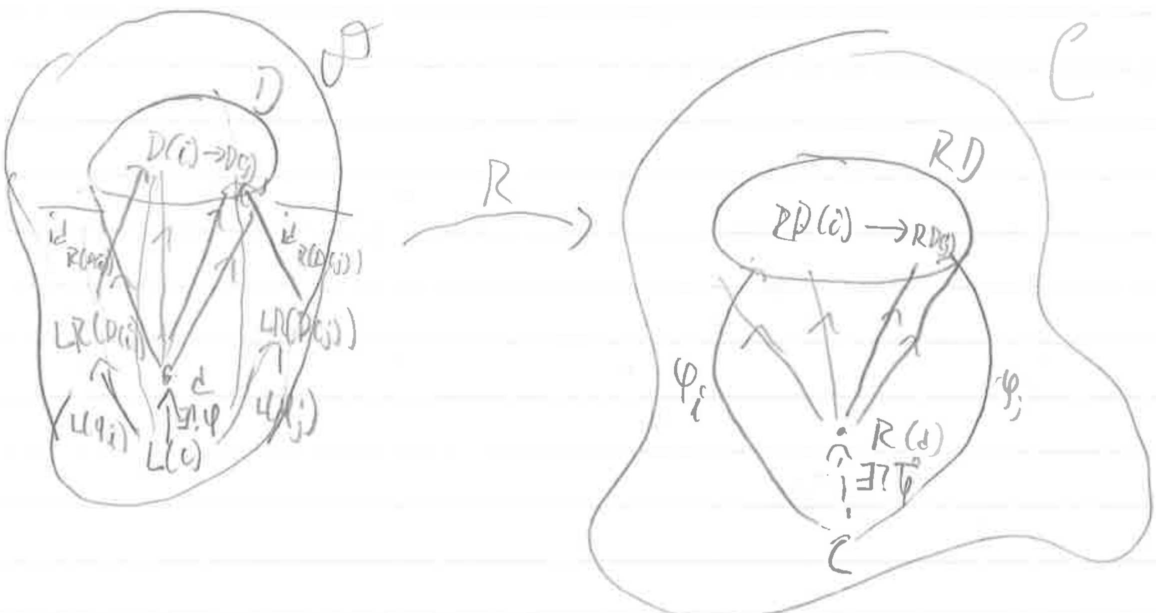
$\quad \| $

$\mathrm{Hom}_{\mathcal{C}^{op}}(-, \mathrm{colim}_{\mathcal{C}}\, \Lambda) = \mathrm{Hom}_{\mathcal{C}^{op}}(-, \lim_{\mathcal{C}^{op}} \Lambda) \ \ldots ?$

General adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$

Show: R preserves limits:

Suppose the diagram $D : \mathcal{I} \longrightarrow \mathcal{D}$



Adjunction: $\mathrm{Hom}(c, R(d)) \leftrightarrow \mathrm{Hom}(L(c), d)$

Consider $R(x) \xrightarrow{id_{R(x)}} R(x) \xrightarrow{\text{adjunction}} L(R(x)) \xrightarrow{\overline{id_{R(x)}}} x$

$\Rightarrow L(c)$ is also a cone below D, so $\exists!\ \varphi : L(c) \to D$, Constant diagrams

$$c^{\mathcal{I}} : \mathcal{I} \to \mathcal{C}$$

Given a diagram $D : \mathcal{I} \to \mathcal{C}$, a cone under D is a natural transformation $c^{\mathcal{I}} \Rightarrow D$

Diagonal functor

$$(-)^{\mathcal{I}} : \mathcal{C} \rightleftarrows \mathcal{C}^{\mathcal{I}} : ?$$
$$c \longmapsto c^{\mathcal{I}}$$

Theorem: $\operatorname{colim}_{\mathcal{I}} \dashv (-)^{\mathcal{I}} \dashv \lim_{\mathcal{I}}$

10/16/17

What is "representation theory"?

Symmetries of an object $\rightsquigarrow$ $\boxed{\text{abstract group}}$

Abstract group $\rightsquigarrow$ $\boxed{\text{Concrete representations of the group}}$

Category of representations

$G \longrightarrow$ functors $[BG \to \mathcal{C}]$
$[BG \to \mathrm{Set}]$

Philosophy:
$G \longrightarrow$ Category of G-Sets
$G\text{-Set}$

Hopefully we don't lose any information, i.e. G-Set determines G up to iso.

Thm (Tanaka Duality):

Let M be a monoid. The Category Set^{BM} of M-sets determines the monoid M as follows:

Set^{BM} is a Concrete Category, meaning we have a faithful functor

$$U : Set^{BM} \longrightarrow Set$$

Forgetful "Underlying set" functor.

Observe that U is an object in the functor Category $Set^{(Set^{BM})}$

Claim: $M \cong End(U)$

Proof: Yoneda trick

Yoneda $\Rightarrow$ $Eval, Yon : Set^{\mathcal{C}} \times \mathcal{C} \longrightarrow Set$

$$(F, C) \longmapsto F(C)$$

$$(F, C) \longmapsto Nat(H^C, F)$$

$$Eval \cong Yon$$

$$M\text{-}Set = Set^{BM}$$

$$M\text{-}Set \times BM \longrightarrow Set$$

$$(F, *) \longmapsto F(*)$$

$$\Rightarrow Eval = U : M\text{-}Set \to Set$$

$$(-, *) \longmapsto (-)(*)$$

$$\mathcal{U} = \mathrm{Eval}(-, *) = \mathrm{Nat}(H^*, -) = H^{H^*}$$

Thus, by Yoneda in Set

$$\mathrm{Hom}(\mathcal{U}, \mathcal{U}) = \mathrm{Hom}(H^{H^*}, H^{H^*})$$

<u>Yoneda:</u> $\mathrm{Hom}(H^c, H^d) = \mathrm{Hom}(c, d)$

$$= \mathrm{Hom}(H^*, H^*) \leftarrow \text{in } BM \to \mathrm{Set}$$

$$= \underset{\substack{\uparrow \\ BM}}{\mathrm{Hom}(*, *)} = M$$

$\square$

Idea: $G \to G\text{-Set}$

<u>Last time</u>: In G-set, we have indecomposable $\Rightarrow$ irreducible

red $\Rightarrow$ decomp.

$$\begin{matrix} Y \\ \cup\!\!\!\mid^{G} \\ X \end{matrix} \rightsquigarrow Y = X \coprod_G (Y \setminus X)$$

$\rightsquigarrow$ Unique irreducible decomposition of G-Sets

$X = \coprod$ orbits

<u>Orbit-Stabilizer Theorem</u>: $X = \coprod_i G/H_i$

$\Rightarrow$ Simple objects have the form G/H, $H \leq G$, and

$G/H \cong_G G/K \iff gHg^{-1} = K$ for some $g \in G$

Simple objects $\longleftrightarrow$ Conj. Classes of Subgroups

Let $[G/H]$ = isomorphism class of G/H as a G-set

Arrows in G-set?

$\varphi : \coprod G/H_i \longrightarrow_G \coprod G/K_j)$

$\mathrm{Hom}(\coprod G/H_i, \coprod G/K_i) = \prod \mathrm{Hom}(G/H_i, \coprod G/K_i)$

$\varphi : G/H_i \longrightarrow \coprod G/K_i$

$G \curvearrowright \varphi(G/H)$ transitively, so $\varphi : G/H \longrightarrow G/K$

$\mathrm{Hom}(G/H, G/K) = \emptyset$, unless $gHg^{-1} \leq K$

To what extent does $\mathrm{Hom}(-,-)$ behave like an inner product on a vector space with basis = Conj. Classes of Subgroups?

Were that the case, we'd want

$$\dim(\mathrm{Hom}(U,V)) = \begin{cases} 1, & U \cong V \\ 0, & U \not\cong V \end{cases}$$

Burnside's Table of Marks:

rows/columns = Conj. Classes of Subgroups

entry in row H, col K = $|\mathrm{Hom}(G/H, G/K)|$.

Is that all we can say?

Louis Solomon 1967:

$$\text{G-Set} \rightsquigarrow \underbrace{B[\text{G-Set}]}_{\text{Grothendieck ring of G-Set, called the Burnside ring}}$$

← Commutative ring structure

Generated by iso. classes $[G/H]$ with operations

$$[G/H] + [G/K] = [G/H \amalg G/K]$$

$$[G/H] \cdot [G/K] = [G/H \times G/K]$$

$B[G]$ = Burnside Ring

What does structure of $B[G]$ tell us about G?

Consider the poset P_G of conjugacy classes of subgroups of G wrt inclusion

Solomon's Theorem: $B[G] \otimes_{\mathbb{Z}} \mathbb{Q} \cong \text{Möb}[P_G]$

Given $x, y \in P_G$, define $xy = \sum_{p \in P} \left(\sum_{p \leq z \leq x, y} \mu(p, z) \right) \cdot p$

General Situation: Let A be an "algebra" of some sort, thought of as a category with one object.

Representations are functors

$$F : A \longrightarrow C$$

Say, $\mathcal{C}^A$ has some kind of $\oplus, \otimes$.

Grothendieck ring $R[A]$ of "Characters"

Question: $[X][Y] = \sum_Z [(?)][Z]$

Structure Constants

$$\underline{M\text{-Set}} \rightsquigarrow \underline{G\text{-Set}} \longrightarrow G\text{-}\boxed{Ab},$$

$\times, \amalg$ $\qquad$ $\oplus, \otimes$

What is a ring?
End(Ab)

Let $A \in Ab$, $End(A)$ = monoid of group endomorphisms $\varphi: A \to A$

We get an extra operation "$\varphi_1 + \varphi_2$"

$$(\varphi_1 + \varphi_2)(a) = \varphi_1(a) + \varphi_2(a)$$

$(End(A), +, \circ)$ is a ring.

It's nice when we can add arrows in a category. Ab is the prime example.

Ab is "Ab-enriched"

---

locally-small category $= \boxed{Set}$-enriched category

any symmetric monoidal category

Monoidal : $+$ coproduct
$\otimes$ tensor

If $\mathcal{V}$ is a Symmetric monoidal category, we can talk about $\mathcal{V}$-enriched categories, and everything we do still works.

10/18/17: Enriched Categories:

Let $\mathcal{V}$ be (some kind) of category
A $\mathcal{V}$-category $\mathcal{C}$ consists of

- A collection $Obj(\mathcal{C})$ of objects
- $\forall x, y \in \mathcal{C}$ a Hom-object

$$Hom_{\mathcal{C}}(x,y) \in \mathcal{V}$$

- $\forall x,y,z \in \mathcal{C}$, an arrow $Hom_{\mathcal{C}}(x,y) \otimes Hom_{\mathcal{C}}(y,z) \longrightarrow Hom_{\mathcal{C}}(x,z)$ in $\mathcal{V}$

We require, at least, $(\mathcal{V}, \otimes, 1) \leftarrow$ a monoidal category
complete

Axioms as before.

Possibilities:

Set-category $\equiv$ category
Ab-category $\equiv$ additive category
$\mathbb{Z}$-linear category

Consider the category $\vec{2} = (0 \to 1)$

$\otimes$	0	1
0	0	0
1	0	1

A $\bar{2}$ - category $\equiv$ preorder/poset

Enriched Yoneda:

If $\mathcal{C}$ is a $\mathcal{V}$-category, we have two canonical embeddings

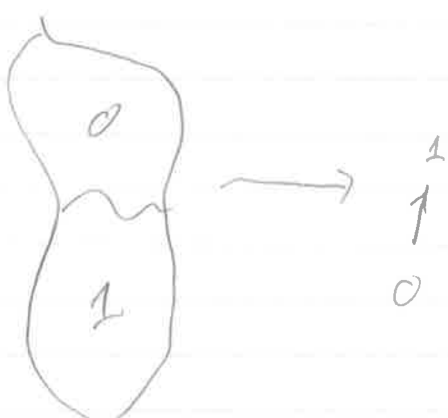
$\mathcal{C} \hookrightarrow \mathcal{V}^{\mathcal{C}^{op}}$ free co-completion

$\mathcal{C} \hookrightarrow (\mathcal{V}^{\mathcal{C}})^{op}$ free completion

Ex: $\mathcal{V} = \bar{2}$

Let P be a poset, i.e. a $\bar{2}$-category. Describe the category $\bar{2}^{P^{op}}$

functors $P^{op} \longrightarrow \bar{2}$ $\equiv$ order ideals in P



$P \hookrightarrow$ order ideals of P

$x \longmapsto P_{\leq x}$

Similarly, $P \hookrightarrow (2^P)^{op}$ = order filters

$x \mapsto P_{\geq x}$

Q: Can we perform the completion + cocompletion simultaneously to embed P into a lattice?

A: Yes! Dedekind cut (1872)

(A,B) with property that

$A \subseteq$ lower bounds of $B = B^\wedge$

$B \subseteq$ upper bounds of $A = A^\vee$

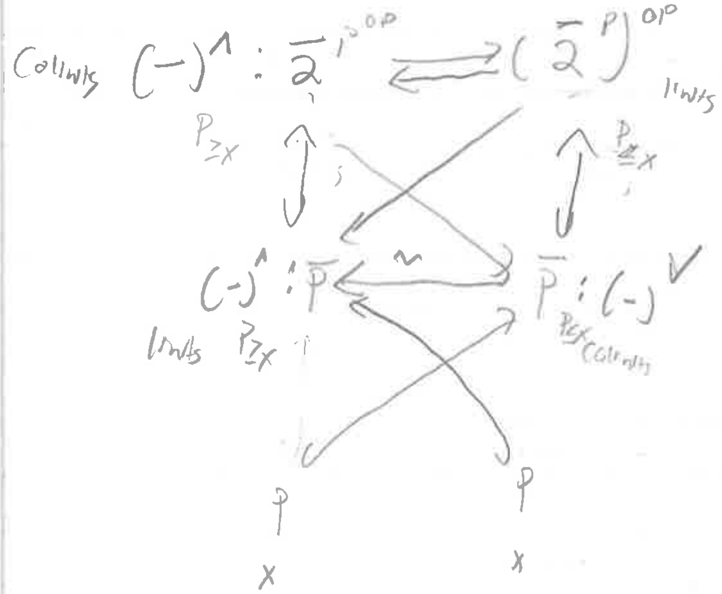
In fact, we claim that for any poset P we obtain an adjunction

$$(-)^\wedge : 2^P \rightleftarrows (2^P)^{op} : (-)^\vee$$

$\forall\ A, B \subseteq P$, we have

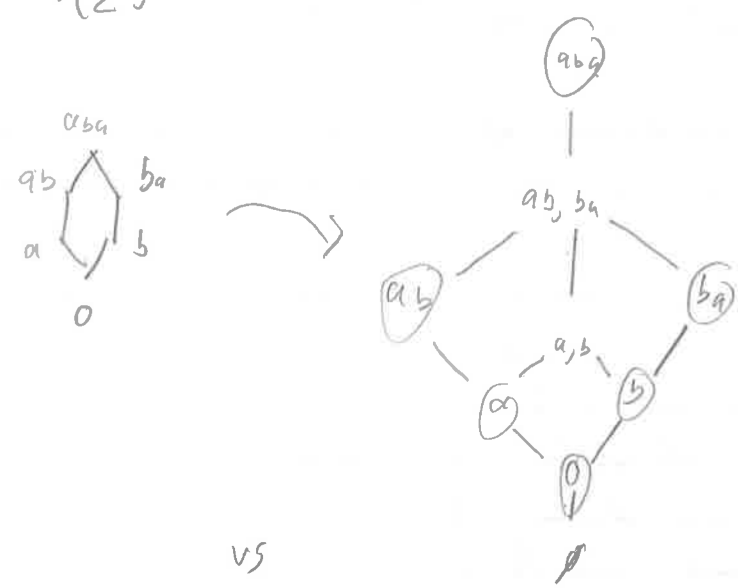
$$\begin{aligned}
A \subseteq B^\wedge &\iff \forall x \in A,\ x \in B^\wedge \\
&\iff \forall x \in A,\ y \in B,\ x \leq y \\
&\iff \forall y \in B,\ \forall x \in A,\ x \leq y \\
&\iff \forall y \in B,\ y \in A^\vee \\
&\iff B \subseteq A^\vee \\
&\iff A^\vee \subseteq_{op} B
\end{aligned}$$

This adjunction (trivially) restricts to the subposets of order ideals and order filters:



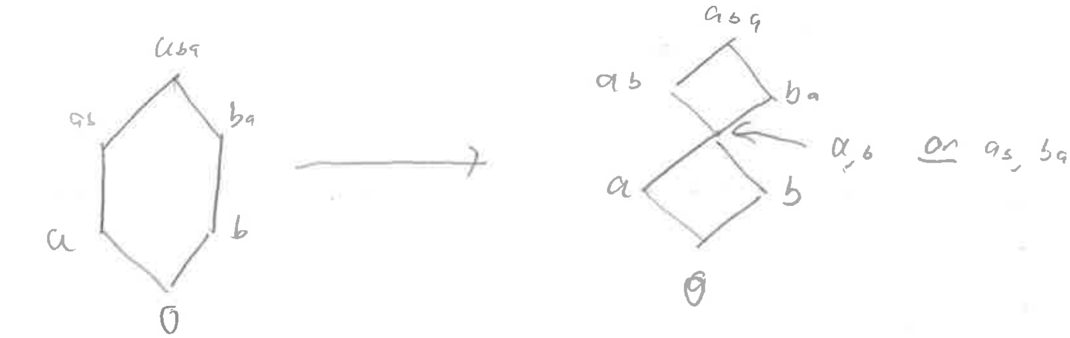
$$\begin{tikzcd} P \arrow[r] & \bar{P} \arrow[ur] \arrow[dr] & \bar{2}^{P^{op}} \\ & & (\bar{2}^P)^{op} \end{tikzcd}$$

ex:



vs

So the Dedekind - MacNeille Completion is



<u>Ab-Categories</u>: Situation where Hom-sets are abelian groups

Category Ab is the prototypical Ab-category. Let $A, B \in Ab$. Then, $Hom_{Ab}(A,B)$ is an abelian group

commutative

$$\varphi_1, \varphi_2 : A \to B, \quad (\varphi_1 + \varphi_2)(a) = \varphi_1(a) + \varphi_2(a)$$

$$(-\varphi)(a) = \varphi(-a)$$

$$0(a) = 0_B$$

$$\begin{tikzcd} A \arrow[rr, bend left, "\text{zero map}"] \arrow[dr, "\exists!"'] & & B \\ & 0 \arrow[ur, "\exists!"'] & \end{tikzcd}$$

<u>Composition</u>: Given $A, B, C \in Ab$

$$Hom(A,B) \times Hom(B,C) \xrightarrow{\circ} Hom(A,C)$$

homom of Abelian groups?

$$(\psi_1+\psi_2)(\varphi_1+\varphi_2)(x) = \psi_1(\varphi_1+\varphi_2) + \psi_2(\varphi_1+\varphi_2)$$
$$= \psi_1(\varphi_1) + \psi_1(\varphi_2) + \psi_2(\varphi_1) + \psi_2(\varphi_2)$$

$$(\psi_1, \varphi_1) + (\psi_2, \varphi_2) \xrightarrow{?} \psi_1 \circ \varphi_1 + \psi_2 \circ \varphi_2$$

NO!

