MTH 785: Fancy Algebra **Fall 2016**
Part I: Adjoint Functors **Drew Armstrong**

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Preface

Once upon a time, my opinion of category theory was the same as my opinion of Facebook: if I ignore it for long enough, hopefully it will go away. It is now my educated opinion that category theory will **not** go away, and in fact the language of category theory will continue to spread until it becomes the default foundation of mathematics. During this transition period there will be three kinds of mathematicians:

1. those young enough to be raised with the new language,
2. those willing to invest enough time and energy to learn the new language,
3. everyone else.

I see this transition as roughly analogous to the process that happened between 1830 and 1930, as Galois' ideas were slowly absorbed into the foundations of mathematics. These notes are written for people like myself who find category theory challenging, but who don't want to get left behind.

My ulterior motive is to gain a better understanding of representation theory by developing its categorical foundations. To be specific, I am interested in

representations of associative algebras.

This is a beautiful subject but it contains many layers of algebraic structure that are often difficult to keep track of. For example, the typical "modern algebraic" (circa 1930) definition of a "representation of an associative algebra" involves **at least three sets** (R, A, M) carrying **at least five binary operations**:

$(R, +_R, \cdot_R)$ is a commutative ring,

$(A, +_A, \cdot_A)$ is a possibly noncommutative ring,

$(M, +_M)$ is an abelian group,

and **at least two homomorphisms** (λ, φ), where:

$\lambda : R \to Z(A)$ is a ring homomorphism sending R into the center of A,

$\varphi : A \to \mathsf{End}(M)$ is a ring homomorphism sending A into the endomorphisms of M.

To make sense out of this subject we need a more sophisticated language than "modern algebra" that will hide the right details at the right times. For this reason the language of category theory has become an indispensible tool in representation theory. Unfortunately, this language has not yet trickled down into introductory textbooks.

There is no best way to introduce category theory. However, any effective introduction will find a balance between the following two contradictory goals:

- go slow enough to build intuition,
- get to nontrivial applications as soon as possible.

My personal preference is to begin by looking at posets and Galois connections between them. Posets provide excellent motivation for the basic definitions of category theory and Galois connections provide excellent intuition for the fundamental concept of

adjoint functors.

The basic theorems on adjoint functors (uniqueness of adjoints and the RAPL theorem) are a bit tricky to prove, but the effort it worth it because they illustrate the main techniques of the subject and they have many interesting applications. We will go as slow as necessary to make all of this clear.

Chapter 1

Adjoint Functors

1.1 From Posets to Categories

Definition of Poset. A *poset* (partially ordered set) is a pair $(P, \leq)$, where:

- P is a set,
- $\leq$ is a binary relation on P satisfying the three axioms of *partial order*:
  - (i) Reflexive: $\forall x \in P,\ x \leq x$
  - (ii) Antisymmetric: $\forall x, y \in P,\ x \leq y\ \&\ y \leq x \implies x = y$
  - (iii) Transitive: $\forall x, y, z \in P,\ x \leq y\ \&\ y \leq z \implies x \leq z$.

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[Remark: What does the symbol "$x = y$" in axiom (ii) mean? Well, we could let "=" be our favorite equivalence relation on the set P (if we have one), or we could just **define** "=" by means of axiom (ii). In practice it doesn't really matter, so I will assume that the symbol "=" is **defined** by (ii). One can then easily check that this "=" satisfies the three properties of an equivalence relation.]

And what does this have to do with category theory? To make the transition I will switch to a nonstandard notation:

$$\begin{aligned} \text{“}x \leq y\text{”} &\iff \text{“}x \to y\text{”} \\ \text{“}x = y\text{”} &\iff \text{“}x \leftrightarrow y\text{”} \end{aligned}$$

The subject of general posets is way too broad, so let's discuss some of the special properties a poset might have.

Definition of Binary Meet/Join. Given $x, y \in P$,

- we say that $u \in P$ is a *least upper bound* of $x, y \in P$ if we have $x \to u$ & $y \to u$, and for all $z \in P$ satisfying $x \to z$ & $y \to z$ we must have $u \to z$.

  It is more convenient to express this definition with a picture. We say that $u \in P$ is a *least upper bound* of x, y if for all $z \in P$ the following picture holds:

$$\begin{tikzcd} u \arrow[r, dashed] & z \\ x \arrow[u] \arrow[ur] & y \arrow[u] \arrow[ul] \end{tikzcd}$$

  [How to read the picture: When the four solid arrows exist, then the dotted arrow necessarily exists. Later on we will want to say that there exists a **unique** such dotted arrow, but right now there is no need to say this.]

- Dually, we say that $\ell \in P$ is a *greatest lower bound* of x, y if for all $z \in P$ the following picture holds:

$$\begin{tikzcd} x & y \\ z \arrow[u] \arrow[ur] \arrow[r, dashed] & \ell \arrow[u] \arrow[ul] \end{tikzcd}$$

Now suppose that $u_1, u_2 \in P$ are two least upper bounds for x, y. Applying the defininition in both directions gives

$$u_1 \to u_2 \quad \text{and} \quad u_2 \to u_1,$$

and then from antisymmetry it follows that "$u_1 = u_2$", which just means that u_1 and u_2 are indistinguishable within the structure of P. For this reason we can speak of **the** least upper bound (or "*join*") of x, y. If it exists, we denote it by

$$x \vee y.$$

Dually, if it exists, we denote **the** greatest lower bound (or "*meet*") by

$$x \wedge y.$$

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Jargon: The definitions of meet and join are called "universal properties". Whenever an object defined by a universal property exists, it is automatically **unique** in a certain canonical sense. However, since the object might **not** exist, maybe it is better to refer to a universal property as a "characterization," or a "prescription," rather than a "definition."

We will see many universal properties in this class. Here's another one right now.

Definition of Top/Bottom Elements. Let P be a poset. We say that $t \in P$ is a *top element* if for all $z \in P$ the following picture holds:

$$z \dashrightarrow t$$

Dually, we say that $b \in P$ is a *bottom element* if for all $z \in P$ the following picture holds:

$$b \dashrightarrow z$$

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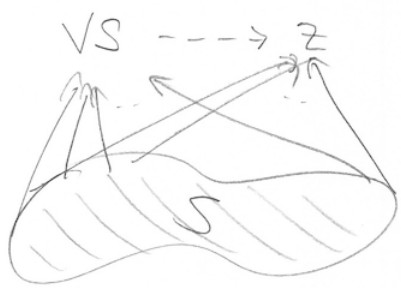
Exercise: Show that top and bottom elements (if they exist) are unique.

We denote **the** top element (if it exists) by 1 and **the** bottom element (if it exists) by 0.

I was careful to phrase the definitions of 0 and 1 so they look like the definitions of $\vee$ and $\wedge$. Here is a more general construction encompassing them both.