This map is a bihomomorphism, so... "Use tensor products"

$$A \oplus B \xrightarrow{\text{bi hom}} C$$
$$? \xrightarrow{\text{hom}} C$$

10/23/17:

Enriched Categories:

Let $(\mathcal{V}, \otimes, 1)$ be a "monoidal category". We can define a "$\mathcal{V}$-category" $\mathcal{C}$ like

- $\forall x, y \in \mathcal{C}$, we have a Hom-object $\mathrm{Hom}_{\mathcal{C}}(x,y) \in \mathcal{V}$
- $\forall x, y, z \in \mathcal{C}$, we have a composition arrow

$$\mathrm{Hom}_{\mathcal{C}}(y,z) \otimes \mathrm{Hom}_{\mathcal{C}}(x,y) \xrightarrow{\circ_{x,y,z}} \mathrm{Hom}_{\mathcal{C}}(x,z)$$

- $\forall\ x \in \mathcal{C}$, an arrow $\mathrm{id}_x : 1 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x,x)$

Satisfying 3 commutative diagrams

Examples: $(\mathcal{V}, \otimes, 1) = (\mathrm{Set}, \times, \{*\})$

$$(\mathcal{V}, \otimes, 1) = (\underset{0}{\circ} \longrightarrow \underset{1}{\circ}, \otimes, 1), \quad \text{01}$$

$\otimes$	0	1
0	0	0
1	0	1

A "01-Category" $\Leftarrow\!\!\Rightarrow$ Preorder

Yoneda: $P \hookrightarrow$ Order ideals of P

$x \longmapsto P_{\leq x}$

$P \hookrightarrow$ Order filters of P

$x \longmapsto P_{\geq x}$

Example: The category of abelian groups is enriched over itself

$(Ab, \otimes, \mathbb{Z})$ is monoidal

For all $A, B \in Ab$, we have seen that $Hom_{Ab}(A, B)$ is an abelian group defined by

$(\varphi + \mu)(a) = \varphi(a) + \mu(a)$

Zero elt defined by

$0_{AB}(a) = 0_B$ $\quad 0 : \mathbb{Z} \to Hom(A, B)$

$\forall A, B, C \in Ab$, we have a composition function of underlying hom sets

$$Hom_{Set}(B, C) \times Hom_{Set}(A, B) \xrightarrow{\circ_{A,B,C}} Hom_{Set}(A, C)$$

$$Hom_{Ab}(B, C) \oplus Hom_{Ab}(A, B) \xrightarrow{\circ_{A,B,C}} Hom_{Ab}(A, C)$$

This is not a group hom. $\ddot{\frown}$

It's "$\mathbb{Z}$-bilinear":

$$(\alpha+\beta)\circ\gamma = (\alpha\circ\gamma)+(\beta\circ\gamma) \checkmark$$
$$\alpha\circ(\beta+\gamma) = (\alpha\circ\beta)+(\alpha\circ\gamma) \checkmark$$

Isomorphism $\mathbb{Z} \cong \mathrm{Hom}_{Ab}(\mathbb{Z},\mathbb{Z})$

$$n \longmapsto (m \longmapsto n\cdot m)$$

We get a multiplication bihomomorphism

$$(-)\cdot m \quad (-)\cdot n \longmapsto (-)\cdot mn$$
$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$$
$$(a, b) \longmapsto ab$$
$$(c, d) \longmapsto cd$$
$$(a,b)+(c,d) \longmapsto (a+c)(b+d) \neq ab+cd$$

What can we do?

Let's suppose that for $B \in Ab$, the functor $\mathrm{Hom}(B,-)$ has a left adjoint $(-)\otimes B$. Then we have a family of bijections

$$\Phi_{A,B,C} : \mathrm{Hom}(A\otimes B, C) \xrightarrow{\sim} \mathrm{Hom}(A, \mathrm{Hom}(B,C))$$

which is natural in $A, B, C \in Ab^{op} \times Ab^{op} \times Ab$

Notation: $[A\otimes B, C] \xrightarrow{\sim} [A,[B,C]]$

Remark: $[A,[B,C]]$ is the set (group) of bihomomorphisms $A\oplus B \longrightarrow C$.

Indeed, given $\varphi: A \longrightarrow [B,C]$, then $\forall\, a \in A$, we get a hom $\varphi(a): B \longrightarrow C$. So we get a bihom by defining

Jacobi-Ind. for Lie Brackets

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

$$\varphi(a,b) := (\varphi(a))(b)$$

$$\begin{aligned} \varphi : A \otimes B &\longrightarrow C \\ (a,b) &\longmapsto \varphi(a,b) \end{aligned}$$

This suggests we should define

$$[A,B;C] = [A,[B,C]] = [B,[A,C]]$$

"bihomomorphisms $A, B \longrightarrow C$"

$\parallel$ adjunction

"homs $A \otimes B \longrightarrow C$"

$$\Phi_{A,B,C} : [A \otimes B, C] \xrightarrow{\sim} [A,B;C]$$

$$\begin{aligned} \Phi_{A,B} : [A\otimes B, A \otimes B] &\xrightarrow{\sim} [A,B;A\otimes B] \\ id_{A\otimes B} &\longmapsto \tau \end{aligned}$$

Furthermore, for __any__ group hom $\varphi : A \otimes B \longrightarrow C$, we have a commutative square

$$\begin{tikzcd}
\varphi \in [A\otimes B, C] & & [A,B;C] \arrow[ll, "\sim"'] \ni \varphi\circ(\tau) \\
{[A\otimes B, A\otimes B]} \arrow[u, "\varphi\circ(-)"] \arrow[rr, "\sim"] & & {[A,B;A\otimes B]} \arrow[u, "\varphi\circ(-)"'] \\
id_{A\otimes B} \arrow[rr, mapsto] & & \tau
\end{tikzcd}$$

$$\left(\text{hom } A\otimes B \xrightarrow{\varphi} C\right) \equiv \left(\text{bihom } A\otimes B \xrightarrow{\tau} A\otimes B \xrightarrow{\varphi} C\right)$$

φ is determined by how it acts on $\mathrm{im}(\tau) \leq A \otimes B$

Notation: $\tau(a,b) =:$ "$a \otimes b$"

Remark: Hom $\varphi: A \otimes B \longrightarrow C$ is completely determined by the values $\varphi(\text{"}a\otimes b\text{"}) \in C$.

Looks like the group $A \otimes B$ is generated (freely?) by the elements "$a \otimes b$" $\forall\ a \in A \quad b \in B$

What is a free group?

Recall: Ab is a "Concrete Category", i.e. we have a "forgetful functor"

$$Set \longleftarrow Ab : U$$

Suppose $\exists$ left adjoint $F \dashv U$

$$F : Set \rightleftarrows Ab : U$$

i.e. a family of bijections

$$\mathrm{Hom}_{Set}(S, U(A)) \longleftrightarrow \mathrm{Hom}_{Ab}(F(S), A)$$

$$(\text{Function } S \longrightarrow U(A)) \equiv (\text{hom } F(S) \to A)$$

Idea: $F(S)$ is the free abelian group gen'd by the "basis" S.

Substituting $A = F(S)$

$$(\text{functions } S \longrightarrow U(F(S))) \equiv (\text{homs } F(S) \longrightarrow F(S))$$

$$\overline{\mathrm{id}_{F(S)}} = \eta \longleftarrow\!\shortmid\ \mathrm{id}_{F(S)} = \varepsilon$$

"insertion of generators"

Universal property of free groups:

For any function $\varphi : S \longrightarrow U(A)$, $\exists!$ homom $\nu : F(S) \longrightarrow A$ extending φ.

$$\begin{tikzcd}
F(S) \arrow[r, "\exists!\,\nu"] & A & & \underline{Ab} \\
U(F(S)) \arrow[r, "U(\nu)"] & U(A) & & \underline{Set} \\
& S \arrow[ul, "\overline{\mathrm{id}_{F(S)}}"] \arrow[u, "\forall\varphi"'] & &
\end{tikzcd}$$

Pf: Suppose such ν exists, w/ $\varphi = U(\nu) \circ \overline{\mathrm{id}_{F(S)}}$

Then $\nu = \nu \circ \mathrm{id}_{F(S)} = \nu \circ \overline{\overline{\mathrm{id}_{F(S)}}} = \overline{U(\nu) \circ \overline{\mathrm{id}_{F(S)}}}$ (Since $F \dashv U$)

$= \overline{\varphi}$, unique ✓

where $\overline{\varphi} : F(S) \longrightarrow A$

Remains to show that $\varphi = U(\overline{\varphi}) \circ \overline{\mathrm{id}_{F(S)}}$

$\overline{\varphi} = \overline{\varphi} \circ \overline{\overline{\mathrm{id}_{F(S)}}}$

$\overline{\varphi} = \overline{U(\overline{\varphi}) \circ \overline{\mathrm{id}_{F(S)}}}$

$\Rightarrow \varphi = U(\overline{\varphi}) \circ \overline{\mathrm{id}_{F(S)}}$ ✓

$\square$

$$\beta X : X \hookrightarrow \prod_{f: X \to [0,1]} [0,1]$$

$$x \longmapsto \prod_{f \in [0,1]^X} f(x)$$

Why do free groups exist?

Theorem (The free functor $F: Set \to Ab$ exists):

Pf: Given a set S, we define two abelian groups:

$$\mathbb{Z}^S := \{ \text{functions } S \to \mathbb{Z} \}$$

$$\cup|$$

$$\mathbb{Z}^{\oplus S} := \{ \text{functions } S \to \mathbb{Z} \text{ with } \underline{\text{finite support}} \}$$

Claim:
$$F: Set \to Ab$$
$$S \longmapsto \mathbb{Z}^{\oplus S}$$
$$g \longmapsto ?$$

is the free functor.

(1) Given $g: S_1 \to S_2$, we need a group hom $F(g): \mathbb{Z}^{\oplus S_1} \to \mathbb{Z}^{\oplus S_2}$

(2) Show $F \dashv U$

Natural basis of $\mathbb{Z}^{\oplus S}$ of "δ-functions"

$$\forall s \in S, \quad \delta_s : S \to \mathbb{Z}$$
$$s \longmapsto 1_{\mathbb{Z}}$$
$$\not s \longmapsto 0_{\mathbb{Z}}$$

Any $f \in \mathbb{Z}^{\oplus S}$ is a finite sum

$$f = \sum_{s \in S} f(s) \delta_s$$

This representation is unique, b/c if we have

$$\left(\sum f(s)\,\delta_s\right) \cdot t = \left(\sum c_s\,\delta_s\right),$$

$$\Rightarrow \underbrace{\left(\sum f(s)\,\delta_s\right)(t)}_{= f(t)} = \underbrace{\left(\sum c_s\,\delta_s\right)(t)}_{= c_t}$$

① Given $g : S_1 \longrightarrow S_2$, define a group hom

$$F(g) : \mathbb{Z}^{\oplus S_1} \longrightarrow \mathbb{Z}^{\oplus S_2}$$

on the basis by

$$F(g)(\delta_s) = \delta_{g(s)}$$

and extend linearly:

$$F(g)\left(\sum_s c_s\,\delta_s\right) = \sum_s c_s\,\delta_{g(s)} \in \mathbb{Z}^{\oplus S_2}$$

✓

② For each group hom $\mathbb{Z}^{\oplus S} \xrightarrow{\varphi} A$, we need a function $S \xrightarrow{\overline{\varphi}} U(A)$

Need: $\varphi \longmapsto \overline{\varphi}$ bijective, & $\overline{\varphi} \circ g = \overline{\varphi \circ F(g)}$
$\overline{\alpha \circ \varphi} = U(\alpha) \circ \overline{\varphi}$

Define $\overline{\varphi} = \varphi(\delta_s)$.

10/25/17 Tensor Products exist, i.e.,

$(Ab, \otimes, \mathbb{Z})$ is a monoidal category

So what?

$(Ab, \otimes, \mathbb{Z}) \longrightarrow$ rings exist!

Q: What is a ring?

A: A ring is a "monoid object" in the monoidal category $(Ab, \otimes, \mathbb{Z})$

Compare: A monoid is a "monoid object" in the monoidal category $(Set, \times, *)$

$$\begin{array}{ccc} (Set, \times, *) & \rightsquigarrow & (Ab, \otimes, \mathbb{Z}) \\ \text{monoid} & \rightsquigarrow & \text{ring} \\ \text{action} & \rightsquigarrow & \text{module} \end{array}$$

Details: Consider an abelian group $A \in Ab$.

The set of endomorphisms $Hom_{Ab}(A, A) = End_{Ab}(A)$ is a monoid under composition

$$(End(A), \circ, id_A)$$

Since Ab is "$\mathbb{Z}$-linear", we also have that $End(A)$ is an abelian group under "pointwise addition"
The resulting structure is called a "ring"

$$(End(A), \circ, id_A, +, 0_A)$$

66 Note that "$\circ$" distributes over addition because "$\circ$" is $\mathbb{Z}$-bilinear

We can axiomatize this:

endomorphisms of an abelian group $\rightsquigarrow$ abstract "ring"

But then we want to go back:

abstract ring $\overset{?}{\rightsquigarrow}$ endomorphisms of something

We already know how to do this for monoids in Set:

- given a monoid M in Set, a representation is a "Set-functor"

$$F : BM \longrightarrow \mathrm{Set}$$

We obtain a Set-category of representations Set^{BM}, which inherits most structure from Set, i.e., it's a topos.

- Given a ring R, i.e. a monoid in Ab, a representation of R is an $\boxed{\text{Ab-functor}}$

$$F : BR \longrightarrow \mathrm{Ab}$$

a.k.a. an R-module.

These naturally form an Ab-category called $R\text{-Mod} = \mathrm{Ab}^{BR}$, which inherits most structure from Ab, i.e. it's an abelian category, that is:

- we have a biproduct $\prod = \oplus = \coprod$ (i.e. matrix algebra)
- we have all kernels + cokernels
- one further axiom

Let M, N be R-modules, and consider an R-linear map

$$\varphi : M \longrightarrow N$$

Kernel:

$$\begin{tikzcd}
Z \arrow[dr, "\forall"] \arrow[drr, bend left, "0"] \arrow[dd, dashed, "\exists!"'] & & \\
& M \arrow[r, "\varphi"] & N \\
\ker(\varphi) \arrow[ur, hook] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Limit

Cokernel:

$$\begin{tikzcd}
& & Z \\
M \arrow[r, "\varphi"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & N \arrow[ur, "\forall"] \arrow[dr, two heads] & \\
& & \operatorname{Coker}(\varphi) = N/\operatorname{im}(\varphi) \arrow[uu, dashed, "\exists!"']
\end{tikzcd}$$

Colimit

For abstract reasons,

$\ker(\varphi) \hookrightarrow M$ is monic

$N \twoheadrightarrow \operatorname{Coker}(\varphi)$ is epic

Consider the following picture:

$$\begin{tikzcd}
\mathrm{Ker}(\varphi) \arrow[r, hook] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & M \arrow[r, "\varphi"] \arrow[d] \arrow[rr, bend left, "0"] \arrow[dr, "\exists!"] & N \arrow[r, two heads] \arrow[dr, "0"] & \mathrm{Coker}(\varphi) \\
& \mathrm{coker}(\mathrm{ker}(\varphi)) \arrow[r, "\exists!\bar{\varphi}"] \arrow[ur, "\exists!"] \arrow[urr] \arrow[d, equal] & \mathrm{Ker}(\mathrm{coker}(\varphi)) \arrow[u] \arrow[ur, "0"'] \arrow[d, equal] & \\
& \mathrm{Coimage}(\varphi) & \mathrm{image}(\varphi) &
\end{tikzcd}$$

We obtain a canonical factorization

$$M \xrightarrow{\varphi} N$$

$$\begin{tikzcd}
M \arrow[r, two heads, "\text{epi}"] \arrow[rrr, bend left, "\varphi"] & \mathrm{Coim}(\varphi) \arrow[r, "\bar{\varphi}"] & \mathrm{im}(\varphi) \arrow[r, hook, "\text{mono}"] & N
\end{tikzcd}$$

<u>Def</u>: The category is <u>abelian</u> if $\bar{\varphi}$ is always an isomorphism

Concretely, $\mathrm{Coim}(\varphi) = M/\mathrm{Ker}(\varphi)$

The category Ab is abelian, so
$R\text{-Mod} = Ab^{BR}$ is also.

The "R-action" is mostly decoration

$Ab = \mathbb{Z}\text{-mod}$

Change of base rings:

$$\begin{tikzcd}
R \arrow[r] & S
\end{tikzcd}$$

$$\begin{tikzcd}
R\text{-Mod} \arrow[r, bend left] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] & S\text{-Mod} \arrow[l] \arrow[l, bend left]
\end{tikzcd}$$

"Morita theory"

Topic: rings $R \rightsquigarrow$ Category R-Mod

$R \leftsquigarrow$? Category R-Mod

To what extent does the category of R-modules determine the ring R?

"Tannaka Duality" for rings?

$$\boxed{\begin{array}{l} U: M\text{-Set} \longrightarrow \text{Set} \\ \text{End}(U) \cong M \end{array}} \qquad \boxed{\begin{array}{l} U: R\text{-Mod} \longrightarrow Ab \\ \text{End}_{Ab}(U) \cong R \end{array}} \checkmark$$

"Set" "Ab"

If you ask the question:
"Does the Ab-category R-Mod determine the ring R?"

Yes: If you include the full $(Ab, \otimes, \mathbb{Z})$-structure.
If functors + nat. transformations preserve $(Ab, \otimes, \mathbb{Z})$, the proof of Tannaka duality goes through verbatim.

To what extent does the Set-category (or $\mathbb{Z}$-linear category, ...) determine R?

Depends if R is commutative!

If R, S commutative, then $R\text{-Mod} \cong S\text{-Mod} \Rightarrow R \cong S$

Otherwise, not.

Example: $R\text{-Mod} \cong Mat_n(R)\text{-Mod}$ for any ring R and any $n \in \mathbb{N}$.

But in general $R \not\cong Mat_n(R)$

10/30/17

Let $(\mathcal{V}, \otimes, 1)$ be a monoidal category.

Let $\mathcal{C}, \mathcal{D}$ be small $\mathcal{V}$-categories

Ex: $\mathcal{V} = (0 \to 1)$, $\mathcal{C}, \mathcal{D}$ posets or $\mathcal{C}, \mathcal{D}$ sets

$\mathcal{V} = Set$, $\mathcal{C}, \mathcal{D}$ monoids

$\mathcal{V} = Ab$, $\mathcal{C}, \mathcal{D}$ ring(oid)s / $\mathbb{Z}$-algebras
$\mathbb{Z}$-mod

We'll add one more

Let R be a commutative ring,

$\mathcal{V} = R\text{-Mod}$, $\mathcal{C}, \mathcal{D}$ R-algebras / R-algebroids

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A $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{D}$ consists of an object function $F: Ob(\mathcal{C}) \to Ob(\mathcal{D})$ and for all $x, y \in \mathcal{C}$ a $\mathcal{V}$-arrow

$$F: \underset{\underset{\mathcal{V}}{\cap}}{\mathcal{C}(x,y)} \longrightarrow \underset{\underset{\mathcal{V}}{\cap}}{\mathcal{D}(F(x), F(y))}$$

that preserve identities: $\forall x \in \mathcal{C}$,

$$\begin{tikzcd} \mathcal{C}(x,x) \arrow[r, "F"] & \mathcal{D}(F(x), F(x)) \\ 1 \arrow[u, "id_x"] \arrow[ur, "id_{F(x)}"'] \arrow[ur, phantom, "\circledR", shift left=3] & \end{tikzcd}$$

and preserves composition: $\forall x, y, z \in \mathcal{C}$,

$$\begin{tikzcd}
\mathcal{D}(F(y),F(z)) \otimes \mathcal{D}(F(x),F(y)) \arrow[r, "\circ"] & \mathcal{D}(F(x),F(z)) \\
\mathcal{C}(y,z) \otimes \mathcal{C}(x,z) \arrow[u, "F \otimes F"] \arrow[r, "\circ"] & \mathcal{C}(x,z) \arrow[u, "F"']
\end{tikzcd}$$

Ex: • $\mathcal{V} = (0 \to 1)$, $\mathcal{C}, \mathcal{D}$ posets, a $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$ is an order-preserving map.

• $\mathcal{V} = Set$, $\mathcal{C}, \mathcal{D}$ monoids, a $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$ is a monoid homo.

• $\mathcal{V} = Ab$, $\mathcal{C}, \mathcal{D}$ rings, $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$ is a ring homom.

Given $\mathcal{V}$-categories $\mathcal{C}$ & $\mathcal{D}$, can we define a category of $\mathcal{V}$-functors $[\mathcal{C} \to \mathcal{D}]$ (aka $\mathcal{D}^{\mathcal{C}}$).

Yes: However we need to define a $\mathcal{V}$-natural transformation

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D}
\end{tikzcd}$$

Instead of a family of "arrows"

$$\Phi_X : F(X) \longrightarrow G(X),$$

we think of a family of arrows

$$\Phi_X : \mathbb{1} \longrightarrow \mathcal{D}(F(X), G(X))$$

<u>Naturality:</u> $\forall\, X, Y \in \mathcal{C}$, we have

$$\begin{tikzcd}
\mathcal{D}(F(X), F(Y)) \arrow[r, "(\Phi_Y)_*"] & \mathcal{D}(F(X), G(Y)) \\
\mathcal{C}(X,Y) \arrow[u, "F"] \arrow[r, "G"'] \arrow[ur, phantom, "\circlearrowright"] & \mathcal{D}(G(X), G(Y)) \arrow[u, "(\Phi_X)^*"']
\end{tikzcd}$$

<u>Definition:</u> If $\mathcal{C}$ is a small $\mathcal{V}$-category (thought of as an "algebroid"), then we define the $\mathcal{V}$-category of "$\mathcal{C}$-modules"

$\mathcal{V}^{\mathcal{C}}$ objects: $\mathcal{V}$-functors $\mathcal{C} \longrightarrow \mathcal{V}$
arrows: $\mathcal{V}$-nat. transformations

<u>Ex:</u> • $\mathcal{V} = (0 \to 1)$, P poset, $\mathcal{V}^P$ = Poset of order ideals

P Set, $\mathcal{V}^P$ = Boolean alg. of subsets

• $\mathcal{V} = \text{Set}$, M a monoid, $\mathcal{V}^M$ = Category of left M-Sets

• $\mathcal{V} = \text{Ab}$, R a ring, $\mathcal{V}^R$ = Category of (left) R-modules

"Change of Base"

Let $\mathcal{C}, \mathcal{D}$ be small $\mathcal{V}$-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor. Then we obtain a functor of module categories in the opposite direction:

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \quad \rightsquigarrow \quad \mathcal{V}^{\mathcal{C}} \xleftarrow[(-)\circ F]{F^*} \mathcal{V}^{\mathcal{D}}$$

Thm/Def: If F^* has a left/right adjoint, we call it the left/right *Kan extension* of F.

$$\begin{tikzcd} \mathcal{V}^{\mathcal{C}} \arrow[r, bend left=60, "\mathrm{Lan}_F(-)"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=60, "\mathrm{Ran}_F(-)"'] & \mathcal{V}^{\mathcal{D}} \arrow[l, "(-)\circ F"'] \end{tikzcd}$$

Under mild completeness/smallness assumptions, the Kan extensions always exist.

Examples: $\mathcal{V} = (0 \to 1)$, S, T sets, $F : S \to T$ is a function.

$$\begin{tikzcd} (0 \to 1)^S \arrow[r, bend left=60, "F"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=60, "F_!"'] & (0 \to 1)^T \arrow[l, "F^{-1}"'] \end{tikzcd}$$

• $\mathcal{V} = Set$, $\quad F : M \hookrightarrow N$

monoid hom.

$$\begin{tikzcd} Set^M \arrow[rr, bend left=60, "\text{induction}"] \arrow[rr, bend right=60, "\text{coinduction}"'] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[rr, phantom, shift right=3, "\perp"] & & Set^N \arrow[ll, "\text{restriction}"'] \end{tikzcd}$$

• $\mathcal{V} = Ab$, $F : R \longrightarrow S$ ring homom

$$\begin{tikzcd} Ab^R \arrow[rr, bend left=60, "\text{extension of scalars}"] \arrow[rr, bend right=60, "\text{Co-extension of scalars}"'] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[rr, phantom, shift right=3, "\perp"] & & Ab^S \arrow[ll, "\text{restriction of scalars}"'] \end{tikzcd}$$

Questions: • To what extent is the structure of a ring R reflected by the structure of the category $R\text{-Mod}$?

$$R \xrightarrow{?} S$$

$$R\text{-Mod} \xrightarrow{\sim} S\text{-Mod}$$

• For which rings R & S do we have equivalent categories $R\text{-Mod} \cong S\text{-Mod}$?

Def: Let $\mathcal{C}, \mathcal{D}$ be Set-categories. An equivalence is a pair of functors
$$F: \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} : G$$
with natural isomorphisms

$$G \circ F \cong \mathrm{id}_{\mathcal{C}} \; ; \; F \circ G \cong \mathrm{id}_{\mathcal{D}}$$
(intuition: homotopy equivalences).

In this case, we say that F & G are quasi-inverse.

Lemma: $F : \mathcal{C} \to \mathcal{D}$ has a quasi-inverse iff
- F is fully faithful (essentially injective on objects & essentially [?])
- Essentially surjective on objects.

Suppose we have an equivalence of categories

$$F : R\text{-Mod} \cong S\text{-Mod} : G$$
So what?

Note: F is an additive, right-exact functor.

Then, there exists a unique (S, R)-bimodule Q s.t.

$$\begin{aligned} F : R\text{-Mod} &\longrightarrow S\text{-Mod} \\ M &\longmapsto Q \otimes_R M \end{aligned}$$

11/1/17 Question: Does the forgetful functor $U: M\text{-Set} \longrightarrow \text{Set}$ have a left-adjoint (free) functor.

Ans: Yes. Given a set $S \in \text{Set}$ define an M-action on the set $\underset{\text{"copower"}}{M \times S} = \{(m, s) : m \in M, s \in S\}$

$\forall\ m' \in M, \quad m' \cdot (m, s) := (m' \circ m, s)$

$M \times S$ is a disjoint union of copies of the left regular action $M \circlearrowright M$, indexed by S.

In particular, the free M-set gen'd by $*$ is just
$F(*) = M^{\circlearrowleft}$

Since $*$ is terminal in Set, we have that

$$\text{Hom}_{\text{Set}}(*, S) \longleftrightarrow S$$

For any M-set $M \circlearrowright S$, let $U(S)$ be the underlying set.
Then,

$$U(S) \longleftrightarrow \text{Hom}_{\text{Set}}(*, U(S)) \longleftrightarrow \text{Hom}_{M\text{-Set}}(F(*), S)$$

$$\longleftrightarrow \text{Hom}_{M\text{-Set}}(M, S) = H^M(S)$$

We have a natural isomorphism $U(-) \cong \text{Hom}_{M\text{-Set}}(M, -)$.

Jargon: The forgetful functor is representable by M.

Now, we apply Yoneda: $\text{Nat}(U, U) = \text{Nat}(H^M, H^M)$
$$= \text{Hom}_{M\text{-Set}}(M, M)$$

$\therefore \text{End}(U) = \text{End}_{M\text{-Set}}(M)$.

Let $\varphi: M \to M$ be any M-equivariant map and let $m' = \varphi(id)$.
Then $\forall m \in M$, $\varphi(m) = \varphi(m \circ id) = m \circ \varphi(id) = m \circ m'$

$$\Rightarrow \boxed{\mathrm{End}_{M\text{-Set}}(M) \cong M^{op}}$$

If $M = G$ is a group then $\mathrm{End}_{G\text{-Set}}(G) \overset{\text{natural}}{\cong} G^{op} \overset{\text{inversion}}{\cong} G$

More generally, for any subgroup $H \leq G$, we have

$$\mathrm{End}_{G\text{-Set}}(G/H) \cong N_G(H)/H$$

We have an adjunction

$$F: \mathrm{Set} \rightleftarrows \underset{\mathbb{Z}\text{-mod}}{\mathrm{Ab}} : U$$

$$\mathrm{End}(U) = \mathrm{End}(H^{\mathbb{Z}}, H^{\mathbb{Z}}) = \mathrm{End}_{Ab}(\mathbb{Z}, \mathbb{Z}) = (\mathbb{Z}, \circ)^{op} = (\mathbb{Z}, \circ)$$
$$= \underset{\text{ring}}{\mathbb{Z}^{op}} = \underset{\text{ring}}{\mathbb{Z}}$$

Ring/algebra of endomorphisms of U.

$$F: \mathrm{Set} \rightleftarrows R\text{-Mod}: U$$

$$\mathrm{End}(U) = \mathrm{Nat}(H^R, H^R) = \mathrm{Hom}_R(R, R) = \mathrm{End}_R(R) = R^{op}$$

First Step: What to look for

$$F : \overset{\text{Ab.}}{\mathbb{Z}\text{-Mod}} \rightleftarrows R\text{-Mod} : U$$

What is F?

Let A be an abelian group, $M \in R\text{-Mod}$. We require

$$\mathrm{Hom}_{\mathbb{Z}}(A, U(M)) = \mathrm{Hom}_R(F(A), M)$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, U(M)) = \mathrm{Hom}_R(F(\mathbb{Z}), M)$$

$$\begin{array}{ccc} \| & & \uparrow \\ U(M) = U(M)^{op} & & \text{Satisfies } \mathrm{Hom}_R(-, M) = M \end{array}$$

So $F : \mathbb{Z} \longrightarrow R$ (well, this <u>should</u> be the case...)

What role play in the category Ab?

<u>Ideas:</u>
- $\mathbb{Z}$ is initial in CRng
- $\mathbb{Z}$ is the unit object for $\otimes_{\mathbb{Z}}$
- $\mathbb{Z}$ is a "progenerator" in Ab:
  1) $\mathbb{Z}$ is projective in Ab $\quad$ $\mathrm{Hom}(\mathbb{Z}, -)$ epi-surjective
  2) $\mathbb{Z}$ is compact in Ab. $\quad$ $\mathrm{Hom}(\mathbb{Z}, -)$ commutes with coproduct
  3) $\mathbb{Z}$ generates Ab.
     $\uparrow$
     Every abelian group $A \in Ab$ has an epi from $\mathbb{Z}^{\oplus S} \longrightarrow A$

<u>Slogan:</u> R is a progenerator of $R\text{-Mod}$.
(not necessarily unique)

Free M-set gen'd by $S \in Set$ is a copower

$$M \times S \quad \text{with} \quad M \otimes (M \times S) = (M \otimes M) \times S$$

So, free R-module gen'd by $A \in Ab$ should be

$$R \text{``}\times\text{''} A \quad \text{with} \quad R \otimes (R \text{``}\times\text{''} A) = (R \otimes R) \text{``}\times\text{''} A$$

How about $R \otimes_{\mathbb{Z}} A$? It's an abelian group. Is it an R-module?

Sure: $r\left(\sum_i n_i (r_i \otimes a_i)\right) := \sum_i n_i \left((r r_i) \otimes a_i\right)$

Claim: $R \otimes_{\mathbb{Z}} A$ with the R-action defined above is the free R-module gen'd by A.

Categorically, it's a "copower":

Let $A, B \in R\text{-mod}$

$$\mathrm{Hom}_R(R \otimes_{\mathbb{Z}} A, B) = \mathrm{Hom}_{\mathbb{Z}}(R, \mathrm{Hom}_R(A, B))$$

$$\begin{tikzcd}
\mathbb{Z}\text{-Mod} \arrow[r, bend left=50, "R \otimes_{\mathbb{Z}} (-)"] \arrow[r, bend right=50, "\mathrm{Hom}_{\mathbb{Z}}(R, -)"'] & R\text{-Mod} \arrow[l, "u"']
\end{tikzcd}$$

$$\varphi : R \longrightarrow A$$

$$(r\varphi)(a) = \varphi(r) \cdot a$$

$F : R\text{-Mod} \longleftrightarrow S\text{-Mod} : G$

$F \dashv G \dashv F$, so F, G exact.

$\exists$ Q (S,R) - bimodule
P (R,S) - bimodule
s.t.
$$F(-) = Q \otimes_R (-)$$
$$G(-) = P \otimes_S (-)$$

$${}_R P_S \otimes_S {}_S Q_R \cong {}_R R_R \qquad {}_S Q_R \otimes_R {}_R P_S \cong {}_S S_S$$

11/6/17

$U : R\text{-Mod} \longrightarrow Ab$

Does this functor know about the ring R?

$$U(M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, U(M)) = \mathrm{Hom}_R(R \otimes_{\mathbb{Z}} \mathbb{Z}, M)$$
$$= \mathrm{Hom}_R(R, M)$$

This is a natural isomorphism $U(-) \cong \mathrm{Hom}_R(R, -)$

Forgetful functor is represented by R.

$$\mathrm{Nat}(U,U) = \mathrm{Nat}(H^R, H^R) = \mathrm{Hom}_R(R,R) = \mathrm{End}_R(R) = R^{op}$$

<u>Ans:</u> $\mathrm{End}(U)^{op} \cong R$, so Yes!

Does the category R-Mod know about the ring R?
<u>Maybe not.</u>

Forgetful functor has a left & a right adjoint

$$\begin{tikzcd} Ab \arrow[rr, bend left=60, "R\otimes_{\mathbb{Z}}(-)"] \arrow[rr, bend right=60, "\mathrm{Hom}_{\mathbb{Z}}(R,\,-)"'] & & R\text{-}Mod \arrow[ll] \end{tikzcd}$$

$\perp$ $\perp$

Given an $A \in Ab$, we define

- $R \curvearrowright R \otimes_{\mathbb{Z}} A$ by

$$r\left(\sum_i r_i \otimes a_i\right) = \sum_i (r r_i) \otimes a_i$$

- $R \curvearrowright \mathrm{Hom}_{\mathbb{Z}}(R, A)$

$\varphi : R \to A$
$r\varphi : R \to A$
$\quad s \mapsto \varphi(sr)$

Need: Given $M \in R\text{-}Mod$, $A \in Ab$,

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, A)) \overset{?}{\cong} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{U}(M), A)$$

$$\mathrm{Hom}_R(R \otimes_{\mathbb{Z}} A, M) \overset{?}{\cong} \mathrm{Hom}_{\mathbb{Z}}(A, \mathcal{U}(M))$$

Claim: The category R-Mod doesn't know about the ring R, but it does know about the commutative ring

$$Z(R) = \{ s \in R \mid rs = sr \ \forall r \in R \}$$

Instead of endomorphisms of $U: R\text{-Mod} \longrightarrow Ab$, we look at endomorphisms of the identity functor

$$\mathrm{id}_{R\text{-Mod}} : R\text{-Mod} \longrightarrow R\text{-Mod}$$

$$\mathrm{End}(\mathrm{id}_{R\text{-Mod}}) \cong Z(R)$$

The Center of a category is $Z(\mathcal{C}) := \mathrm{End}(\mathrm{id}_{\mathcal{C}})$.