Definition of Arbitrary Meet/Join. For any subset of elements of a poset $S \subseteq P$ we say that the element $\bigvee S \in P$ is its *join* if for all $z \in P$ the following diagram is satisfied:



Dually, we say that $\bigwedge S \in P$ is the *meet* of S if for all $z \in P$ the following diagram is satisfied:

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These diagrams are a bit more impressionistic but I suppose you can understand them now. (If all solid arrows exist, then the dotted arrow exists.) If the objects $\bigvee S$ and $\bigwedge S$ exist then they are uniquely characterized by their universal properties, hence I didn't cheat when I gave them special names.

[Remark: The universal properties in these diagrams will be called the "limit" and "colimit" properties when we move from posets to categories. Note that a limit/colimit diagram looks like a "cone over S". This is one example of the link between category theory and topology.]

Note that all of our definitions so far are included in this single (pair of) definition(s):

$$\bigvee\{x,y\} = x \vee y \quad \& \quad \bigwedge\{x,y\} = x \wedge y$$
$$\bigvee \varnothing = 0 \quad \& \quad \bigwedge \varnothing = 1.$$

Exercise: Show that $\bigvee\{x\} = x$ and $\bigwedge\{x\} = x$.

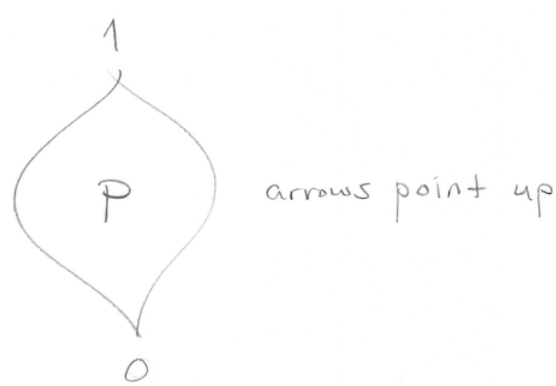
Thus, 0 is an example of a join and 1 is an example of a meet. You have probably seen this idea in the form of the following convention:

The sum of no numbers is 0*; the product of no numbers is* 1.

But maybe you didn't realize that this convention can be formalized with the langauge of universal properties.

Special Kinds of Posets:

Bounded. Let $\mathcal{P}$ be a poset. We say that P is *bounded* if it has a top and a bottom element. I will draw bounded posets like this:



[Unbounded posets are much harder to draw.]

Lattice. If $\mathcal{P}$ is a poset in which the elements $\bigvee S$ and $\bigwedge S$ exist for all **finite** subsets $S \subseteq P$ then we call P a *lattice*. Since the empty set is finite, every lattice is bounded.

Complete Lattice. If in addition the elements $\bigwedge S$ and $\bigvee S$ exist for all **infinite** subsets $S \subseteq P$ then we call P a *complete lattice*.

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You might think that the existence of meets and joins are independent, but this is not so.

Exercise: Show that if P contains all meets/joins, then it also contains all joins/meets, respectively. Thus there is no reason to distinguish between "complete" and "cocomplete" lattices. For general categories the two notions will be independent.

Examples of Lattices:

Boolean Lattice. Let U be any set and let 2^U be the set of all subsets of U. This is a (complete) lattice with the following structure:

$$\begin{array}{rcl} \text{“}\rightarrow\text{”} & = & \text{“}\subseteq\text{”} \\ 0 & = & \varnothing \\ 1 & = & U \\ \vee & = & \cup \\ \wedge & = & \cap. \end{array}$$

Any lattice of this form is called a *Boolean lattice*.

If $U = \{1,2,3\}$ then we draw the "Hasse diagram" of the Boolean lattice 2^U as follows:

$$\begin{tikzcd}
 & & \{1,2,3\} & & \\
\{1,2\} \arrow[urr] & & \{1,3\} \arrow[u] & & \{2,3\} \arrow[ull] \\
\{1\} \arrow[u] \arrow[urr] & & \{2\} \arrow[ull] \arrow[urr] & & \{3\} \arrow[ull] \arrow[u] \\
 & & \varnothing \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

If the set U carries some algebraic structure then we may be interested in the subposet of 2^U consisting of subsets with special structure, e.g.,

- subgroups / normal subgroups

- subrings / ideals
- submodules

Each such subposet has a meet operation given by intersection. Each such subposet also has a join operation, **however** this join is not just the union of subsets. (For example, the union of subgroups is not a group; we must take the "group closure" of the union.) So, while each such subposet is itself a lattice, we will not call it a **sublattice** of 2^U.

From Analysis. The real interval $[0,1] \subseteq \mathbb{R}$ with the usual partial order "$\leq$" is a complete lattice with structure:

$$\begin{aligned} 0 &= 0 \\ 1 &= 1 \\ \vee &= \text{supremum} \\ \wedge &= \text{infimum.} \end{aligned}$$

The sublattice $[0,1] \cap \mathbb{Q}$ is not complete.

From Number Theory. The set $\mathbb{N} = \{0, 1, 2, \ldots, \}$ is a lattice under the divisibility relation:

$$\text{“}a \to b\text{”} \quad \Longleftrightarrow \quad \text{“}b|a\text{”}$$

It has structure given by:

$$\begin{aligned} 0 &= 0 \\ 1 &= 1 \\ \vee &= \text{greatest common divisor} \\ \wedge &= \text{least common multiple.} \end{aligned}$$

This lattice is isomorphic to the lattice of subgroups of the abelian group $(\mathbb{Z}, +, 0)$. To see this, recall that every subgroup of $\mathbb{Z}$ has the form $(n) := n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$ for some $n \in \mathbb{N}$ and that

$$(a) \subseteq (b) \quad \Longleftrightarrow \quad b|a.$$

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Now I will let you in on the secret of today's lecture. Everything I have told you so far is a special case of category theory. Here is a dictionary between the poset-theoretic and the category-theoretic terminology:

$$
\begin{aligned}
\text{poset } P &= \text{category } P \\
\text{elements of } P &= \text{objects of } P \\
1 &= \text{final object} \\
0 &= \text{initial object} \\
x \wedge y &= \text{categorical product} \\
x \vee y &= \text{categorical coproduct (or sum)} \\
\bigwedge S &= \text{categorical limit of the ``diagram'' } S \subseteq P \\
\bigvee S &= \text{categorical colimit of the ``diagram'' } S \subseteq P.
\end{aligned}
$$

Thus, products and final objects are examples of "limits," while coproducts and initial objects are examples of "colimits." This explains our previous use of the word "complete."

Now it's time to define the word "category." Posets provide excellent intuition for categories, but a general category differs from a poset in two important ways:

(1) **Posets Are Small.** A poset has a **set** of objects/elements, but a general category can have "more" objects. For example, the category of all sets is very important, but you probably know that there is no such thing as the "set of all sets."

A category with a set of arrows (and hence also a set of objects) is called a "small category." Thus a poset is an example of a small category.

(2) **Posets Are Thin.** For elements x, y in a poset we used the nonstandard notation:

$$\text{``}x \leq y\text{''} \iff \text{``}x \to y\text{''}$$

Thus, an ordered pair of objects in a poset can have either **zero** or **one** arrow between them.

If $\mathcal{C}$ is a general category and $x, y \in \mathcal{C}$ are two objects, then there can exist a whole **set** of arrows from x to y (and maybe even more). We let

$$\mathsf{Hom}_{\mathcal{C}}(x, y)$$

denote the collection of arrows from x to y. If there are many arrows then we will give them Greek names, like $\alpha, \beta, \gamma, \ldots$, and to indicate these names we will use the notation

$$x \xrightarrow{\alpha} y \quad \text{or} \quad \alpha : x \to y.$$

The reflexive property of posets becomes the following rule:

- Every object $x \in \mathcal{C}$ has a special "identity arrow" $\mathsf{id}_x \in \mathsf{Hom}_{\mathcal{C}}(x, x)$.

And the transitive property gets modified as follows:

- Given objects $x, y, z \in \mathcal{C}$ and arrows $\alpha \in \mathsf{Hom}_{\mathcal{C}}(x,y)$, $\beta \in \mathsf{Hom}_{\mathcal{C}}(y,z)$, there exists a "composite arrow" $\beta \circ \alpha \in \mathsf{Hom}_{\mathcal{C}}(x,z)$. Composition of arrows (when defined) is associative and behaves correctly with the identity arrows.

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Here is the official definition.

Definition of Category. A category $\mathcal{C}$ consists of the following data:

- A collection[1] $\mathsf{Obj}(\mathcal{C})$ of *objects*. We will write "$x \in \mathcal{C}$" to mean that "$x \in \mathsf{Obj}(\mathcal{C})$."
- For each ordered pair $x, y \in \mathcal{C}$ there is a collection $\mathsf{Hom}_{\mathcal{C}}(x,y)$ of *arrows*. We will write "$\alpha : x \to y$" to mean that "$\alpha \in \mathsf{Hom}_{\mathcal{C}}(x,y)$." Each collection $\mathsf{Hom}_{\mathcal{C}}(x,x)$ has a special element called the *identity arrow* $\mathsf{id}_x : x \to x$. We let $\mathsf{Arr}(\mathcal{C})$ denote the collection of all arrows in $\mathcal{C}$.
- For each ordered triple of objects $x, y, z \in \mathcal{C}$ there is a function

$$\circ : \mathsf{Hom}_{\mathcal{C}}(x,y) \times \mathsf{Hom}_{\mathcal{C}}(y,z) \to \mathsf{Hom}_{\mathcal{C}}(x,z),$$

  which is called *composition of arrows*. If $\alpha : x \to y$ and $\beta : y \to z$ then we denote the composite arrow by $\beta \circ \alpha : x \to z$.

If each collection of arrows $\mathsf{Hom}_{\mathcal{C}}(x,y)$ is a **set** then we say that the category $\mathcal{C}$ is *locally small*. If in addition the collection $\mathsf{Obj}(\mathcal{C})$ is a **set** then we say that $\mathcal{C}$ is *small*.[2]

This data is required to satisfy two axioms, which we express in graphical form.

(i) Identitiy: For each arrow $\alpha : x \to y$ the following diagram commutes:

$$\begin{tikzcd} x \arrow[loop left, "\mathsf{id}_x"] \arrow[r, "\alpha"] & y \arrow[loop right, "\mathsf{id}_y"] \end{tikzcd}$$

(ii) Associative: For all arrows $\alpha : x \to y$, $\beta : y \to z$, $\gamma : z \to w$, the following diagram commutes:

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta\circ\alpha"'] & y \arrow[r, "\beta"] \arrow[rr, bend left, "\gamma\circ\beta"] & z \arrow[r, "\gamma"] & w \end{tikzcd}$$

We say that $\mathcal{C}' \subseteq \mathcal{C}$ is a *subcategory* if $\mathsf{Obj}(\mathcal{C}') \subseteq \mathsf{Obj}(\mathcal{C})$ and if for all $x, y \in \mathsf{Obj}(\mathcal{C}')$ we have $\mathsf{Hom}_{\mathcal{C}'}(x,y) \subseteq \mathsf{Hom}_{\mathcal{C}}(x,y)$. We say that the subcategory is *full* if each inclusion of hom sets is an equality. ///

[1] This is a non-technical term. I pray that no one ever gives it a technical definition.

[2] Even if $\mathcal{C}$ is not locally small we will still refer to the collections $\mathsf{Hom}_{\mathcal{C}}(x,y)$ as "hom sets." Sorry.

To understand the pictures in the definition we need an auxiliary definition. I will give two versions of this definition; one to look at now and one to look at later.

Naive Definition of Diagram. Let $\mathcal{C}$ be a category. A *diagram* $D \subseteq \mathcal{C}$ is a collection of objects in $\mathcal{C}$ with some arrows between them. Repetition of objects and arrows is allowed. ///

Fancy Definition of Diagram. Let $\mathcal{I}$ be any small category, which we think of as an "index category." Then any functor $D : \mathcal{I} \to \mathcal{C}$ is called a *diagram of shape* $\mathcal{I}$ *in* $\mathcal{C}$. (The word "functor" will defined later.) ///

We say that the diagram D *commutes* if for all pairs of objects x, y in D, any two directed paths in D from x to y yield the same arrow under composition.[3]

Exercise: Prove that a poset is the same thing as a small category $\mathcal{P}$ in which, for all $x, y \in \mathcal{P}$, we have $|\mathsf{Hom}_{\mathcal{P}}(x,y)| \in \{0,1\}$.

Identity arrows generalize the reflexive property of posets, and composition of arrows generalizes the transitive property of posets. But what happened to the antisymmetric property? Well, it's the same issue we had before: we should really **define** equivalence of objects in terms of antisymmetry.

Definition of Isomorphism. Let $\mathcal{C}$ be a category. We say that two objects $x, y \in \mathcal{C}$ are *isomorphic in* $\mathcal{C}$ if there exist arrows $\alpha : x \to y$ and $\beta : y \to x$ such that the following diagram commutes:

$$\begin{tikzcd} x \arrow[loop left, "\mathsf{id}_x"] \arrow[r, bend left, "\alpha"] & y \arrow[l, bend left, "\beta"] \arrow[loop right, "\mathsf{id}_y"] \end{tikzcd}$$

In this case we write $x \cong_{\mathcal{C}} y$, or just $x \cong y$ if the category is understood. ///

If $\gamma : y \to x$ is any other arrow satisfying the same diagram as β, then by the axioms of identity and associativity we must have

$$\gamma = \gamma \circ \mathsf{id}_y = \gamma \circ (\alpha \circ \beta) = (\gamma \circ \alpha) \circ \beta = \mathsf{id}_x \circ \beta = \beta.$$

This allows us to refer to β as **the inverse** of the arrow α. We use the notations $\beta = \alpha^{-1}$ and $\beta^{-1} = \alpha$.

Exercise: Check that isomorphism is an equivalence relation on the collection of objects. This generalizes the equivalence relation induced on a poset $(P, \leq)$ by the antisymmetry property.

[3]This is the naive definition. I haven't seen a fancy definition of "commutative diagram" that completely satisfies me. The most general version I've seen is in Grothendieck's Tohoku paper, where he defines "diagram schemes" with "commutativity conditions."