Let $\eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow \mathrm{id}_{\mathcal{C}}$

$\forall X \in \mathcal{C}$, $\eta_X : X \longrightarrow X$

$\forall f : X \longrightarrow Y$, we have

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] & Y \\ X \arrow[u, "f"] \arrow[r, "\eta_X"'] \arrow[ur, phantom, "\circlearrowright"] & X \arrow[u, "f"'] \end{tikzcd}$$

i.e., $\eta_Y \circ f = f \circ \eta_X$

Taking $X = Y$, $\forall f : X \longrightarrow X$ (i.e., $f \in \mathrm{End}_{\mathcal{C}}(X)$), we must have, $\forall X \in \mathcal{C}$, $\eta_X \circ f = f \circ \eta_X$

In summary, we obtain a homom of monoids

$$(-)_X : \mathrm{End}(\mathrm{id}_{\mathcal{C}}) \longrightarrow Z(\mathrm{End}_{\mathcal{C}}(X))$$

injective?
surjective?

Suppose $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is <u>faithful</u>, then for all Y, Z

$$\begin{array}{ccc} \mathrm{Hom}(Y,Z) & \hookrightarrow & \mathrm{Hom}(\mathrm{Hom}(X,Y), \mathrm{Hom}(X,Z)) \\ g_1 & \longmapsto & g_1 \circ f \\ \shortparallel & \Longleftarrow & \shortparallel \quad \forall f \\ g_2 & \longmapsto & g_2 \circ f \end{array}$$

$g_1 \circ f = g_2 \circ f \ \forall f \Rightarrow g_1 = g_2$, or equivalently, $g_1 \neq g_2 \Rightarrow \exists f, g_1 \circ f \neq g_2 \circ f$

"Left-cancellable relative to X"

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, bend left, "\eta"] \arrow[r, bend right, "\eta'"'] & Y \end{tikzcd}$$

"$\eta \neq \eta' \Rightarrow \exists f, \eta \circ f \neq \eta' \circ f$" $\Longleftrightarrow$ "$\forall f, \eta \circ f = \eta' \circ f \Rightarrow \eta = \eta'$"

<u>Claim:</u> If $\mathrm{Hom}(X, -)$ is faithful, then

$$(-)_X : \mathrm{End}(\mathrm{id}_{\mathcal{C}}) \hookrightarrow Z(\mathrm{End}_{\mathcal{C}}(X))$$

is <u>injective</u> (and hence $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ is <u>commutative</u>)

Pf: Suppose $\eta, \eta' : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$ with $\eta_X = \eta'_X$.

WTS $\eta = \eta'$.

By naturality, we have, $\forall\ f : X \longrightarrow Y$,

$$\eta_Y \circ f \overset{\text{naturality}}{=} f \circ \eta_X = f \circ \eta'_X \overset{\text{naturality}}{=} \eta'_Y \circ f, \text{ i.e.}$$

$$\eta_Y \circ f = \eta'_Y \circ f \quad \forall f \text{ (out of } X)$$

Since $\mathrm{Hom}(X,-)$ is faithful, this implies $\eta_Y = \eta'_Y$.

Since Y was arbitrary, $\eta = \eta'$. $\square$

Question: When is $(-)_X : \mathrm{End}(\mathrm{id}_{\mathcal{C}}) \hookrightarrow Z(\mathrm{End}_{\mathcal{C}}(X))$ surjective?

Let $\mathcal{C} = R\text{-Mod}$, $X = {}_R R$.

Claim: That X is a generator and that $(-)_X$ is surjective.

Check: 1) X is a generator, i.e.,

$\mathrm{Hom}_R(R,-) : R\text{-Mod} \longrightarrow Ab$ is faithful?

$$\varphi_m : R \longrightarrow M$$
$$1 \longmapsto m$$

$$\varphi_m(r) = \varphi_m(r1) = r\varphi_m(1) = rm$$

$\mathrm{Hom}_R(R,-) \cong U(-)$

Is the forgetful functor faithful?

$$\begin{tikzcd} R \arrow[r, "\varphi_m"] & M \arrow[r, bend left, "\psi"] \arrow[r, bend right, "\psi'"'] \arrow[r, phantom, "\neq"] & N \end{tikzcd}$$

$\psi \neq \psi' \Rightarrow \exists\, m \in M,\ \psi(m) \neq \psi'(m)$

$$\begin{aligned} \psi \circ \varphi_m(1) &= \psi(m) \\ &\neq \\ \psi' \circ \varphi_m(1) &= \psi'(m) \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\psi \circ \varphi_m \\ &\neq \\ &\psi' \circ \varphi_m \end{aligned}$$

$\Rightarrow \operatorname{Hom}_R(R, -)$ is faithful.

2) $(-)_X$ is surjective:

Know $(-)_R : \operatorname{End}(\mathrm{id}_{R\text{-Mod}}) \hookrightarrow Z(\operatorname{End}_R(R)) = Z(R^{op})$
$= Z(R)$

Surjective? Given $r \in Z(R)$, we need to define $\eta : \mathrm{id}_{R\text{-Mod}} \Rightarrow \mathrm{id}_{R\text{-Mod}}$

s.t. $\eta_R : R \longrightarrow R$
$s \longmapsto sr = rs$

Idea: Given $M \in R\text{-Mod}$, define $\eta_M : M \longrightarrow M$
$m \longmapsto rm$

Since $r \in Z(R)$, this is an R-linear homomorphism.

Is this family of arrows natural in M?

Given $\varphi: M \to N$ R-linear,

$$\begin{tikzcd} N \arrow[r, "r'"] & N \\ M \arrow[u, "\varphi"] \arrow[r, "r'"'] \arrow[ur, phantom, "\circlearrowright"] & M \arrow[u, "\varphi"'] \end{tikzcd}$$

by R-linearity of φ, $\forall r' \in R$.

If we take $r' = r \in Z(R)$, we get

$$\begin{tikzcd} N \arrow[r, "r_N"] & N \\ M \arrow[u, "\varphi"] \arrow[r, "r_M"'] \arrow[ur, phantom, "\circlearrowright"] & M \arrow[u, "\varphi"'] \end{tikzcd}$$

i.e. naturality

$\square$

<u>Conclusion:</u> For any ring R, $End(id_{R\text{-Mod}}) \cong Z(R)$

<u>Corollary:</u> If $R\text{-Mod} \simeq S\text{-Mod}$, then $Z(R) \cong Z(S)$

<u>Corollary:</u> If R, S are commutative, $R\text{-Mod} \simeq S\text{-Mod} \iff R \cong S$.

Question: R, S non-commutative?

$\mathrm{Hom}_R(R, -)$ faithful

11/8/17

Let K be a commutative ring.

Idea: We can replace $\mathbb{Z}$ by K in any categorical construction

$(Ab, \otimes, \mathbb{Z})$
$(K\text{-Mod}, \otimes_K, K)$ monoidal category

Recall: The tensor product of abelian groups linearizes $\mathbb{Z}$-bilinear functions

$$A \oplus B \longrightarrow C$$
$$A \otimes_{\mathbb{Z}} B \longrightarrow C$$

$$\mathrm{Hom}(A \otimes_{\mathbb{Z}} B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C)) = \mathrm{Hom}(B, \mathrm{Hom}(A, C))$$

$$\begin{tikzcd} A \oplus B \arrow[r] \arrow[d, "\exists!"'] & C \quad K\text{-bilinear map} \\ A \otimes_K B \arrow[r] & C \quad K\text{-linear map} \end{tikzcd}$$

Furthermore, K-Mod is enriched over itself:

$\varphi_1, \varphi_2 : A \longrightarrow B$, $\varphi_1 + \alpha \varphi_2 : A \to B$ is still K-linear

A ring is a "monoid in $\mathbb{Z}$-Mod"

A K-algebra is a "monoid in K-Mod"

Let's say R, S are K-algebras.

Define an R,S-bimodule as an abelian group M carrying a left R-action and a right S-action, such that:

$\forall r \in R, s \in S, m \in M, \ (rm)s = r(ms)$ "associativity"

---

left R-module: $R \to \mathrm{End}_{\mathbb{Z}}(M)$
right S-module: $S^{op} \to \mathrm{End}_{\mathbb{Z}}(M)$

$$\left.\begin{array}{l} R \longrightarrow \mathrm{End}_{\mathrm{Mod}\text{-}S}(M) \\ S^{op} \longrightarrow \mathrm{End}_{R\text{-}\mathrm{Mod}}(M) \end{array}\right\} \text{Commutativity of actions}$$

$$R \otimes_{\mathbb{Z}} S^{op} \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$$

$\uparrow$

Is this a ring?

Yes: Define a ring structure on the generators
$$(r_1 \otimes s_1)(r_2 \otimes s_2) = (r_1 r_2) \otimes (s_2 s_1)$$

It turns out that $\otimes_{\mathbb{Z}}$ is the coproduct in the category of commutative rings.

$\mathrm{CRing}^{op} = \mathrm{Aff}$ "affine schemes"

$\otimes_{\mathbb{Z}}$ is the product of schemes

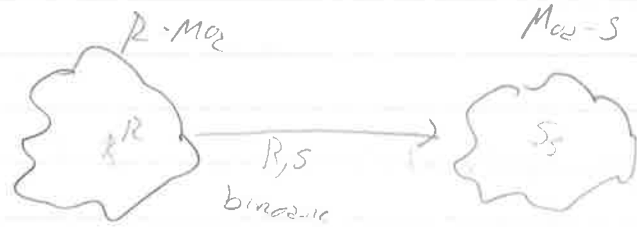
$\otimes_K$:
$$\begin{tikzcd} & K_1 \arrow[dr] & \\ K \arrow[ur] \arrow[dr] & & K_1 \otimes_K K_2 \\ & K_2 \arrow[ur] & \end{tikzcd}$$
Colim

fiber product of schemes

Def: Let R, S be K-algebras. An (R,S)-bimodule is a functor

$$R \otimes_K S^{op} \longrightarrow K\text{-Mod}$$

Behaves like a Hom bifunctor



Tensor product of modules over noncommutative rings.

Let A_R be right R-module
$_R B$ be left R-Module

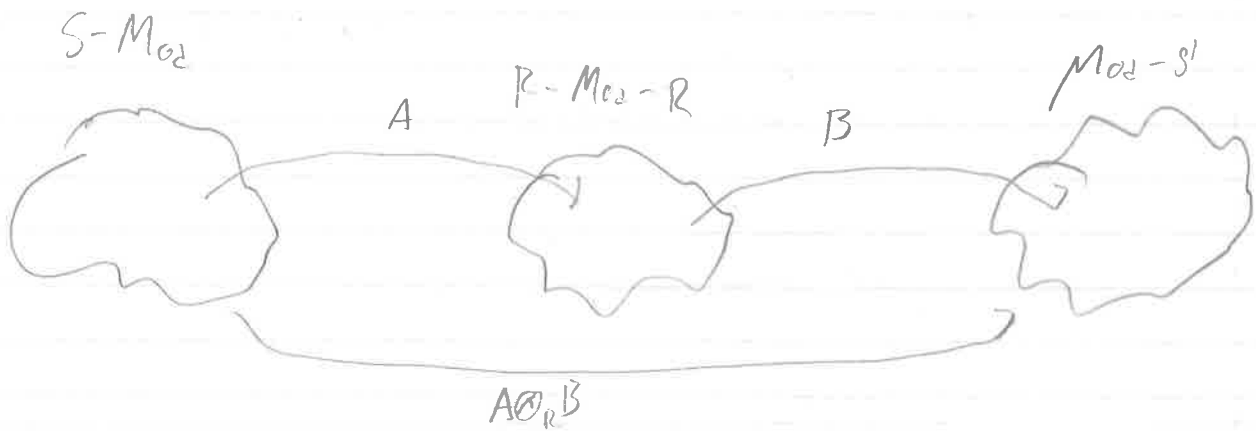
A $\mathbb{Z}$-bilinear map $A \oplus B \xrightarrow{\varphi} C$ is R-balanced if
$(a,b) \longmapsto \varphi(a,b)$

$\forall\ a \in A,\ b \in B,\ r \in R,$

$$\varphi(ar, b) = \varphi(a, rb)$$

The tensor product $_S A_R \otimes_R {}_R B_{S'}$, as an abelian group, turns $\mathbb{Z}$-bilinear & R-balanced maps into $\mathbb{Z}$-linear maps,

$_S A_R \otimes_R {}_R B_{S'}$ is an (S, S')-bimodule



Adjunctions:

Given A_R, ${}_R B_S$, ${}_S C$, we have

this is a right R-module

$$\mathrm{Hom}_S(A \otimes_R B, C) = \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

Given ${}_R A$, ${}_S B_R$, C_S:

$$\mathrm{Hom}_S(B \otimes_R A, C) = \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

Observation: $(-) \otimes_R B : \text{Mod-}R \longrightarrow \text{Mod-}S$

$B \otimes_R (-) : R\text{-Mod} \longrightarrow S\text{-Mod}$

both are cocontinuous.

Thm (Eilenberg-Watts): Let $F: R\text{-Mod} \longrightarrow S\text{-Mod}$ be any (additive) cocontinuous functor.

Then, $F(-) \cong Q \otimes_R (-)$, for some (S, R)-bimodule Q.

Proof: Idea: Let ${}_S Q_R = {}_S F(R)$

$F(R)$ is a left S-module by definition. Why is it a right R-module

$$F: R\text{-Mod} \longrightarrow S\text{-Mod}$$

$$\begin{tikzcd} F: \mathrm{End}_R(R) \arrow[r] \arrow[d, equal] & \mathrm{End}_S(F(R)) \\ R^{op} \arrow[ur] & \end{tikzcd}$$

We need, for each $M \in R\text{-Mod}$, an S-linear map

$$\tau_M : F(R) \otimes_R M \xrightarrow{\sim} F(M)$$

<u>Adjunction:</u> $\mathrm{Hom}_S(F(R) \otimes_R M, F(M)) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(F(R), F(M)))$

$$\tau_M \qquad\qquad M \xrightarrow{\sim} \mathrm{Hom}_R(R, M) \xrightarrow{F} \mathrm{Hom}_S(F(R), F(M))$$

<u>Observe:</u> $\tau_R : F(R) \otimes_R R \xrightarrow{\sim} F(R)$

Since $F(R) \otimes_R (-)$ is co-continuous, it's an isomorphism at any free module $R^{\oplus I}$

$$\tau_{R^{\oplus I}} : F(R) \otimes_R R^{\oplus I} \xrightarrow{\sim} F(R^{\oplus I})$$

$$F(R) \otimes_R R^{\oplus I} = F(R \otimes_R R^{\oplus I}) = F(R^{\oplus I}) \qquad (\text{since } F \text{ is cocontinuous})$$

To show that τ_M is an isom, consider a free resolution of M:

$$R^{\oplus J} \to R^{\oplus I} \to M \to 0$$

$$\begin{tikzcd}
F(R^{\oplus J}) \arrow[r] & F(R^{\oplus I}) \arrow[r] & F(M) \arrow[r] & 0 \arrow[r] & 0 \\
F(R) \otimes R^{\oplus J} \arrow[r] \arrow[u, "\tau_{R^{\oplus J}}", "\wr"'] & F(R) \otimes R^{\oplus I} \arrow[r] \arrow[u, "\tau_{R^{\oplus I}}", "\wr"'] & F(R) \otimes M \arrow[r] \arrow[u, "\tau_M"] & 0 \arrow[r] \arrow[u, "0", "\wr"'] & 0 \arrow[u, "0", "\wr"']
\end{tikzcd}$$

The middle map is an iso by the 5-lemma. $\square$

11/13/17 Let R, S be any rings. Last time, we showed (Eilenberg – Watts) that

$$R\text{-Mod} \underset{\substack{\uparrow \\ \mathbb{Z}\text{-linear}}}{\cong} S\text{-Mod} \iff \exists \text{ bimodules } {}_RP_S,\ {}_SQ_R \text{ with}$$

$${}_RP_S \otimes_S {}_SQ_R \cong {}_RR_R$$

$$\text{and } {}_SQ_R \otimes_R {}_RP_S \cong {}_SS_S$$

Because the second condition is symmetric in R + S, we find

$$R\text{-Mod} \cong S\text{-Mod} \iff \text{Mod-}R \cong \text{Mod-}S$$

In either case, we write $R \approx S$ and say that R + S are <u>Morita equivalent.</u>

<u>Recall</u>: If $R \approx S \Rightarrow Z(R) \underset{\substack{\uparrow \\ \text{as rings}}}{\cong} Z(S)$

If R, S <u>commutative</u>, $R \approx S \iff R \cong S$.

<u>Big Theorem (Morita)</u>: $R \approx S \iff \exists$ progenerator $P \in R\text{-Mod}$ s.t. $S \cong \text{End}_R(P)^{op}$

$\iff \exists$ progenerator $Q \in S\text{-Mod}$ s.t. $R \cong \text{End}_S(Q)^{op}$

(<u>Stronger version</u>: Let $\mathcal{C}$ be a cocomplete abelian category. Then $\mathcal{C} \cong R\text{-Mod}$ iff $\exists$ progenerator $P \in \mathcal{C}$ with $R \cong \text{End}_{\mathcal{C}}(P)^{op}$).