[Remark: Two objects x, y may belong simultaneously to two different categories, say $\mathcal{C}$ and $\mathcal{D}$. In this case, the statement $x \cong_{\mathcal{C}} y$ means that "x and y are equivalent from the $\mathcal{C}$ point of view," while $x \cong_{\mathcal{D}} y$ means that "x and y are equivalent from the $\mathcal{D}$ point of view." In the philosophy of category theory there is **no absolute point of view**, thus there is no way to say whether x and y are "really the same."]

Examples of Categories:

Posets/Preorders. We have already seen that a poset $\mathcal{P}$ is just a small category satifying $|\mathsf{Hom}_{\mathcal{P}}(x, y)| \in \{0, 1\}$ for each ordered pair $x, y \in \mathcal{P}$. More precisely, such a category is called a *preorder*. If we throw away all but one object from each $\mathcal{P}$-isomorphism class, then it becomes a *poset*. The resulting poset is called the *skeleton* of the preorder.

Sets/Numbers. Let Set denote the collection of all sets, together with all functions between them. This is called the *category of sets*. (Note that it is not small.) The collection of all **finite sets and injective functions** is a (non-full) subcategory of Set. The skeleton of this subcategory is the poset category $(\mathbb{N}, \leq)$ of *finite cardinal numbers*:

$$0 \to 1 \to 2 \to 3 \to \cdots.$$

Monoids/Groupoids/Groups. A category with one object is called a *monoid*. A monoid in which each arrow is invertible is called a *group*. A small category in which each arrow is invertible is called a *groupoid*. Thus we can give the following fancy definition of groups:

a group is a groupoid with one object.

The skeleton of a groupoid is not very interesting.

Concrete Categories. Subcategories of Set are called *concrete categories*. Given a concrete category $\mathcal{C} \subseteq \mathsf{Set}$ we can think of its objects as special kinds of sets and its arrows as special kinds of functions. Some famous examples of conrete categories are:

- Grp = groups & homomorphisms
- Ab = abelian groups & homomorphisms
- Rng = rings & homomorphisms
- CRng = commutative rings & homomorphisms

Note that $\mathsf{Ab} \subseteq \mathsf{Grp}$ and $\mathsf{CRng} \subseteq \mathsf{Rng}$ are both **full** subcategories. In general, the arrows of a concrete category are called *morphisms* or *homomorphisms*. This explains our notation $\mathsf{Hom}_{\mathcal{C}}$ (but some people prefer $\mathsf{Mor}_{\mathcal{C}}$).

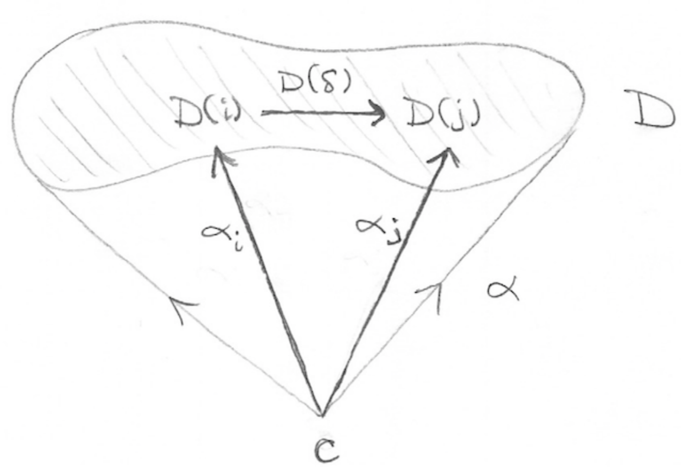
Homotopy. The most famous example of a **non-concrete** category is the *fundamental groupoid* $\pi_1(X)$ of a topological space X. Here the objects are points and the arrows are homotopy classes of continuous directed paths. The skeleton is the set $\pi_0(X)$ of path components (really a *discrete category*, i.e., in which the only arrows are the identities). Categories like this are the reason we prefer the name "arrow" instead of "morphism." ///

Those are the basics. Now we should discuss the general categorical version of meet and join. This definition is **not** obvious, but I will work hard to convince you that it is interesting and useful.

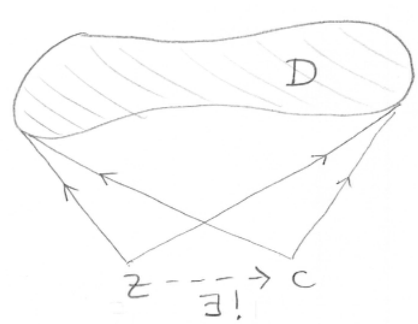
Definition of Limit/Colimit. Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$ (thus D is a functor and $\mathcal{I}$ is a small "index" category). A *cone under D* consists of

- an object $c \in \mathcal{C}$,
- a collection of arrows $\alpha_i : x \to D(i)$, one for each index $i \in \mathcal{I}$,

such that for each arrow $\delta : i \to j$ in $\mathcal{I}$ we have $\alpha_j = D(\delta) \circ \alpha_i$. I visualize this as follows:



The cone $(c, (\alpha_i)_{i\in\mathcal{I}})$ is called a *limit of the diagram D* if, for **any** cone $(z, (\beta_i)_{i\in\mathcal{I}})$ under D, the following picture holds:

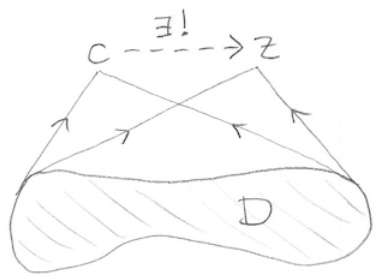


[This picture means that **there exists a unique arrow** $\upsilon : z \to c$ such that, for each arrow $\delta : i \to j$ in $\mathcal{I}$ (including the identity arrows), the following diagram commutes:

$$\begin{tikzcd}
D(i) \arrow[rr, "D(\delta)"] & & D(j) \\
 & c \arrow[ul, "\alpha_i"'] \arrow[ur, "\alpha_j"] & \\
 & z \arrow[u, dashed, "\upsilon"'] \arrow[uul, bend left, "\beta_i"] \arrow[uur, bend right, "\beta_j"'] &
\end{tikzcd}$$

When $\delta = \mathsf{id}_i$ this diagram just says that $\beta_i = \alpha_i \circ \upsilon$. We do not assume that D itself is commutative. Thus the picture of the cones above is not fully commutative, but I still find it quite helpful for intuition.]

Dually, a *cone over* D consists of an object $c \in \mathcal{C}$ and a set of arrows $\alpha_i : D(i) \to c$ satisfying $\alpha_i = \alpha_j \circ D(\delta)$ for each arrow $\delta : i \to j$ in $\mathcal{I}$. This cone is called a *colimit of the diagram* D if, for **any** cone $(z, (\beta_i)_{i \in \mathcal{I}})$ over D, the following picture holds:



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Exercise: State the precise sense in which a limit or colimit (if it exists) is **unique**.