Prototype: Let $P = R^{\oplus n} \in R\text{-Mod}$

Then since P is a progenerator, we have $R \simeq S$, where
$S^{op} = \mathrm{End}_R(R^{\oplus n}) = \mathrm{Mat}_{n \times n}(R)$

Strong example: If K is a field, then $K \simeq S \iff S = \mathrm{Mat}_{n\times n}(K)$

Proof: for our purposes, say $P \in R\text{-Mod}$ is a progenerator if

$$\mathrm{Hom}_R(P, -) : R\text{-Mod} \longrightarrow \mathrm{Ab}.$$

- is faithful (generator)
- preserves coproducts (compact)
- preserves epimorphisms (projective)

Remark: $\mathrm{Hom}(P, -)$ necessarily preserves limits, because it's right-adjoint to $P \otimes (-)$

Fact: Given $\varphi : M \longrightarrow N$,

$$\begin{tikzcd} N \arrow[r, "id_N"] & N \\ M \arrow[u, "\varphi"] \arrow[r, "\varphi"'] \arrow[ur, phantom, "\ulcorner"] & N \arrow[u, "id_N"'] \end{tikzcd}$$

φ is epi $\iff$ this square is a pushout / fiber sum

(a colimit construction

Let $F : R\text{-Mod} \to S\text{-Mod}$ be an equivalence, since R is a progenerator in $R\text{-Mod}$, we see that $F(R)$ is a progenerator in $S\text{-Mod}$
Furthermore, we see that $R^{opp} \cong \mathrm{End}_R(R) \cong \mathrm{End}_S(F(R))$

$$\Updownarrow$$

$$R \cong \mathrm{End}_S(F(R))^{op}$$

Conversely, let $P \in R\text{-Mod}$ be a progenerator with $S \cong \mathrm{End}_R(P)^{op}$

To show that $R \sim S$, we observe that ${}_RP$ is a right S-module. Indeed, $S^{op} = \mathrm{End}_R(P)$ has a natural left action on P.

Thus, ${}_RP_S$ is an R,S bimodule. By Eilenberg-Watts, it suffices to find a bimodule ${}_SQ_R$ s.t.

$$ {}_RP_S \otimes_S {}_SQ_R \cong {}_RR_R $$

$$ {}_SQ_R \otimes_R {}_RP_S \cong {}_SS_S $$

Claim that $Q = \mathrm{Hom}_R(P, R) = P^*$ will work

$$\boxed{\begin{array}{l} \underline{\text{intuition}}: P \otimes P^* = R \\ \quad P^* \otimes P = \mathrm{Mat}_n(R) \end{array}}$$

$\underline{\text{General remark}}$: Given bimodules ${}_RA_S$ + ${}_RB_T$, $\mathrm{Hom}_R({}_RA_S, {}_RB_T)$ is an (S,T)-bimodule.

Let $\varphi : {}_RA_S \longrightarrow {}_RB_T$
$\quad a \longmapsto a\varphi$

R-linearity $\iff (ra)\varphi = r(a\varphi)$

Given $a \in A$, $s \in S$, $t \in T$, $\varphi : A \longrightarrow B$, define $s\varphi$ and φt so that

$a(s\varphi) = (as)\varphi$
$a(\varphi t) = (a\varphi)t$

In particular, ${}_S Q_R = \mathrm{Hom}_R({}_R P_{S,R} R_R)$.

Need to show:

(1) ${}_R P_S \otimes_S {}_S Q_R \cong {}_R R_R$

${}_R P_{\mathrm{End}(P)} \otimes_{\mathrm{End}(P)} \mathrm{Hom}_R(P, R) \cong {}_R R_R$

Note that $P \times \mathrm{Hom}(P, R) \longrightarrow R$
$(p, \varphi) \longmapsto p^{\varphi}$

By definition, this $\mathbb{Z}$-bilinear map is $\mathrm{End}(P)$-balanced, so it defines a unique map

$${}_R P \otimes_{\mathrm{End}(P)} \mathrm{Hom}(P,R)_R \xrightarrow{\Phi} {}_R R_R$$

$$\Phi\Big(\sum_i p_i \otimes \varphi_i\Big) := \sum_i p_i \varphi_i$$

Need to show that this is bijective.

Idea: • Φ is surjective b/c P is a generator

$P^{\oplus I} \twoheadrightarrow R$,

$\mathrm{Hom}(P^{\oplus I}, R) = \mathrm{Hom}(P, R)^{\oplus I}$

• Φ is injective b/c of two further "associative" properties
$(P \otimes Q \otimes P)$
$(Q \otimes P \otimes Q)$ $\rightsquigarrow$ $\Phi(\text{Something}) = 0$
$\Rightarrow$ Something $= 0$.

(2) ${}_{End(P)}Hom(P,R)_R \otimes_R {}_R P_{End(P)} \xrightarrow{\Psi} End(P)$

$$(\varphi, p) \longmapsto \varphi p$$

s.t. $\forall p'$, $p'(\varphi p) = (p'\varphi)p$

Ψ is onto b/c P R f.g. and projective

Ψ is injective b/c of associativity conditions

$\square$

Prototype: $R\text{-Mod} \simeq Mat_n(R)\text{-Mod}$

Another point of view:

Let R be any ring. We say $e \in R$ is a full idempotent if

- $e^2 = e$
- $ReR = R$

In this case, we get $R \approx eRe$

much simpler

$Re \otimes_{eRe} eR = R$, $eR \otimes_R Re = eRe$

eg: $R = Mat_n(S)$

$$e = E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}$$

$ReR = R$

$eRe \approx R$
$\parallel$
S

11/15/17 Big Theorem (Morita):

An equivalence $R\text{-Mod} \cong S\text{-Mod}$
Comes with canonical data $({}_RP_S, {}_SQ_R)$ s.t.

- ${}_RP_S \otimes_S {}_SQ_R \cong {}_RR_R$
- ${}_SQ_R \otimes_R {}_RP_S \cong {}_SS_S$

and

$$Q\otimes(-) : R\text{-Mod} \rightleftarrows S\text{-Mod} : P\otimes(-)$$

- ${}_RP \in R\text{-Mod}$, ${}_SQ \in S\text{-Mod}$ are progenerators

- $R \cong End_S(P_S) \cong End_S({}_SQ)^{op}$

$S \cong End_R({}_RP)^{op} \cong End_R(Q_R)$

- ${}_SQ_R \cong Hom_S({}_RP_S, {}_SS_S)$

${}_SQ_R \cong Hom_R({}_RP_S, {}_RR_R)$

${}_RP_S \cong Hom_S({}_SQ_R, {}_SS_S)$

${}_RP_S \cong Hom_S({}_SQ_R, {}_RR_R)$

TY Lam calls this the "Morita Context"

Next topic: Matrices

In any category with a biproduct $\oplus$, we have a notion of "Matrices"

$$\mathrm{Hom}_{\mathcal{C}}(A\oplus B, C\oplus D) = \mathrm{Hom}_{\mathcal{C}}(A,C) \prod \mathrm{Hom}_{\mathcal{C}}(A,D) \prod \mathrm{Hom}_{\mathcal{C}}(B,C) \prod \mathrm{Hom}_{\mathcal{C}}(B,D)$$

$$\varphi_{AC} \qquad \varphi_{AD} \qquad \varphi_{BC} \qquad \varphi_{BC}$$

$$[\varphi] = \begin{bmatrix} \varphi_{AC} & \varphi_{AD} \\ \varphi_{BC} & \varphi_{BD} \end{bmatrix}$$

$$\begin{tikzcd}
A\oplus B \arrow[r, "\varphi"] \arrow[r, bend left=60, "\varphi_{AD}"] \arrow[r, bend right=40, "\varphi_{AC}"'] & C\oplus D \arrow[r, "\pi"] \arrow[r, bend left=60, "\pi_{DE}"] \arrow[r, bend right=40, "\pi_{CE}"'] & E\oplus F
\end{tikzcd}$$

$$(\pi\circ\varphi)_{AE} = \pi_{CE}\circ\varphi_{AC} + \pi_{DE}\circ\varphi_{AD}$$

$$[\pi\circ\varphi] = [\pi]\cdot[\varphi] \qquad (\text{composition in } \mathcal{C}^{op})$$

Special Case: R a ring, $\varphi \in \mathrm{Hom}(R^{\oplus n}, R^{\oplus n})$. Each component is in $\mathrm{End}_R(R) = R^{op}$,

$$\mathrm{Hom}_R(R^{\oplus n}, R^{\oplus n}) = \mathrm{End}_R(R^{\oplus n}) = (R^{\oplus n})^{op}$$

$$= \mathrm{Mat}_n(R^{op}) = \mathrm{Mat}_n(R)^{op}$$

We say that an R-module S is simple if it has no nontrivial submodule

Schur's Lemma: In this case $End_R(S)$ is a division k-algebra

Pf: Let $\varphi : S \to S$ be an R-lin map. Since S is simple

$Ker(\varphi) \in \{0, S\}$ + $im(\varphi) \in \{0, S\}$

$ker \varphi = S \iff im\, \varphi = 0 \iff \varphi = 0$

$\underbrace{Ker\, \varphi = 0 \iff Im\, \varphi = S}_{\varphi \text{ is bijection}} \iff \varphi \neq 0$

Special case: If k is an alg. closed field, then

S simple $\Rightarrow End_R(S) = k$

What's this good for

General $M, N \in R$-Mod, we define

$\langle M, N \rangle_R = \dim_k Hom_R(M, N)$

Schur's lemma: S simple $\iff \langle S, S \rangle_R = 1$

S, T simple $\Rightarrow \langle S, T \rangle_R = \begin{cases} 1, & S \cong T \\ 0 & \text{else} \end{cases}$

From now on we need to impose some finiteness conditions

- A ring R is left/right/two-sided **Artinian** if every descending chain of left/right/two-sided ideals stabilizes.

Motivational examples:

- Finite rings
- Finite-dim'l algebras over a field.

$$|G/H| = |G|/|H|, \quad \dim(U/V) = \dim(U) - \dim(V)$$

Fact: If R is artinian, then $\forall\, a, b \in R$, $ab = 1 \iff ba = 1$

Pf: Suppose $ab = 1$. Then

$$R \supseteq bR \supseteq b^2R \supseteq \cdots$$

$$\exists\ b^kR = b^{k+1}R$$

$$\Rightarrow \exists\, c \in R \text{ s.t. } b^k = b^{k+1}c$$

$$\Rightarrow a^k b^k = a^k b^{k+1} c$$

$$\Rightarrow 1 = bc$$

Finally, $a = a1 = a(bc) = (ab)c = 1c = c$. $\square$

If R is artinian, then every f.g. R-module has a **unique** decomposition as a sum of indecomposables:

$$M \cong J_1 \oplus J_2 \oplus \cdots \oplus J_m \qquad \text{"Krull-Remak-Schmidt-etc"}$$

J is for "Jordan Block"

A Jordan block is an indecomposable f.d. $K[x]$-module

If in addition to Artinian, we have indecomp $\Longrightarrow$ simple,
then every f.g. R-module has a unique decomp. into simples

$$M \cong S_1 \oplus S_2 \oplus \cdots \oplus S_m$$

In this case, we say that R is a semisimple (Artinian) algebra

Warning: The word "semisimple" has several seemingly-different definitions

- "Semisimple" = "diagonalizable"
- "Semisimple" = "radical is zero"

$A \qquad Jac(A) = 0$

$A/Jac(A)$ is always semisimple

Big Theorem (Artin-Wedderburn Thm): Let R be an Artinian algebra. R is semisimple $\Leftrightarrow$ R is a direct product of matrix rings over division algebras

$$R \cong Mat_{n_1}(D_1) \oplus \cdots \oplus Mat_{n_n}(D_n)$$

Ex: Let G be a finite group.

$\mathbb{C}G$ is semisimple (Maschke's Thm)
Hence

$$\mathbb{C}G = Mat_{n_1}(\mathbb{C}) \oplus \cdots \oplus Mat_{n_m}(\mathbb{C})$$

11/27/17

Idempotents $(e^2 = e)$:

From commutative geometry...

Let R be a commutative ring with zero divisors.
There are two ways this can happen:

- If R has nontrivial nilpotents $(x^n = 0,\ x \neq 0)$
- R has more than one minimal prime ideal.

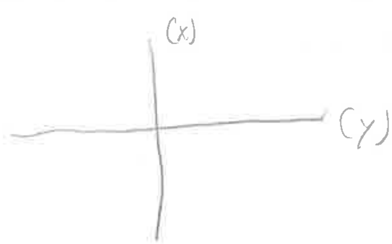
Proof: Suppose R has no nontrivial nilpotents, i.e. $\sqrt{0} = 0$.

In general, $\underset{\underset{0}{\shortparallel}}{\sqrt{0}} = \bigcap_{\substack{P \subseteq R \\ \text{prime}}} P = \bigcap_{\substack{\text{minimal} \\ \text{primes}}} P$

If there's only one minimal prime P, then $P = 0$. Hence R is a domain.

$\square$

examples: $K[x,y]/(xy)$



has minimal primes (x) & (y),
giving two irreducible ("connected") components

$K[x,y]/(y^2)$

has nilpotent y.

$\varepsilon^2 = 0$

By the Chinese remainder theorem, we have that $\dfrac{K[x,y]}{(xy)} \cong \dfrac{K[x,y]}{(x)} \oplus \dfrac{K[x,y]}{(y)}$ (direct sum of rings)

in $R \oplus S$, $(r_1, s_1) \cdot (r_2, s_2) = (r_1 r_2, s_1 s_2)$

Where are the zerodivisors?

$e = (1,0)$, $f = (0,1)$ $\Rightarrow$ $ef = (1,0)\cdot(0,1) = (0 \cdot 1, 1 \cdot 0) = (0,0)$

Also have $e^2 = (1^2, 0^2) = (1,0) = e$
$f^2 = (0^2, 1^2) = (0,1) = f$

and $e + f = (1,1) = 1_R$

Let R be any ring, and consider a left R-module $R \circlearrowright M$.

Let $E = \mathrm{End}_R(M)$. Then M decomposes as a direct sum iff E has a nontrivial idempotent.

Pf: If $M = M_1 \oplus M_2$, let $e_1 = \pi_1 \oplus 0 = i_1 \circ \pi_1$
$e_2 = 0 \oplus \pi_2 = i_2 \circ \pi_2$

Clearly, $e_1 + e_2 = id$, and $e_1 \circ e_2 = e_2 \circ e_1 = 0$, and $e_i^2 = e_i$, $i = 1, 2$ ✓

Conversely, Suppose $e = E \setminus \{0, id\}$ is idempotent: $e^2 = e$,

Then $f = id - e$ is also a nontrivial map, and also idempotent:

$f^2 = (id - e) \circ (id - e) = id - 2e + e^2 = id - 2e + e = id - e = f$.

Furthermore we have $e \circ f = f \circ e = 0$:

$e \circ f = e \circ (id - e) = e - e^2 = e - e = 0$.
$f \circ e = (id - e) \circ e = e - e^2 = e - e = 0$.

Claim: $M \underset{R}{\cong} e(M) \oplus f(M)$

Indeed, 1): $e(M)$ + $f(M)$ are R-submodules: $\forall\, m \in M$, $r \in R$,

$r(e(m)) = e(r(m))$ ✓

2): Consider any $m \in M$: $m = id(m) = (e + f)(m) = e(m) + f(m) \in e(M) \oplus f(M)$,

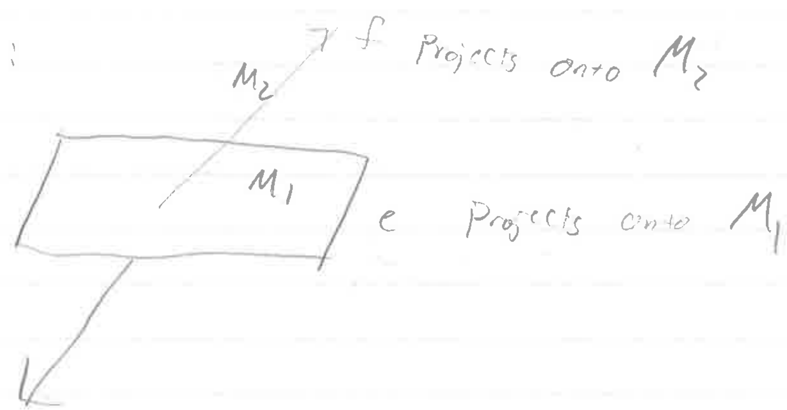
3) $e(M) \cap f(M) = 0$.

Consider any $m \in e(M) \cap f(M)$. Then $m = e(n_1) = f(n_2)$ for $n_1, n_2 \in M$.

$e(n_1) = f(n_2) \Rightarrow e(e(n_1)) = e(f(n_2)) \Rightarrow m = e(n_1) = 0$.

$\square$ 105

Picture:



By induction, we can write $M \cong M_1 \oplus \cdots \oplus M_n$ iff there exists a complete system of orthogonal idempotents

- $\mathrm{id} = e_1 + e_2 + \cdots + e_n$ (complete)
- $e_i^2 = e_i$ (idempotent)
- $e_i \circ e_j = e_j \circ e_i = 0 \quad \forall i \neq j$ (orthogonal).

Furthermore, the summand M_i is indecomposable iff idempotent e_i is primitive, i.e., cannot be written as

$e_i = f + g$

where f, g are idempotent, $fg = gf = 0$, and $f, g \notin \{0, e_i\}$.

"CSOPOI"

Let's apply these ideas to the "regular representation" $R \curvearrowright R$

$M = R, \quad E = \mathrm{End}_R({}_R R) = R^{op}$

Thus, idempotent endomorphisms are just idempotent elts $e \in R$, whose image is the principal left ideal

$$\operatorname{im}(\cdot e) = Re = \{re : r \in R\}.$$

Thus, every decomposition of R has the form

$${}_R R \cong Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$$

for some orthogonal idempotents

- $1_R = e_1 + e_2 + \cdots + e_n$
- $e_i^2 = e_i \quad \forall i$
- $e_i e_j = e_j e_i = 0 \quad \forall i \neq j.$

Question: When is Re_i irreducible?

In other words, when is Re a minimal left ideal of R?

Brauer's Lemma: Let R be any ring, and let $I \subseteq R$ be any minimal left ideal. Then we have either

- $I^2 = 0$
- $I = Re$ for some idempotent $e \in R$.

What about homomorphisms between the summands?

$$\operatorname{Hom}_R(Re_i, Re_j) = e_i R e_j$$

Lemma: Let M be any left R-module, and $e \in R$ be any idempotent.

Then, $\mathrm{Hom}_R(Re, M) \cong eM \subseteq$ submodule iff $e \in Z(R)$

Pf: Consider an R-map $\varphi: Re \longrightarrow M$.

Then $e(\varphi(e)) = \varphi(e^2) = \varphi(e) \in eM$

Define a map $\mathrm{Hom}_R(Re, M) \longrightarrow eM$

$$\varphi \xmapsto{\mathbb{Z}\text{-lin}} \varphi(e)$$

$$\begin{pmatrix} \varphi_n : Re \longrightarrow M \\ re \longmapsto rn \end{pmatrix} \longleftarrow\!\!\!\shortmid\; n$$

$$\begin{aligned} r'\varphi_n(re) &= r'rn \\ &= (r'r)n \\ &= \varphi_n(r're) \end{aligned}$$

$$\varphi_n(e) = \varphi_n(1e) = 1 \cdot n = n$$

$re = r'e \Rightarrow rn = r'n$?

$n = em$ for some m

$rn = rem = r'em = r'n$.

$\square$

Corollary: $\mathrm{Hom}_R(Re_i, Re_j) \cong e_i R e_j$.

$\mathrm{Hom}_R(Re, Re) = \mathrm{End}_R(Re) \cong eRe \leftarrow$ a ring

Remark: Let e be an idempotent, Then eRe is a ring, even though

$eRe \subseteq R$ is not a subring (unit of eRe is e, not 1)

In many cases, $R \simeq eRe$

"Morita"

$$R \cong Re \oplus Rf$$

$$\mathrm{End}_R(R) = \mathrm{End}_R(Re \oplus Rf)$$

$$R^{op} = eRe \oplus eRf \oplus fRe \oplus fRf$$

$$R^{op} = \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$$

12/4/17

Recall: If R is an artinian ring (think f.d. algebra over a field), then $R \cong Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$ as R-modules

where $1 = e_1 + \cdots + e_n$, $e_i^2 = 0$ $\forall i$, and $e_i e_j = e_j e_i = 0$ $\forall i$.

If Re is an indecomposable R-module, we say that the idempotent e is primitive.

Krull-Schmidt: The indecomposable summands are unique up to isomorphism

By taking endomorphisms, we get

$$\begin{array}{ccc} \mathrm{End}_R(R) & = & \mathrm{End}_R(Re_1 \oplus \cdots \oplus Re_n) \\ \| & & \| \\ R^{op} & \cong & \bigoplus_{i,j} e_i R e_j \end{array} \quad \text{(as abelian groups)}$$

Where e_iRe_i is a ring with unit e_i, and e_iRe_j is, in general, just an abelian group.

$$e_iRe_i \circlearrowright e_iRe_j \circlearrowleft e_jRe_j$$

The summands Re_i are not necessarily <u>simple</u>.

If they are simple, then we say that R is a <u>semisimple</u> ring, and many nice properties follow.

<u>Schur's Lemma</u>: If $R \cong Re_1 \oplus \cdots \oplus Re_n$ with each summand simple, then

$$e_iRe_j = \begin{cases} \text{a division ring } D_i, & \text{if } Re_i \cong Re_j \\ 0, & \text{if } Re_i \not\cong Re_j \end{cases}$$

Combine the "isotypic components"!

$$R^{op} = \bigoplus_{i,j} e_iRe_j = \bigoplus_i D_i^{\oplus n_i}$$

$\big\downarrow$ take $\mathrm{End}_R(-)$ of both sides

$$R \cong \bigoplus_i \mathrm{End}(D_i^{\oplus n_i}) = \bigoplus_i \mathrm{Mat}_{n_i}(D_i^{op})$$

<u>Artin-Wedderburn Thm</u>:

110 Artinian semisimple rings $\Longleftrightarrow$ direct sum of matrix algebras

More generally, let $R = Re_1 \oplus \cdots \oplus Re_n$ be indecomposable summands.

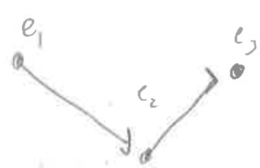
Division rings are local rings, in the sense that (0) is the unique maximal ideal.

More generally, let $e \subseteq R$ be such that $e^2 = e$.

Then e primitive $\iff$ Re is indecomp $\iff$ $\underbrace{End_R(Re) = eRe}_{\text{has no nontrivial idempotents}}$ $\underset{\text{if } Re \text{ is injective}}{\overset{\Longleftarrow}{\Longrightarrow}}$ eRe local

To go further, one would discuss projective/injective modules + quivers.

Given a digraph,



Path algebra $= \sum$ paths (with concatenation)

$\mathbb{C}SP \otimes I$ = vertices

Easiest example: Let $R = Mat_n(F)$ over a field F.

What are the idempotents of R?

Theorem: If $P^2 = P$ has rank $k \leq n$, then there exist two $n \times k$ matrices A, B, both of rank k such that

$$P = A(B^T A)^{-1} B^T. \quad \text{(oblique projection)}$$

If moreover P is symmetric, we can take $A = B$: $\boxed{P = A(A^T A)^{-1} A^T}$ (orthogonal projection)

Pf: Since $P^2 = P$, we know that $F^n \cong \operatorname{im} P \oplus \operatorname{im}(I-P)$

Claim: $\operatorname{im}(I-P) = \ker(P)$
$\ker(I-P) = \operatorname{im}(P)$

- $\operatorname{im}(I-P) \subseteq \ker(P)$: $P((I-P)x) = (P-P)x = 0x = 0$
- $\ker(P) \subseteq \operatorname{im}(I-P)$: by rank-nullity, they have the same dimension. ✓

Same argument for $\ker(I-P) = \operatorname{im}(P)$. ✓

Now choose bases $(u_1, \dots, u_k) = \operatorname{im} P = \ker(I-P)$
$(v_1, \dots, v_k) = (\ker P)^\perp = (\operatorname{im}(I-P))^\perp$

Form matrices

$$A = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{pmatrix} \quad n \times k \qquad \operatorname{col}(A) = \operatorname{im} P$$

$$B = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix} \quad n \times k \qquad \operatorname{col}(B) = (\ker P)^\perp$$

Compute P: For all $x \in F^n$, we have $Px \in \operatorname{im} P \Rightarrow Px \in \operatorname{col}(A)$

$\Rightarrow Px = Ay$ for some y.

We also have $(I-P)x \in \operatorname{im}(I-P) = \ker(P)$

$\Rightarrow (I-P)x \perp v_i \ \forall i$

 $\Rightarrow B^T(I-P)x = 0$

$\Rightarrow B^T x - B^T P x = 0 \Rightarrow B^T x = B^T P x \Rightarrow B^T x = B^T \underset{n\times k}{A} \underset{k\times 1}{y}$

Solve for y:

Claim $k\times k$ matrix $B^T A$ is invertible.

Indeed, j^{th} col $B^T A = B^T(j^{th}$ col $A)$

Col relations on $B^T A \longleftrightarrow$ Col. relations on $A \longleftrightarrow \emptyset$ ✓

$\Rightarrow y = (B^T A)^{-1} B^T x$

$\Rightarrow Px = A(B^T A)^{-1} B^T x \quad \forall x$, so

$$\boxed{P = A(B^T A)^{-1} B^T x}$$

$\square$

Primitive Idempotents?

In general, the collection $\mathrm{Idem}(R)$ of idempotents is a poset.

$Re \subseteq Rf \Leftrightarrow e \leq f \Leftrightarrow ef = fe = e$

The minimal elements are primitive.

$P_1 \leq P_2 \Leftrightarrow$ $\mathrm{im}(P_1) \subseteq \mathrm{im}(P_2)$
$\mathrm{Ker}(P_1) \supseteq \mathrm{Ker}(P_2)$

P is minimal / Primitive $\iff P$ has rank 1

Primitive idempotents of $Mat_n(F)$ have the form

$p = \vec{a}(\vec{b}^T\vec{a})^{-1}\vec{b}^T$ for $\vec{a}, \vec{b} \in F^n$

$= \dfrac{\vec{a}\vec{b}^T}{\vec{b}^T\vec{a}} \begin{matrix} n\times n \\ 1\times 1 \end{matrix}$

WLOG assume $\vec{b}^T\vec{a} = 1$

Conclusion: Primitive idempotents have the form $(col)(row) = \vec{a}\vec{b}^T$.

$R = Mat_n(F)$

Let $I = P_1 + \cdots + P_n$

$I = \vec{a}_1\vec{b}_1^T + \cdots + \vec{a}_n\vec{b}_n^T$, $\quad P_i^2 = P_i \iff \vec{a}_i\vec{b}_i^T\vec{a}_i\vec{b}_i^T = \vec{a}_i\vec{b}_i$, so WMA $\vec{a}_i\vec{b}_i^T = 1$.

• $P_iP_j = 0 \iff a_i b_i^T a_j b_j^T = 0$, so assume $b_i^T a_j = 0 \quad \forall\, i \neq j$

I.e., we have $\vec{b}_i^T\vec{a}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ (dual bases)

$$I = AB = \sum_i (i^{\text{th}} \text{ col of } A)(i^{\text{th}} \text{ row } B)$$
$$\phantom{I = AB = \sum_i (i^{\text{th}} \text{ col of } A)(} i^{\text{th}} \text{ col } B^T$$

$$\Downarrow$$

$$I = \sum_i a_i b_i^T$$

Choose invertible $A \in Mat_n(F)$

Let $a_i = i^{\text{th}}$ col of A
$b_i = i^{\text{th}}$ col of $(A^{-1})^T$

Then $P_i = a_i b_i^T$ is a CSOPO I.

---

ex:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad (A^{-1})^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \checkmark$$

$Mat_n(F)$ has one simple module, namely F^n, and

$$R = Mat_n(F) \cong (F^n)^{\oplus n}$$

$$R \curvearrowright \underset{\text{simple}}{F^n}$$

12/6/17

Let R be a semisimple Artinian ring, so

$${}_R R \cong S_1 \oplus \cdots \oplus S_n$$

where each S_i is a simple left R-module.

Then, if S is any nonzero simple R-module, $S \cong S_i$ for some i, and hence, R has finitely many nonisomorphic simples.

Pf: Take $0 \neq s \in S$ and define an R-map

$$\varphi : R \longrightarrow S$$
$$r \longmapsto rs$$

Let $\varphi_i = \varphi|_{S_i} : S_i \longrightarrow S$.

Since $\varphi \neq 0$, we must have $\varphi_i \neq 0$ for some i.

By Schur's lemma, $S_i \xrightarrow[\varphi_i]{\neq 0} S \Rightarrow S_i \cong_{\varphi_i} S$.

$\square$

Let R be a commutative K-algebra for some alg. closed field K. Then every simple R-module has dimension 1.

Pf: Let S be a nonzero simple. Choose a nonzero $s \in S$ and define $\varphi: R \to S$ as before. Since $\text{im}(\varphi) \subseteq S$ is nonzero, we must have $\text{im}(\varphi) = S$.

By the first isomorphism thm $S \cong R/\text{Ker}(\varphi)$

Since S has no submodules, $\text{Ker}(\varphi)$ is a max'l ideal of R.
$\Rightarrow S$ is a field extension of K

Since k is alg. closed we must have $S \cong k$ as k-vector spaces.

Hence $\dim S = 1$.

$\square$

Ex: Let $R = K[x]$ with K alg. closed.

R is not artinian: $(x) \supset (x^2) \supset (x^3) \supset \cdots$

A finite-dim'l $K[x]$ module $\longleftrightarrow$ square matrix up to conjugation

Simple module $\longleftrightarrow$ 1×1 matrix

Semisimple $\longleftrightarrow$ diagonalizable

$K[x]$ itself is not semisimple

Indecomposables $\longleftrightarrow$ Jordan blocks

More generally, let $R = \underline{K\langle x_1, \dots, x_n \rangle}$
relations

f.d. R-module $\longleftrightarrow$ Square matrices $X_1, X_2, \dots, X_n$
(up to iso)

satisfying the relations
(up to simultaneous conjugation)

Let M be a $K[x]$-module.

For all $f(x) = \sum a_i x^i \in K[x]$, we have

$$f(x) \cdot m = \left(\sum a_i x^i\right) \cdot m$$

$$= \sum a_i (x \cdot m)^i$$

1. K acts on $M \Rightarrow M$ is a K-vector space
2. x can be any endomorphism of M, preserving K-linearity

$\Rightarrow x$ is a K-linear endomorphism of a vector space

Example: Given a finite group G, define the group algebra

$$KG = K\langle x_g : g \in G \rangle \Big/ (x_g x_h = x_{gh},\ x_{id} = 1)$$

A f.d. KG-module $\longleftrightarrow$ Square matrices $\{X_g : g \in G\}$ satisfying group relations

$\longleftrightarrow$ group hom $\underbrace{X : G \longrightarrow GL_n(K)}_{\text{matrix representation}}$

Thm (Maschke): For a finite group G and alg closed field K (even char $K \nmid \# G$), the group algebra is semisimple.

In other words, indecomposable $\Rightarrow$ irreducible.

Pf: Let V be any f.d. KG-module. Let $0 \subsetneq U \subsetneq V$ be any nontrivial submodule. Goal: Find a complement $V = U \oplus_{KG} U'$.

Let U'' be any vector space complement of U,

$$V = U \oplus_K U''$$

Problem: U'' is not G-stable

Let P be the projection onto U in the direction of U'':

$$P^2 = P, \quad \text{im } P = U, \ \ker P = U''$$

Now, define a new projection by

$$\overline{P} = \frac{1}{|G|} \sum_{g \in G} g P g^{-1}$$

$$\overline{P}|_U = \text{id}, \text{ and } \text{im } \overline{P} \subseteq U \Rightarrow \overline{P}^2 = \overline{P}$$
(since $P|_U = \text{id}$)

Furthermore, $\text{im } \overline{P} = U$. $\ker(\overline{P}) = U'$

Define $U' := \ker(\overline{P})$

Then we get $V = \text{im } \overline{P} \oplus \ker \overline{P}$. Claim that U' is G-stable.

Indeed, $\forall h \in G, x \in U'$,

$$\bar{P}(hx) = \frac{1}{|G|}\sum_{g\in G} (g)P(g^{-1}h)x$$

Let $\ell = h^{-1}g$

$$\bar{P}(hx) = \frac{1}{|G|}\sum_{\ell\in G}(h\ell)\bar{P}\ell^{-1}x = h\cdot\frac{1}{|G|}\Big[\sum_{\ell\in G}\ell\bar{P}\ell^{-1}\Big]x = h(\bar{P}x) = 0 \qquad \square$$

Remark: If G is compact, this trick still works:

$$\bar{P} = \frac{1}{\mathrm{Vol}(G)}\int_{g\in G} gPg^{-1} \qquad \text{(Haar measure)}$$

GL_n, SL_n not compact

U, O, SO compact

Since kG is semisimple, we find that every irrep is a summand of the regular representation

$$kG = \bigoplus_i S_i^{\oplus n_i},$$

where the S_i are a complete list of nonisomorphic simple modules.

Q: How to compute the multiplicities?

A: Claim: $n_i = \dim_k S_i$.

Consider the hom space

$$\mathrm{Hom}_{kG}(S_i, kG) = \mathrm{Hom}(S_i, \bigoplus_j S_j^{\oplus n_j}) = \bigoplus_j \mathrm{Hom}_j(S_i, S_j)^{\oplus n_j} = \underbrace{\mathrm{Hom}(S_i, S_i)}_{\text{division ring finite over } k}{}^{\oplus n_i}$$

For now, assume K is alg. closed.

$$\Rightarrow \mathrm{Hom}(S_i, S_j) = \begin{cases} 0, & i \neq j \\ K, & i = j \end{cases}$$

$$\Rightarrow \quad \mathrm{Hom}_{KG}(S_i, KG) = K^{n_i}.$$

That is we can recover the multiplicities:

$$n_i = \dim_K(\mathrm{Hom}_{KG}(S_i, KG))$$

More generally,

$$\dim_K(\mathrm{Hom}_{KG}(S_i, V)) = \text{multiplicity of } S_i \text{ inside } V.$$

Define a "$\mathbb{Z}$-bilinear form"

$$\begin{aligned} KG\text{-mod} \times KG\text{-mod} &\longrightarrow \mathbb{Z} \\ U, V &\longmapsto \dim_K(\mathrm{Hom}_{KG}(U, V)) \\ & \qquad \langle U, V \rangle \end{aligned}$$

With properties:

$\langle S_i, S_j \rangle = \delta_{ij}$ "Simples are orthonormal"

$\langle S_i, V \rangle =$ multiplicity of S_i in V.

$$\langle \bigoplus_i U_i, \bigoplus_j V_j \rangle = \underbrace{\sum_{i,j}}_{\text{"}\mathbb{Z}\text{-bilinearity"}} \langle U_i, V_j \rangle$$

Let $Rep_k(G)$ be the free $\mathbb{Z}$-module generated by the simple KG-modules

$$\begin{tikzcd}
(KG\text{-Mod})^2 \arrow[d] \arrow[dr, "{\langle , \rangle}"] & \\
Rep_k(G)^2 \arrow[r, "{\langle , \rangle}"'] & \mathbb{Z}
\end{tikzcd}$$

$\mathbb{Z}$-bilinear

What other structure does this have?

12/11/17 Let k be an algebraically closed field, R a f.d. associative k-algebra. Suppose

$${}_R R \cong \bigoplus_{i=1}^{n} S_i^{\oplus d_i}$$

where S_i simple.

<u>Last time</u>: Every simple R-module is isomorphic to S_i for some i

$\Rightarrow$ R has finitely many simple R-modules (up to isomorphism)

How many?