When the (unique) limit or colimit of the diagram $D : \mathcal{I} \to \mathcal{C}$ exists, we denote it by

$$(\mathsf{lim}_{\mathcal{I}} D, (\varphi_i)_{i \in I}) \quad \text{or} \quad (\mathsf{colim}_{\mathcal{I}} D, (\varphi_i)_{i \in I}),$$

respectively. Sometimes we omit the *canonical arrows* φ_i from the notation and refer to the **object** $\mathsf{lim}_{\mathcal{I}} D \in \mathcal{C}$ as "the limit of D." However, we should not forget that the arrows are part of the structure, i.e., **the limit is really a cone**. Maybe the following words will help you remember this:

$$\begin{aligned}
\text{limit of } D &= \text{highest cone under } D, \\
\text{colimit of } D &= \text{lowest cone over } D.
\end{aligned}$$

Special Kinds of Limits/Colimits:

Final/Initial Objects. Consider the empty diagram $\varnothing \subseteq \mathcal{C}$, which is indexed by the empty category. The limit or colimit of this diagram (if it exists) is called the *final object of* $\mathcal{C}$ or the *initial object of* $\mathcal{C}$, respectively. Each comes equipped with the canonical empty set of arrows, which we will never mention again.

If an object $z \in \mathcal{C}$ is both final and initial, we call it the *zero object*. In this case there is also a unique *zero arrow* between any two objects, defined by the following commutative diagram:

$$\begin{tikzcd} x \arrow[r, "\exists!"'] \arrow[rr, bend left, "\text{zero}"] & z \arrow[r, "\exists!"'] & y \end{tikzcd}$$

Products/Coproducts. Let $\mathcal{I}$ be the category with $\mathsf{Obj}(\mathcal{I}) = \{1,2\}$ and $\mathsf{Arr}(\mathcal{I}) = \{\mathsf{id}_1, \mathsf{id}_2\}$. Then a diagram $D(\mathcal{I}) = D \subseteq \mathcal{C}$ is really just a choice of two objects $D(1), D(2) \in \mathcal{C}$, **which are allowed to be equal**.

The limit of D is called the *categorical product*. It consists of an object $D(1)\prod D(2) \in \mathcal{C}$ and two arrows $\pi_1 : D(1)\prod D(2) \to D(1)$ and $\pi_2 : D(1)\prod D(2) \to D(2)$, called *canonical projections*. It is defined by the following picture:

$$\begin{tikzcd} D(1) & & D(2) \\ & D(1)\prod D(2) \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\ & z \arrow[u, dashed, "\exists!"'] \arrow[uul, bend left] \arrow[uur, bend right] & \end{tikzcd}$$

Dually, the colimit of D is called the *categorical coproduct* (or *categorical sum*). It consists of an object $D(1)\coprod D(2) \in \mathcal{C}$ and two arrows $\iota_1 : D(1) \to D(1)\coprod D(2)$ and $\iota_2 : D(2) \to D(1)\coprod D(2)$, called *canonical injections*. It is defined by the following picture:

$$\begin{tikzcd} & z & \\ & D(1)\coprod D(2) \arrow[u, dashed, "\exists!"'] & \\ D(1) \arrow[ur, "\iota_1"'] \arrow[uur, bend left] & & D(2) \arrow[ul, "\iota_2"] \arrow[uul, bend right] \end{tikzcd}$$

[How to read the pictures: For all objects $z \in \mathcal{C}$ and all solid arrows, there exists a **unique dotted arrow** making the diagram commute.]

Kernels/Cokernels. See the Exercises.

///

Examples from our Favorite Categories:

Posets. Let $\mathcal{P}$ be a poset. We have already seen that the product/coproduct in $\mathcal{P}$ (if they exist) are the meet/join, respectively, and that the final/initial objects in $\mathcal{P}$ (if they exist) are the top/bottom elements, respectively. The only poset with a zero object is the **one element poset**.

Sets. The *empty set* $\varnothing \in \mathsf{Set}$ is an initial object and the *one point set* $* \in \mathsf{Set}$ is a final object. Note that two sets are isomorphic in Set precisely when there is a bijection between them, i.e., when they have the same cardinality. Since initial/final objects are unique up to isomorphism, we can identify the initial object with the cardinal number 0 and the final object with the cardinal number 1. There is no zero object in Set.

Products and coproducts exist in Set. The product of $S, T \in \mathsf{Set}$ consists of the *Cartesian product* $S \times T$ together with the canonical projections $\pi_S : S \times T \to S$ and $\pi_T : S \times T \to T$. The coproduct of $S, T \in \mathsf{Set}$ consists of the *disjoint union* $S \coprod T$ together with the canonical injections $\iota_S : S \to S \coprod T$ and $\iota_T : T \to S \coprod T$. After passing to the skeleton, the product and coproduct of sets become the *product* and *sum* of cardinal numbers.

[Note: The "external disjoint union" $S \coprod T$ is a formal concept. The familiar "internal disjoint union" $S \sqcup T$ is only defined when there exists a set U containing both S and T as subsets. Then the union $S \cup T$ is the join operation in the Boolean lattice 2^U; we call the union "disjoint" when $S \cap T = \varnothing$.]

Groups. The trivial group $1 \in \mathsf{Grp}$ is a zero object, and for any groups $G, H \in \mathsf{Grp}$ the zero homomorphism $1 : G \to H$ sends all elements of G to the identity element $1_H \in H$. The product of groups $G, H \in \mathsf{Grp}$ is their *direct product* $G \times H$ and the coproduct is their *free product* $G * H$, along with the usual canonical morphisms.

Let $\mathsf{Ab} \subseteq \mathsf{Grp}$ be the full subcategory of abelian groups. The zero object and product are inherited from Grp, but we give them new names: we denote the zero object by $0 \in \mathsf{Ab}$ and for any $A, B \in \mathsf{Ab}$ we denote the zero arrow by $0 : A \to B$. We denote the Cartesian product by $A \oplus B$ and we rename it the *direct sum*. The big difference between Grp and Ab appears when we consider coproducts: it turns out that the product group $A \oplus B$ is **also** the coproduct group. We emphasize this fact by calling $A \oplus B$ the *biproduct* in Ab. It comes equipped with **four canonical homomorphisms** $\pi_A, \pi_B, \iota_A, \iota_B$ satisfying the usual properties, as well as the following commutative diagram:

$$\begin{tikzcd}
A \arrow[rr, "\mathsf{id}_A"] \arrow[dr, "\iota_A"] \arrow[dd, "0"'] & & A \\
& A \oplus B \arrow[ur, "\pi_A"] \arrow[dl, "\pi_B"] & \\
B & & B \arrow[ll, "\mathsf{id}_B"] \arrow[ul, "\iota_B"'] \arrow[uu, "0"']
\end{tikzcd}$$

This diagram is the ultimate reason for **matrix notation**. The universal properties of product and coproduct tell us that each endomorphism $\varphi : A \oplus B \to A \oplus B$ is uniquely determined by its four *components* $\varphi_{ij} := \pi_i \circ \varphi \circ \iota_j$ for $i, j \in \{A, B\}$, so we can represent it as a matrix:

$$\begin{pmatrix} \varphi_{AA} & \varphi_{AB} \\ \varphi_{BA} & \varphi_{BB} \end{pmatrix}.$$

Then the composition of endomorphisms becomes **matrix multiplication**.