<u>Trick</u>: Take endomorphisms

$$\underset{\substack{\parallel \\ R^{op}}}{\mathrm{End}_R(R)} = \bigoplus_{i=1}^{n} \mathrm{End}_R(S_i^{\oplus d_i}) = \bigoplus_{i=1}^{n} \mathrm{Mat}_{d_i}(\mathrm{End}(S_i))$$

direct product of rings

$$\Rightarrow R \cong \bigoplus_{i=1}^{n} \mathrm{Mat}_{d_i}(\mathrm{End}(S_i)^{op})$$

Claim: Since K alg. closed, $\mathrm{End}(S_i) \cong K$

Pf: Let S be any f.d. simple R-module, where R is a K-algebra.

Any nonzero R-map $\varphi: S \longrightarrow S$ must have an eigenvalue, say $\lambda \in K$

In other words, the R-map $\varphi - \lambda I$ is <u>not</u> invertible.

$\Rightarrow$ Since S is simple we have $\varphi - \lambda I = 0 \Rightarrow \varphi = \lambda I$.

$\therefore$ $\mathrm{End}_R(S) = \text{Scalars} \cong K$. $\square$

$R = \bigoplus_{i=1}^{n} \mathrm{Mat}_{d_i}(K)$, where $\mathrm{Mat}_{d_i}(K) \cong \mathrm{End}_R(S_i^{d_i})$

Claim: • $d_i = \dim_K(S_i)$

• $n = \dim_K(Z(R))$

$R \cong \bigoplus_{i=1}^{n} \mathrm{Mat}_{d_i}(K)$

$\Downarrow$

$Z(R) \cong \bigoplus_{i=1}^{n} Z(\mathrm{Mat}_{d_i}(R))$

Lemma: For a general ring R, $Z(\mathrm{Mat}_d(R)) \cong Z(R)$

Pf: Let $r \in Z(R)$. Then, $rI \in Z(\mathrm{Mat}_d(R))$. ✓

Let $A \in Z(\mathrm{Mat}_d(R))$, $E_{ij} = 1$ in (i,j), zeroes elsewhere.

$E_{ij} A = A E_{ij}$

$$\begin{array}{cc} & \begin{matrix} & & j & & \end{matrix} \\ i & \begin{pmatrix} & 0 & & 0 & \\ a_{j1} & a_{j2} \cdots & a_{jj} & \cdots & a_{jn} \\ & 0 & & 0 & \end{pmatrix} \end{array} = \begin{array}{cc} & \begin{matrix} & & j & \end{matrix} \\ i & \begin{pmatrix} & 0 & a_{1i} & 0 \\ & & \vdots & \\ \cdots & \cdots & a_{ii} & \cdots \\ & & \vdots & \\ & 0 & a_{ni} & 0 \end{pmatrix} \end{array}$$

$\Rightarrow a_{ij} = 0 \quad \forall\, i \neq j \quad + \quad a_{jj} = a_{ii} \quad \Longrightarrow \quad A$ is scalar.

(In particular, A is a central scalar matrix) $\square$

$\Rightarrow Z(R) \cong \bigoplus_{i=1}^{n} Z(Mat_{d_i}(k))$

$\Rightarrow Z(R) \cong k^{\oplus n}$

$\Rightarrow \dim(Z(R)) = n = \#$ non-isomorphic simple R-modules.

$R \cong \bigoplus_{i=1}^{n} S_i^{\oplus d_i}$

$\dim(R) = \sum_{i=1}^{n} d_i \cdot \dim(S_i)$, but $\dim_k(R) = \sum_{i=1}^{n} d_i^2$

$\Rightarrow \boxed{\sum_{i=1}^{n} d_i \cdot \dim(S_i) = \sum_{i=1}^{n} d_i^2} \quad \overset{?}{\Longrightarrow} \quad d_i = \dim_k(S_i)$

When $\underline{n=1}$: $R = S^d \longrightarrow d \cdot \dim_k(S)$
$R = Mat_d(k) \longrightarrow d^2$ $\Big\} \Rightarrow \dim_k(S) = d$.

$$R \cong \bigoplus_{i=1}^{n} Mat_{d_i}(k)$$

$\uparrow$ product of $\underline{\text{rings}}$

$\underline{\text{General fact:}}$ Let R, R' be rings. Then, every $(R \oplus R')$-module has the form $M \oplus M'$ where $M \in R\text{-Mod}$, $M' \in R'\text{-Mod}$.

Indeed, $M = \underbrace{(1,0)M}_{R\text{-module}} \oplus \underbrace{(0,1)M}_{R'\text{-module}}$

The nonisomorphic simples of $R \cong \bigoplus_{i=1}^{n} Mat_{d_i}(k)$ are $S_1, S_2, \dots, S_n$.

$\underline{\text{Claim:}}$ S_i is a simple $Mat_{d_i}(k)$-module

What are the simple $Mat_d(k)$-modules?

$\underline{\text{A}}$: Every simple module over $Mat_d(k)$ is isomorphic to k^d.

$Mat_d(k) \circlearrowright (d \times 1)$ column vectors

$\Rightarrow S_i \cong k^{d_i} \Rightarrow \dim_k S_i = d_i$ $\square$

Special Case: Group algebras

$$kG = \left\{ \sum_{g \in G} \lambda_g g \mid \lambda_g \in k \right\}$$

If $\operatorname{Char} k \nmid |G|$, we saw that

kG is semisimple, say

$$kG = \bigoplus_{i=1}^{n} S_i^{d_i},$$

where $n = \dim_k Z(kG) = \#$ Conjugacy Classes

Claim:

Proof: Let $\sum \lambda_g g \in Z(kG)$

Then $\forall\, h \in G$, $\sum \lambda_g g = h(\sum \lambda_g g) h^{-1}$

$$= \sum \lambda_g hgh^{-1}$$

$$\sum_g \lambda_g g = \sum_g \lambda_g hgh^{-1} \qquad \text{Let } l = hgh^{-1}, \quad g = h^{-1} l h$$

$$\sum_g \lambda_g g = \sum_g \lambda_{h^{-1}gh} hgh^{-1}$$

$$\Rightarrow \lambda_g = \lambda_{h^{-1}gh} \quad \forall\, g, h \in G.$$

126 $\Rightarrow \sum \lambda_g g \in Z(kG) \iff \lambda_{h^{-1}gh} = \lambda_g \quad \forall\, g, h \in G$

If we think of kG as k^G

$$\sum \lambda_g g \longleftrightarrow (g \mapsto \lambda_g)$$

$\Rightarrow Z(kG) \longleftrightarrow$ Class functions (functions constant on conjugacy classes)

Basis: Characteristic functions for each class

$\Rightarrow \dim(Z(kG)) = \#$ Conjugacy Classes.

$\square$

Problem: Can we somehow construct the simple kG-modules from the conjugacy classes?

Equivalently: Can we find a complete system of central idempotents for $Z(kG)$?

Maybe the class characteristic functions?

$e_1 = 1_G$

$G = S_3 = \{\boxed{1}, \boxed{(12), (13), (23)}, \boxed{(123), (132)}\}$

$e = (12) + (13) + (23)$

$$e^2 = \big(((12)+(13)+(23))((12)+(13)+(23))\big) = (12)^2 + (13)^2 + (23)^2 + (12)(13) + (12)(23) + (13)(12) + (13)(23) + (23)(12) + (23)(13)$$

$$= (12)^2 + (13)^2 + (23)^2 + (132) + (123) + (123) + (132) + (132) + (123)$$

$= 3\cdot\underline{1} + 3(123) + 3(132)$

The easiest case: Let $e = \frac{1}{|G|}\sum g$

Claim that e is a central idempotent.

Last time: e is central. ✓

$$e^2 = \frac{1}{|G|^2}\left(\sum_g g\right)\left(\sum_h h^{-1}\right) = \frac{1}{|G|^2}\sum_{g,h} gh^{-1} = \frac{1}{|G|^2}\left(|G|\sum g\right)$$

$$= \frac{1}{|G|}\sum g = e. \quad ✓$$

For any kG module V, e projects onto the fixed subspace

$$e : V \twoheadrightarrow V^G$$

$\boxed{\text{Lemma}}$

Theorem: For each simple module, S, we will define the class function

$$\begin{aligned} G &\longrightarrow k \\ g &\longmapsto tr(g|_S) \end{aligned}$$

$tr(gh) = tr(hg)$

$\Rightarrow tr(h^{-1}gh) = tr(ghh^{-1}) = tr(g)$

These "Characters" are a basis for the class functions.

12/13/17 What is a Character?

kG-Mod (nice k)

$$\Downarrow$$

Grothendieck ring

$$\Updownarrow$$

ring of "Characters"

Let $kG \circlearrowright M$ finite dim'l

Then, $\forall\ r \in kG$, we get a k-linear endomorphism $r: M \longrightarrow M$, $m \longmapsto rm$

The "Character" of r on M is just the trace

$$\operatorname{tr}_M(r) \in k$$

Fact: The ~~whole category~~ representation theory is determined by the character values

$$\operatorname{tr}_S(g) \text{ for } g \in G,\ S \text{ simple module}$$

$$\begin{array}{c|c} & g \\ \hline S & \operatorname{tr}_S(g) \end{array}$$

"Character table"

"Orthogonal matrix"

Recall: A semisimple f.d. algebra R can be written

$$R = \bigoplus_{i=1}^{d} S_i^{n_i} \quad \text{as } R\text{-modules}$$

$\Rightarrow R^{op} \cong \bigoplus_{i=1}^{d} \mathrm{End}_R(S_i^{n_i}) = \bigoplus_{i=1}^{d} \mathrm{Mat}_{n_i}(\mathrm{End}_R(S_i))$

$\Rightarrow R \cong \bigoplus_{i=1}^{d} \mathrm{Mat}_{n_i}(K)$ as rings

CSOCPOI ← Complete System of Centrally primitive Orthogonal idempotents

where $n_i = \dim_K(S_i)$

Recall: Given $e \in R \setminus \{0,1\}$, $e^2 = e$, we get $R \cong Re \oplus R(1-e)$ as left R-modules

For any left R-module $R \curvearrowright M$, we get

$$M = eM \oplus (1-e)M$$

as K-vector spaces.

If $e \in Z(R)$, $R = eR \oplus (1-e)R$ as rings

$$M = eM \oplus (1-e)M \quad \text{as modules}$$

TY Lam "First Course on Noncommutative rings"

Decomposition as rings

$$R = R_1 \oplus \cdots \oplus R_d$$

is the same thing as a CSOCPOI ← "Centrally-primitive"

$1 = b_1 + \cdots + b_d$

130 Moreover: The elements b_i are unique.

We have $R_i = b_i R = \mathrm{End}_R(S_i^{n_i})$

Each b_i is centrally-primitive, not necessarily primitive.

Decomposing further:

$$b_i = e_{i_1} + \cdots e_{i_{n_i}}$$

where $b_i R e_{i_j} \cong S_i$ for all j.

Let G be a finite group, K alg. closed field of char. 0.

$kG \longleftrightarrow$ functions $G \to k$

$$\sum_{g \in G} \lambda_g g \qquad \lambda : G \to k, \quad g \mapsto \lambda_g$$

$Z(kG) \longleftrightarrow$ Class functions $G \to k$

$$\sum \lambda_g g \in Z(kG) \iff \lambda_g = \lambda_{h^{-1}gh} \quad \forall g, h \in G$$

Goal of representation theory: Find a CSOCPOI for kG.

A certain nice basis for the center $Z(kG)$.

$Z(kG) \longleftrightarrow$ Class functions $G \to k$

kG has a basis of "delta functions"

$$\delta_g : G \to k, \quad h \mapsto \delta_{g,h}$$

For any Conjugacy Class $C \subseteq G$, define

$$\delta_C = \sum_{g \in C} \delta_g = \sum_{g \in C} g$$

$\dim_k Z(kG) = \#$ Conj. Classes $= d$

δ_C are a basis for $Z(kG)$, but this basis does not interact well with the structure of kG.

The correct basis to choose is the unique {SOCPOI}

$$b_1, \dots, b_d \in Z(kG).$$

How to compute the central idempotents?

Theorem: $$b_i = \sum_g b_i(g) g = \frac{(\dim_k S_i)^2}{|G|} \sum \chi_{S_i}(g^{-1}) \cdot g$$

$$\Rightarrow b_i(g) = \frac{(\dim_k(S_i))}{|G|} \underbrace{\chi_{S_i}(g^{-1})}_{tr_{S_i}(g^{-1})}$$

ex: $\mathbb{C}S_3 = \mathbb{C}\langle 1, (12), (13), (23), (123), (132) \rangle$

$Z(\mathbb{C}S_3) = \mathbb{C}\langle 1, (12)+(13)+(23), (123)+(132) \rangle$

$= \mathbb{C}\langle \frac{1}{6}(1+(12)+(13)+(23)+(123)+(132)), \frac{1}{6}(1-(12)-(13)-(23)+(123)+(132)),$

$\frac{2}{6}(2 - 2(123) - 2(132)) \rangle$

Char table of S_3:

trivial	1	1	1
sign	1	-1	1
refl.	2	0	-2

One vector space

$$\begin{array}{ccc} & Z(G) & \\ \swarrow & & \searrow \\ \text{basis of } \delta\text{-functions} & & \text{basis of characters (traces)} \\ \langle\ ,\ \rangle_* & & \langle\ ,\ \rangle_{tr} \end{array}$$

How to multiply functions $G \longrightarrow k$:

$$b_i = \sum_g b_i(g)\cdot g$$

$$b_i b_j = \Big(\sum_g b_i(g) g\Big)\Big(\sum_h b_j(h) h\Big) = \sum_k \Big(\sum_{k=gh} b_i(g) b_j(h)\Big) k$$

Convolution: $(b_i * b_j)(k) = \sum_{gh=k} b_i(g) b_j(h)$

equivalently

$$\boxed{(b_i * b_j)(h) = \sum_g b_i(g)\, b_j(hg^{-1})}$$

$$(b_i * b_i)(h) = \boxed{\sum_g b_i(g)\, b_i(hg^{-1}) = b_i(h)} \quad \forall h \quad \text{(idempotence)}$$

$$(b_i * b_j)(h) = \sum_g b_i(g)\, b_j(hg^{-1}) = 0 \quad \forall h \quad \text{(orthogonality)}$$

Put $h=1$, get a bilinear form:

$$\begin{aligned} kG \times kG &\longrightarrow k \\ \lambda, \mu &\longmapsto (\lambda * \mu)(1) = \sum_g \lambda(g)\mu(g^{-1}) \end{aligned}$$

Restricting to $Z(kG) \times Z(kG)$, this gives an inner product $\langle\, , \rangle_*$ on central idempotents, and

The central idempotents are an orthogonal basis

Given $z \in Z(kG)$, for any simple module S and any multiplicity n,

$$z \circlearrowright kG = \bigoplus S_i^{n_i}$$

$$\downarrow$$

$z \circlearrowright S_i^{n_i}$ acts as a scalar matrix

$$z \in Z(\mathrm{End}(S_i^{n_i})) = Z\left(\mathrm{Mat}_{n_i}(\mathrm{End}_k(S_i))\right) = Z(\mathrm{Mat}_n(k)) = k$$

Thm: $z|_{S^n} = \frac{tr_{S^n}(z)}{n} \cdot I$

Pf: $\mathrm{Tr}(kI_n) = nk \Rightarrow k = \frac{\text{Trace}}{n}$ $\square$

In particular, for our b_i: Say b_S Projects onto S^n

$$tr_{S^n}(b_T) = \begin{cases} \dim_k S, & T \cong S \\ 0, & \text{else} \end{cases}$$

$$\downarrow$$

$$Z(kG) \times Z(kG) \longrightarrow k$$
$$b_S, \; b_T \longmapsto tr_{S^n}(b_T)$$

Extend tr to

$$\langle \, , \rangle_{tr} : Z(kG) \times Z(kG) \longrightarrow k$$
$$b_S, \; z \longmapsto tr_{S^n}(z)$$

Last thing to do:

For each simple S, define a character

$$\chi_S(g) = tr_S(g)$$

Prop: $\forall$ simples S, T,

$$\langle b_S, \chi_{T^*} \rangle = \langle b_S, b_T \rangle_{tr}$$

By non-degeneracy:

$$\text{Scalar}\, \chi_{S^*} = b_S$$

12/18/17 Today, we'll work over $\mathbb{C}$.

Recall: $\lambda : G \to \mathbb{C}$, $\lambda = \sum_{g \in G} \lambda(g) \cdot g$

$\lambda \in Z(\mathbb{C}G)$ iff $\lambda(g) = \lambda(hgh^{-1})$ $\forall g, h \in G$.

$Z(\mathbb{C}G) = \mathbb{C}$-algebra of class functions under convolution

$\lambda, \mu : G \to \mathbb{C}$

$$(\lambda * \mu)(h) = \sum_{g \in G} \lambda(g)\, \mu(hg^{-1})$$

The center has a distinguished basis of primitive idempotents

$e_1, e_2, \ldots, e_c$,

where $c = \#$ conjugacy classes in G.

Problem: Compute the e_i.

Key: For each f.d. $\mathbb{C}G$-module V ($\mathbb{C}G \curvearrowright V$), we have a class function $\chi_V \in Z(\mathbb{C}G)$ defined by

$$\chi_V : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \mathrm{Tr}(g \curvearrowright V)$$

We call χ_V the "character" of the module V.

Q: How do the idempotents e_i and the characters χ_V relate?