Rings. We let Rng denote the category of rings with unity, together with their homomorphisms. The initial object is the *ring of integers* $\mathbb{Z} \in \mathsf{Rng}$ and the final object is the *zero ring* $0 \in \mathsf{Rng}$, i.e., the unique ring in which $0_R = 1_R$. There is no zero object. The product of two rings $R, S \in \mathsf{Rng}$ is the *direct product* $R \times S \in \mathsf{Rng}$ with componentwise addition and multiplication. The coproduct is something like a "free product" but it is difficult to describe.

Let $\mathsf{CRng} \subseteq \mathsf{Rng}$ be the full subcategory of commutative rings. The initial/final objects and product in CRng are inherited from Rng. The difference between Rng and CRng again appears when considering coproducts. The coproduct of $R, S \in \mathsf{CRng}$ is denoted by $R \otimes_{\mathbb{Z}} S$ and is called the *tensor product over* $\mathbb{Z}$. We'll come back to this later.

1.2 Galois Connections

Every good definition should be legitimized by a theorem, and the definition of limits has a great theorem called RAPL ("right adjoints preserve limits"). To discuss the theorem I need to define right/left adjoint functors, but before doing so I will motivate their definition by considering the special case of adjoint functors between posets.

Definition of Galois Connection. Let $(\mathcal{P}, \leq_{\mathcal{P}})$ and $(\mathcal{Q}, \leq_{\mathcal{Q}})$ be posets, and consider two set functions $* : \mathcal{P} \rightleftarrows \mathcal{Q} : *$. We will denote these by $p \mapsto p^*$ and $q \mapsto q^*$ for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$. This pair of functions is called a *Galois connection* if, for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, we have

$$p \leq_{\mathcal{P}} q^* \iff q \leq_{\mathcal{Q}} p^*.$$

///

Basic Properties of Galois Connections. Let $* : \mathcal{P} \rightleftarrows \mathcal{Q} : *$ be a Galois connection. For all elements x of $\mathcal{P}$ or $\mathcal{Q}$ we will use the notations $x^{**} := (x^*)^*$ and $x^{***} := (x^{**})^*$.

(1) For all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ we have

$$p \leq_{\mathcal{P}} p^{**} \quad \text{and} \quad q \leq_{\mathcal{Q}} q^{**}.$$

(2) For all elements $p_1, p_2 \in \mathcal{P}$ and $q_1, q_2 \in \mathcal{Q}$ we have

$$p_1 \leq_{\mathcal{P}} p_2 \Longrightarrow p_2^* \leq_{\mathcal{Q}} p_1^* \quad \text{and} \quad q_1 \leq_{\mathcal{Q}} q_2 \Longrightarrow q_2^* \leq_{\mathcal{P}} q_1^*.$$

(3) For all elements $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ we have

$$p^{***} = p^* \quad \text{and} \quad q^{***} = q^*.$$

///

Proof. Since the definition of a Galois connection is symmetric in $\mathcal{P}$ and $\mathcal{Q}$, we will simplify the proof by using the notation

$$x \leq y^* \iff y \leq x^*$$

for all elements x, y such that the inequalities make sense. To prove (1) note that for any element x we have $x^* \leq x^*$ by the reflexivity of partial order. Then from the definition of Galois connection we obtain

$$(x^*) \leq (x)^* \implies (x) \leq (x^*)^* \implies x \leq x^{**}.$$

To prove (2) consider elements x, y such that $x \leq y$. From (1) and the transitivity of partial order we have

$$x \leq y \leq y^{**} \implies x \leq y^{**}.$$

Then from the definition of Galois connection we obtain

$$(x) \leq (y^*)^* \implies (y^*) \leq (x)^* \implies y^* \leq x^*.$$

To prove (3) consider any element x. On the one hand, part (1) tells us that

$$(x^*) \leq (x^*)^{**} \implies x^* \leq x^{***}.$$

On the other hand, part (1) tells us that $x \leq x^{**}$ and then part (2) says that

$$(x) \leq (x^{**}) \implies (x^{**})^* \leq (x)^* \implies x^{***} \leq x^*.$$

Finally, the antisymmetry of partial order says that $x^{***} = x^*$, which we interpret as isomorphism of objects in the poset category. □

The following definition captures the essence of these three basic properties.

Definition of Closure in a Poset. Given a poset $(\mathcal{P}, \leq)$, we say that a function $\mathsf{cl} : \mathcal{P} \to \mathcal{P}$ is a *closure operator* if it satisfies the following three properties:

(i) Extensive: $\forall p \in \mathcal{P}$, $p \leq \mathsf{cl}(p)$

(ii) Monotone: $\forall p, q \in \mathcal{P}$, $p \leq q \Longrightarrow \mathsf{cl}(p) \leq \mathsf{cl}(q)$

(iii) Idempotent: $\forall p \in \mathcal{P}$, $\mathsf{cl}(\mathsf{cl}(p)) = p$.

///

[Remark: If $\mathcal{P} = 2^U$ is a Boolean lattice, and if the closure $\mathsf{cl} : 2^U \to 2^U$ also preserves finite unions, then we call it a *Kuratowski closure*. Kuratowski proved that such a closure is equivalent to a topology on the set U.]

If $* : \mathcal{P} \to \mathcal{Q} : *$ is a Galois connection, then the basic properties above immediately imply that the compositions $** : \mathcal{P} \to \mathcal{P}$ and $** : \mathcal{Q} \to \mathcal{Q}$ are closure operators.

Proof: Property (ii) follows from applying property (2) twice and property (iii) follows from applying $*$ to property (3). □

Fundamental Theorem of Galois Connections. Any Galois connection $* : \mathcal{P} \rightleftarrows \mathcal{Q} : *$ determines two closure operators $** : \mathcal{P} \to \mathcal{P}$ and $** : \mathcal{Q} \to \mathcal{Q}$. We will say that the element $p \in \mathcal{P}$ (resp. $q \in \mathcal{Q}$) is $**$*-closed* if $p^{**} = p$ (resp. $q^{**} = q$). Then the Galois connection restricts to an **order-reversing bijection** between the subposets of $**$-closed elements. ///

Proof: Let $\mathcal{Q}^* \subseteq \mathcal{P}$ and $\mathcal{P}^* \subseteq \mathcal{Q}$ denote the images of the functions $* : \mathcal{Q} \to \mathcal{P}$ and $* : \mathcal{P} \to \mathcal{Q}$, respectively. I claim that the restriction of the connection to these subsets defines an order-reversing bijection:

$$\begin{array}{ccc} * : \mathcal{P} & \rightleftarrows & \mathcal{Q} : * \\ \cup\!\mathsf{I} & & \cup\!\mathsf{I} \\ * : \mathcal{Q}^* & \longleftrightarrow & \mathcal{P}^* : * \end{array}$$

Indeed, consider any $p \in \mathcal{Q}^*$, so that $p = q^*$ for some $q \in \mathcal{Q}$. Then by properties (1) and (3) of Galois connections we have

$$(p)^{**} = (q^*)^{**} \implies p^{**} = q^{***} \implies p^{**} = q^* \implies p^{**} = p.$$

Similarly, for all $q \in \mathcal{P}^*$ we have $q^{**} = q$. The bijections reverse order because of property (2).