Punchline: If $S_1, \ldots, S_c$ are the simple $\mathbb{C}G$-modules, then we have

$$\boxed{e_i = \frac{\dim S_i}{|G|} \sum_g \chi_{S_i}(g^{-1})\, g}$$

Recall: e_i is the projection of $\mathbb{C}G$ is the projection of $\mathbb{C}G$ onto its summand $S_i^{\dim(S_i)}$

Hence, $Tr(e_i) = \dim(S^{\dim(S_i)}) = (\dim S_i)^2$

On the other hand, $Tr(e_i) = Tr\left(\sum e_i(g)\cdot g\right) = \sum e_i(g)\, Tr(g \curvearrowright \mathbb{C}G)$

Note: $Tr(g \curvearrowright \mathbb{C}G) = \begin{cases} |G|, & g=1 \\ 0, & g\neq 1 \end{cases}$

$\Rightarrow Tr(e_i) = |G|\, e_i(1)$, So $e_i(1) = \dfrac{(\dim S_i)^2}{|G|}$

This checks out: $\chi_{S_i}(1) = Tr(1 \curvearrowright S_i) = \dim(S_i)$

(indeed, $\chi_V(1) = Tr(1 \curvearrowright V) = \dim(V)$ $\forall$ $\mathbb{C}G$-modules V).

Recall: $\mathbb{C}G = \bigoplus S_i^{\dim(S_i)}$

One component is easy:

$\mathbb{C}G \curvearrowright \mathbb{C}$ as the identity.

Thus, $\mathbb{C}G = \underbrace{\boxed{\mathbb{C}}}_{\text{"trivial representation"}} \oplus \bigoplus_{i \geq 2} S_i^{\dim S_i}$

The idempotent of the trivial rep is $\boxed{e_1 = \frac{1}{|G|} \sum_{g\in G} g}$

Thus, for any $\mathbb{C}G \curvearrowright V$, we have $V = \mathbb{C}^d \oplus$ other stuff, and $e_1 \curvearrowright V$ is the projection $V \longrightarrow \mathbb{C}$

$e_1 \curvearrowright V$ is the projection onto the fixed subspace V^G

$\Rightarrow Tr(e_1 \curvearrowright V) = \dim V^G$.

$$Tr(e_1 \curvearrowright V) = \frac{1}{|G|} \sum Tr(g \curvearrowright V)$$

$$\Rightarrow \boxed{\dim V^G = \frac{1}{|G|} \sum_g \chi_V(g)}$$

More generally, we will see that

$$\boxed{\dim(\mathrm{Hom}_G(W,V)) = \frac{1}{|G|} \sum \chi_V(g) \overline{\chi_W(g)}}$$

Recall: If S, S' are simple, $\dim \mathrm{Hom}_G(S,S') = \begin{cases} 1, & S \cong S' \\ 0, & S \not\cong S' \end{cases}$

Corollary: Let $\mathbb{C}G \curvearrowright S$ be simple. Then for any $\mathbb{C}G \curvearrowright V$, we have

$V = S^{d?} \oplus$ other stuff

$$\dim(\mathrm{Hom}_G(S,V)) = \dim(\mathrm{Hom}_G(S, \oplus T_i)) = d.$$

$$= \frac{1}{|G|} \sum_g \chi_V(g) \overline{\chi_S(g)}$$

Let $\mathbb{C}G \curvearrowright V$. Then, in particular, we have $\mathbb{C} \curvearrowright V$, and we can consider the dual space $V^* = Hom_{\mathbb{C}}(V, \mathbb{C})$.

Q: $\mathbb{C}G \curvearrowright V^*$?

A: Let R be an arbitrary f.d. $\mathbb{C}$-algebra

$R \curvearrowright V$
$R^{op} \curvearrowright V^*$

We define this by $f: V \longrightarrow \mathbb{C}$
$\forall r \in R, v \in R, \ (rf)(v) = f(rv)$

Sadly, $(rr'f)(v) = ((r'f)(rv)) = f(r'rv)$ $\frown$

$\mathbb{C}G \neq \mathbb{C}G^{op}$

However, we do have a specific isomorphism

$$\mathbb{C}G \xrightarrow{\cong} \mathbb{C}G^{op}$$
$$g \longmapsto g^{-1}$$

$$\sum_g \lambda(g)\, g \longmapsto \sum_g \lambda(g)\, g^{-1} = \sum_g \lambda(g^{-1})\, g$$

We define $\mathbb{C}G \curvearrowright V^*$ by $(gf)(v) = f(g^{-1}v)$ "Contragredient action"

$$\Rightarrow \left(\sum \lambda(g)\, gf\right)(v) = f\left(\sum \lambda(g)\, g^{-1}v\right)$$

Corollary: $\chi_V(g) = \chi_{V^*}(g^{-1})$ (equivalently, $\chi_{V^*}(g) = \chi_V(g^{-1})$)

Remark specific to $\mathbb{C}$: Let $\mathbb{C}G \curvearrowright V$. Then $\forall g \in G$, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

WHY?

Proof: Trace is invariant under conjugation, so we are free to choose a nice basis for V.

First, choose an arbitrary basis and let $\langle x, y \rangle = \sum_i x_i \bar{y}_i$

Sadly, $\langle gx, gy \rangle \underset{\text{WANT}}{\neq} \langle x, y \rangle$.

Define a new inner product $\langle x, y \rangle_G = \frac{1}{|G|} \sum_g \langle gx, gy \rangle$

Then for all $g \in G$, $x, y \in V$, we have $\langle gx, gy \rangle_G = \langle x, y \rangle_G$

$$\langle g^* g x, y \rangle_G = \langle x, y \rangle_G \quad \forall y$$

Choose an orthonormal basis for $\langle \, , \, \rangle_G$.
WRT this basis,

$$g^* g \, x = x \qquad \forall x$$

$$\Rightarrow g^* g = \mathrm{id}$$

(where g^* is the conjugate transpose of g).

In particular, we have, in this basis, $Tr(g^{-1} \curvearrowright V) = Tr(\bar{g}^T \curvearrowright V)$

$$= Tr(\bar{g} \curvearrowright V)$$
$$= \overline{Tr(g \curvearrowright V)}.$$

$\blacksquare$

In particular, $g^* g = \mathrm{id} \Rightarrow g$ is diagonalizable (basis independent).

So far, we have shown $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$.

$$\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$$

$$\boxed{\chi_{V \otimes W}(g) = \chi_V(g) \chi_W(g)} \quad ?$$

$$\mathrm{Hom}(\mathbb{C}G, \mathrm{Hom}(V\otimes_{\mathbb{C}} W, V\otimes_{\mathbb{C}} W)) \cong \mathrm{Hom}(\mathbb{C}G \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} W, V\otimes_{\mathbb{C}} W)$$

Let $\mathbb{C}G \curvearrowright V, W$.
How does $\mathbb{C}G \curvearrowright V\otimes_{\mathbb{C}} W$?

Define: $G \times V\otimes_{\mathbb{C}} W \longrightarrow V \otimes_{\mathbb{C}} W$
$(g, v\otimes w) \longmapsto gv \otimes gw$, and extend $\mathbb{C}$-linearly

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$\mathrm{Hom}_G(W, V) \cong V \otimes W^*$

"There are subtleties here"

(Could also note that $\mathbb{C}G \curvearrowright V^* \otimes_{\mathbb{C}} W$)

Proof of dim $\mathrm{Hom}_G(W,V) = \frac{1}{|G|} \sum \chi_V(g)\overline{\chi_W(g)}$

$$\mathrm{Hom}_G(W,V) \cong \mathrm{Hom}_{\mathbb{C}}(W,V)^G = (V\otimes W^*)^G$$
$$= \text{image of } e_1 \curvearrowright V\otimes W^*$$

$$\Rightarrow \dim(\mathrm{Hom}_G(W,V)) = \frac{1}{|G|}\sum \chi_{V\otimes W^*}(g)$$

$$= \frac{1}{|G|}\sum \chi_V(g)\chi_{W^*}(g)$$
$$= \frac{1}{|G|}\sum \chi_V(g)\overline{\chi_W(g)}.$$
$\square$

In general, for class functions $\lambda, \mu \in Z(\mathbb{C}G)$, we define

$$\langle \lambda, \mu \rangle = \frac{1}{|G|}\sum_g \lambda(g)\overline{\mu(g)}$$

We have shown that χ_S for S simple satisfy
$$\langle \chi_S, \chi_{S'} \rangle = \begin{cases} 1, & S \cong S' \\ 0, & \text{else} \end{cases}$$

They span because we know that # Conj. Classes = dim $Z(\mathbb{C}G)$ = # non-iso simples

Thus, the χ_S form a basis for $Z(\mathbb{C}G)$ (indep b/c they're orthogonal).

$$e_i \overset{?}{\sim} \chi_{S_i}$$

Show that χ_{S_i} are idempotents:

$$(\chi_{S_i} * \chi_{S_i})(g) = ?$$

12/20/17 If R is a k-algebra, then R-Mod is a "k-linear category"

If $R = \mathbb{C}G$, then the category $\mathbb{C}G$-Mod has a lot of extra structure.

"Tensor categories"

"Hopf algebras"

If H is a Hopf k-algebra

$(H, \mu, u, \Delta, \epsilon, S)$

$$\begin{tikzcd} H\otimes H\otimes H \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] & H\otimes H \arrow[d, "\mu"] \\ H\otimes H \arrow[r, "\mu"'] & H \end{tikzcd}$$ (Associativity of μ)

$u: k \longrightarrow H$

$\varepsilon: H \longrightarrow k$

$\Delta: \mathbb{C}G \longrightarrow \mathbb{C}G \otimes \mathbb{C}G$
$g \longmapsto g \otimes g$

$\varepsilon: \mathbb{C}G \longrightarrow \mathbb{C}$
$\sum \lambda(g) g \longmapsto \sum \lambda(g)$

$S: \mathbb{C}G \xrightarrow{\cong} \mathbb{C}G^{op}$
$g \longmapsto g^{-1}$

If H is a Hopf algebra, H-mod is a "tensor category"

H-Mod has

- tensor product $\otimes$

  $G \curvearrowright V, W$
  $G \curvearrowright V \otimes_k W$, $g(v \otimes w) = (gv) \otimes (gw)$

Say that $x \in H$ is "grouplike" if $\Delta(x) = x \otimes x$

- dual representations:

  $G \curvearrowright V$
  $G \curvearrowright V^*_{\mathbb{C}}$, $(g\alpha)(v) = \alpha(g^{-1}v)$ (Antipode!)

- H-Mod is enriched over itself:

  $G \curvearrowright V, W$
  $\Rightarrow G \curvearrowright \mathrm{Hom}_{\mathbb{C}}(V, W)$

  $(gf)(v) := g(f(g^{-1}(v)))$ "conjugation"

<u>Primitive elements</u>: $x \in H$ is <u>Primitive</u> if $x \otimes 1 + 1 \otimes x$

$\mathrm{Prim}(H) \subseteq H$ is a Lie algebra

$H = \mathcal{U}_q(\mathfrak{g})$ ← "where this is"

<u>Invariants</u>: $G \curvearrowright V$

$$V^G = \{x \in V,\ gx = x \quad \forall g \in G\}$$

Many natural isomorphisms

$$\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)^G$$
$$\varphi g = g\varphi \qquad\qquad g\varphi g^{-1} = \varphi$$

$$\mathrm{Hom}_{\mathbb{C}}(V, W) \cong_G V^* \otimes_{\mathbb{C}} W \quad !$$

<u>Pf</u>:
$$\begin{tikzcd}
V^* \times W \arrow[r, "\text{bilinear}"'] \arrow[d] & \mathrm{Hom}_{\mathbb{C}}(V, W) \\
V^* \otimes W \arrow[ur, "\text{linear}"', "\exists! ?"] &
\end{tikzcd}$$
$$(\alpha, w) \longmapsto (v \longmapsto \alpha(v)w)$$
$$\alpha \otimes w$$

<u>Bases</u>: $\alpha_i(v_j) = \delta_{ij}$

w_j

$$\alpha_i \otimes w_j$$
$$(v \longmapsto \alpha_i(v) w_j)$$

$$v_k \longmapsto \alpha_i(v_k) w_j = \delta_{ik} w_j$$

This is exactly the matrix with a 1 in the $(i,j)^{th}$ position & 0's elsewhere which is a basis for $Hom_{\mathbb{C}}(V,V)$.

$\square$

Trace comes into the picture.

Lemma: $\dim V^G = \frac{1}{|G|} \sum_g \underbrace{\boxed{Tr(g \circlearrowright V)}}_{\chi_V(g)}$

Theorem: $\underbrace{\frac{1}{|G|} \sum_g \chi_V(g) \overline{\chi_W(g)}}_{\langle \chi_V, \chi_W \rangle \text{ on } \mathbb{C}G} = \dim_{\mathbb{C}} Hom_{\mathbb{C}G}(V,W)$

Pf: $\dim_{\mathbb{C}} Hom_{\mathbb{C}G}(V,W) = \dim Hom_{\mathbb{C}}(V,W)^G = \dim(V^* \otimes W)^G$

lemma

$= \frac{1}{|G|} \sum_g \chi_{V^* \otimes W}(g)$

$= \frac{1}{|G|} \sum_g \chi_{V^*}(g) \chi_W(g)$

$= \frac{1}{|G|} \sum_g \overline{\chi_V(g)} \chi_W(g)$

$= \frac{1}{|G|} \sum_{g^{-1}} \overline{\chi_V(g^{-1})} \chi_W(g^{-1})$

$= \frac{1}{|G|} \sum_{g^{-1}} \chi_V(g^{-1}) \overline{\chi_W(g^{-1})}$

$= \frac{1}{|G|} \sum_g \chi_V(g) \overline{\chi_W(g)}$ $\square$

Note, this implies that

$\langle \chi_V, \chi_W \rangle = \langle \chi_W, \chi_V \rangle$

Corollary: For simple $\mathbb{C}G \curvearrowright S_i, S_j$

$$\langle \chi_{S_i}, \chi_{S_j} \rangle = \dim \mathrm{Hom}_{\mathbb{C}G}(S_i, S_j) = \delta_{ij}.$$

$\square$

The simple characters are an orthonormal subset of $Z(\mathbb{C}G)$.

Since $\dim(Z(\mathbb{C}G)) = \#$ conj. classes of G
$= \#$ non-isom. simples.

$\Rightarrow$ Simple characters are an orthonormal basis for $Z(\mathbb{C}G)$.

Question: Are simple characters idempotent?

$$(\chi_{S_i} * \chi_{S_j})(h) = \sum_g \chi_{S_i}(g)\, \chi_{S_j}(hg^{-1}) = ?$$

Let $e_i = \sum e_i(g) g$ be the idempotent that projects onto the $S_i^{\dim(S_i)}$-component of $\mathbb{C}G$.

Let $\rho = \sum_i \chi_i(1) \chi_i$, so $\rho(g) = \mathrm{Tr}(g \curvearrowright \mathbb{C}G) = \begin{cases} |G|, & g = \mathrm{id} \\ 0, & g \neq \mathrm{id} \end{cases}$

$$\chi_j(e_i g) = \delta_{ij}\, \chi_i(g)$$

$$|G| \cdot e_i(g) = \rho(e_i g^{-1}) = \sum_j \chi_j(1)\, \chi_j(e_i g^{-1}) = \chi_i(1)\, \chi_i(g^{-1})$$

$$\Rightarrow \boxed{e_i(g) = \frac{\chi_i(1)}{|G|} \chi_i(g^{-1})}$$

Corollary 1: $\frac{\chi_i(1)}{|G|}\chi_i(h) = e_i(h) = (e_i * e_i)(h)$

$$= \sum_g e_i(h) e_i(hg^{-1})$$

$$= \left(\frac{\chi_i(1)}{|G|}\right)^2 \sum_g \chi_i(g^{-1})\chi_i(gh^{-1})$$

$$\Rightarrow \boxed{\chi_i(h) = \frac{\chi_i(1)}{|G|}\sum_g \chi_i(g^{-1})\chi_i(gh^{-1})}$$

$$\Rightarrow 1 = \frac{1}{|G|}\sum \chi_i(g^{-1})\chi_i(g) = \langle \chi_i, \chi_i \rangle.$$

Corollary 2: $i \neq j$

$$0 = (e_i * e_j)(h) = \left(\frac{\chi_i(1)\chi_j(1)}{|G|^2}\right) \sum_g \chi_i(g^{-1})\chi_j(gh^{-1})$$

$$\Rightarrow \boxed{0 = \sum_g \chi_i(g^{-1})\chi_j(gh^{-1})}$$

$$\Rightarrow 0 = \langle \chi_i, \chi_j \rangle$$

The Character table:

Let $g_j \in C_j$ be conj. class representatives

$$\begin{array}{cc} & g_j \\ \chi_i & \boxed{\quad \chi_i(g_j) \quad} \end{array}$$

Same matrix

Orthogonality?

Modify Char table

$$T = \left(\frac{\chi_i(g_j)}{\sqrt{|G|/|C_j|}} \right)$$

The rows of T are orthonormal!

$$\sum_k \frac{\chi_i(g_k)\overline{\chi_j(g_k)}}{\left(\sqrt{|G|/|C_k|}\right)^2} = \frac{1}{|G|}\sum_k |C_k|\, \chi_i(g_k)\overline{\chi_j(g_k)}$$

$$= \frac{1}{|G|}\sum_g \chi_i(g)\overline{\chi_j(g)} = \langle \chi_i, \chi_j \rangle = \delta_{ij} \quad \square$$

<u>Corollary 3</u>: $TT^* = I \Rightarrow T^*T = I$ (by "a miracle of basic linear algebra")

$$\Rightarrow \forall i,j, \quad \delta_{ij} = \sum_k \frac{\chi_k(g_i)\overline{\chi_k(g_j)}}{\sqrt{|G|/|C_i|}\cdot\sqrt{|G|/|C_j|}}$$

$$\Rightarrow \sum_k \chi_k(g_i)\overline{\chi_k(g_i)} = \frac{|G|}{|C(g)|}$$

$$\Rightarrow \sum_{\text{Simple } S} |\chi_S(g)|^2 = \frac{|G|}{|C(g)|} = |Z(g)| \quad \text{(Orbit-Stabilizer)}$$

ex:

	1	3	2
	1	(12)	(123)
triv	1	1	1
sgn	1	-1	1
refl	2	0	-1

$1^2+1^2+2^2=6$ ✓

$1^2+(-1)^2+0=6/3$ ✓

$1^2+1^2+(-1)^2=6/2$ ✓