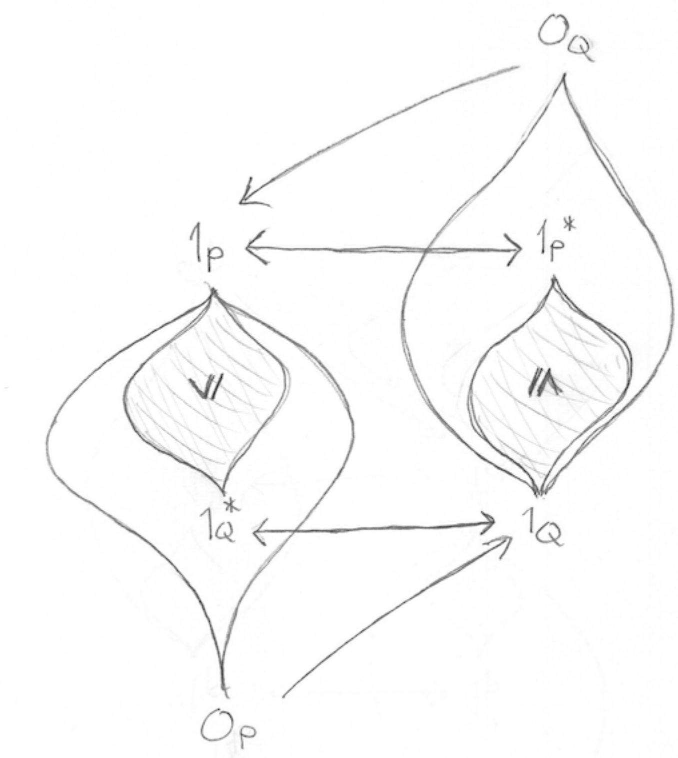
Finally, note that $\mathcal{Q}^*$ and $\mathcal{P}^*$ are exactly the subsets of $**$-closed elements in $\mathcal{P}$ and $\mathcal{Q}$, respectively. Indeed, we have seen above that every element of $\mathcal{Q}^*$ is $**$-closed. Conversely, if $p \in \mathcal{P}$ is $**$-closed then we have

$$p = p^{**} \implies p = (p^*)^*,$$

and it follows that $p \in \mathcal{Q}^*$. Similarly, every element of $\mathcal{P}^*$ is $**$-closed. □

Thus, a Galois connection is something like a "loose bijection." It's not necessarily a bijection but it becomes one after we "tighten it up." Sort of like tightening your shoelaces.

I'm not good at drawing unbounded posets, so let's assume that our posets have top and bottom elements: $1_{\mathcal{P}}, 0_P \in \mathcal{P}$ and $1_{\mathcal{Q}}, 0_{\mathcal{Q}} \in \mathcal{Q}$. In this case, I visualize a Galois connection $* : \mathcal{P} \rightleftarrows \mathcal{Q} : *$ with in the following picture:



The shaded subposets here consist of the $**$-closed elements. They are supposed to look (anti-) isomorphic. The unshaded parts of the posets get "tightened up" into the shaded subposets. Note that the top elements are $**$-closed. Indeed, property (2) tells us that $1_{\mathcal{P}} \leq_{\mathcal{P}} 1_{\mathcal{P}}^{**}$ and then from the universal property of the top element we have $1_{\mathcal{P}}^{**} = 1_{\mathcal{P}}$. (In detail: Since $p \leq_{\mathcal{P}} 1_P$ for all $p \in \mathcal{P}$ we must have $1_{\mathcal{P}}^{**} \leq_{\mathcal{P}} 1_{\mathcal{P}}$, and then antisymmetry gives $1_{\mathcal{P}}^{**} = 1_{\mathcal{P}}$.) Similarly, we have $1_{\mathcal{Q}}^{**} = 1_{\mathcal{Q}}$.

But the bottom elements are **not necessarily $**$-closed**. Instead, we have the important fact that $0_{\mathcal{P}}^* = 1_{\mathcal{Q}}$ and $0_{\mathcal{Q}}^* = 1_{\mathcal{P}}$. To see this, recall from the definition of Galois connection that for all $q \in \mathcal{Q}$ we have

$$0_{\mathcal{P}} \leq_{\mathcal{P}} q^* \iff q \leq_{\mathcal{Q}} 0_{\mathcal{P}}^*.$$

Since the left hand side is always true, so is the right hand side. But then from the universal property of the top element in $\mathcal{Q}$ we conclude that $0_{\mathcal{P}}^* = 1_{\mathcal{Q}}$. The following exercise generalizes this phenomenon.

Exercise: Prove that a Galois connection sends joins to meets. That is, if $* : \mathcal{P} \rightleftarrows \mathcal{Q} : *$ is a Galois connection, then for all subsets $S \subseteq \mathcal{P}$ and $T \subseteq \mathcal{Q}$ we have

$$(\vee_{\mathcal{P}} S)^* = \wedge_{\mathcal{Q}} S^* \quad \text{and} \quad (\vee_{\mathcal{Q}} T)^* = \wedge_{\mathcal{P}} T^*.$$

As a consequence of this, the arbitrary meet of $**$-closed elements (if it exists) is still $**$-closed. We will see, however, that the join of $**$-closed elements is not necessarily $**$-closed.

(And hence not all Galois connections induce topologies.) ///

Galois connections between Boolean lattices have a particularly nice form, which is closely related to the *universal quantifier* "$\forall$".

Galois Connections of Boolean Lattices. Let U, V be sets and let $\sim \subseteq U \times V$ be any subset (called a *relation*) between U and V. As usual, we will write "$u \sim v$" in place of the statement "$(u, v) \in \sim$", and we read this as "u is related to v." Then for all $S \in 2^U$ and $T \in 2^V$ we define

$$S^\sim := \{v \in V : \forall s \in S,\, s \sim v\} \in 2^V,$$
$$T^\sim := \{u \in U : \forall t \in T,\, u \sim t\} \in 2^U.$$

I claim that the pair of functions $S \mapsto S^\sim$ and $T \mapsto T^\sim$ is a Galois connection,

$$\sim : 2^U \rightleftarrows 2^V : \sim .$$

To see this, note that for all subsets $S \in 2^U$ and $T \in 2^V$ we have

$$\begin{aligned} S \subseteq T^\sim &\iff \forall s \in S,\, s \in T^\sim \\ &\iff \forall s \in S,\, \forall t \in T,\, s \sim t \\ &\iff \forall t \in T,\, \forall s \in S,\, s \sim t \\ &\iff \forall t \in T,\, t \in S^\sim \\ &\iff T \subseteq S^\sim. \end{aligned}$$

Moreover, one can prove that **any** Galois connection between 2^U and 2^V arises in this way from a unique relation. ///

Exercise. Prove this.

The most famous Galois connections arise from simple relations between two sets.

Examples of Galois Connections:

Orthogonal Complement. Let V be a vector space over field K and let V^* be the *dual space*, consisting of linear functions $\alpha : V \to K$. We define the relation $\perp \subseteq V^* \times V$ by

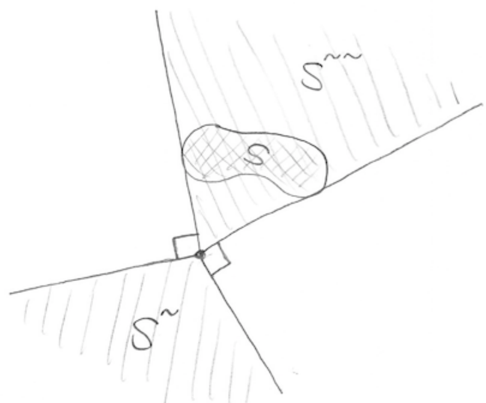
$$\alpha \perp v \iff \alpha(v) = 0.$$

The resulting $\perp\perp$-closed subsets are precisely the linear subspaces on both sides. Thus the Fundamental Theorem of Galois Connections gives us an order-reversing bijection between the subspaces of V^* and the subspaces of V.

Convex Complement. Let V be a Euclidean space, i.e., a real vector space with an inner product $\langle -,- \rangle : V \times V \to \mathbb{R}$. We define the relation $\sim \subseteq V \times V$ by

$$u \sim v \quad \Longleftrightarrow \quad \langle u, v \rangle \le 0.$$

For all $S \subseteq V$ the operation $S \mapsto S^{\sim\sim}$ gives the *cone genrated by* S, thus the $\sim\sim$-closed sets are precisely the cones. Here is a picture:



Original Galois Connection. Let L be a field and let G be a finite group of automorphisms of L, i.e., each $g \in G$ is a function $g : L \to L$ preserving addition and multiplication. We define a relation $\sim \subseteq G \times L$ by

$$g \sim \ell \quad \Longleftrightarrow \quad g(\ell) = \ell.$$

Define $K := L^{\sim}$ to be the "subfield fixed by G." The original *Fundamental Theorem of Galois Theory* says that the $\sim\sim$-closed subsets of G are precisely the **subgroups** and the $\sim\sim$-closed subsets of L are precisely the **subfields containing** K. [There's a bit more to it, but those are the basics.]

Hilbert's Nullstellensatz. Let K be a field and consider the ring of polynomials $K[x] := K[x_1, \ldots, x_n]$ in n commuting variables. For each polynomial $f(x) := f(x_1, \ldots, x_n) \in K[x]$ and for each n-tuple of field elements $\alpha := (\alpha_1, \ldots, \alpha_n) \in K^n$, we denote the evaluation by $f(\alpha) := f(\alpha_1, \ldots, \alpha_n) \in K$. Now we define a relation $\sim \subseteq K[x] \times K^n$ by

$$f(x) \sim \alpha \quad \Longleftrightarrow \quad f(\alpha) = 0.$$

By definition, the closure operator $\sim\sim$ on subsets of K^n is called the *Zariski closure*. From the Fundamental Theorem we know that every Zariski closed sets $X \subseteq K^n$ has the form $J^{\sim}$ for some set of polynomials $J \subseteq K[x]$. Let $X_i := J_i^{\sim}$ be an arbitrary family of closed sets. Then from the previous exercise we have

$$\cap_i X_i = \cap_i J_i^{\sim} = (\cup_i J_i)^{\sim},$$

and it follows that an arbitrary intersection of closed sets is closed. What about a union of closed sets? Note that for each index j we have $\cap_i J_i \subseteq J_j$ and hence $X_j = J_j^{\sim} \subseteq (\cap_i J_i)^{\sim}$. Since the closed set $(\cap_i J_i)^{\sim}$ is an upper bound of the closed sets X_j, it must contain their union:

$$\cup_i X_i = \cup_i J_i^{\sim} \subseteq (\cap_i J_i)^{\sim}.$$

In general this inclusion is strict. However, in this special case we can show that **finite** unions are still closed. To see this we first note that

- for any set of polynomials $J \subseteq K[x]$, its closure $J^{\sim\sim} \subseteq K[x]$ is an ideal.

Indeed, consider any $f, g \in J^{\sim\sim}$ and $h \in K[x]$. For any point $\alpha \in J^{\sim}$ we have by definition that $f(\alpha) = 0$ and $g(\alpha) = 0$. But then we must also have

$$(f + gh)(\alpha) = f(\alpha) + g(\alpha)h(\alpha) = 0 + 0\alpha = 0,$$

from which it follows that $f + gh \in J^{\sim\sim}$. Then we can show that

- for any two closed sets $X_1 = J_1^{\sim}$ and $X_2 = J_2^{\sim}$ we have

  $$X_1 \cup X_2 = J_1^{\sim} \cup J_2^{\sim} = (J_1^{\sim\sim} \cap J_2^{\sim\sim})^{\sim},$$

  so that the union $X_1 \cup X_2$ is closed.

Indeed, since $J^{\sim} = (J^{\sim\sim})^{\sim}$ for all sets $J \subseteq K[x]$, we already know that

$$J_1^{\sim} \cup J_2^{\sim} = (J_1^{\sim\sim})^{\sim} \cup (J_2^{\sim\sim})^{\sim} \subseteq (J_1^{\sim\sim} \cap J_2^{\sim\sim})^{\sim}.$$

To prove the other inclusion, consider any point $\alpha \in K^n$ that is **not** in the set $(J_1^{\sim\sim})^{\sim} \cup (J_2^{\sim\sim})^{\sim}$. This means that $\alpha \notin (J_1^{\sim\sim})^{\sim}$ and $\alpha \notin (J_2^{\sim\sim})^{\sim}$, so by definition there exist polynomials $f_1 \in J_1^{\sim\sim}$ and $f_2 \in J_2^{\sim\sim}$ such that $f_1(\alpha) \neq 0$ and $f_2(\alpha) \neq 0$. Since K is an **integral domain** this implies that $(f_1 f_2)(\alpha) = f_1(\alpha) f_2(\alpha) \neq 0$, so the polynomial $f_1 f_2$ also does not vanish at α. Finally, since $J_1^{\sim\sim}$ and $J_2^{\sim\sim}$ are **ideals** we conclude that $f_1 f_2 \in J_1^{\sim\sim} \cap J_2^{\sim\sim}$ and it follows that α is not in the set $(J_1^{\sim\sim} \cap J_2^{\sim\sim})^{\sim}$.

We have seen that the binary union (and, by induction, any finite union) of Zariski closed sets is closed, hence the Zariski closure defines a topology on K^n, naturally called the *Zariski topology*. The $\sim\sim$-closed subsets of the ring $K[x]$ are harder to pin down. We saw above that the closed subsets of $K[x]$ are ideals, but which ideals? If the field K is **algebraically closed** then Hilbert's *Nullstellensatz* says that the $\sim\sim$-closed subsets of $K[x]$ are precisely the *radical ideals* (i.e., ideals closed under taking arbitrary roots). This is much harder to prove.

1.3 From Galois Connections to Adjunctions

To make the transition from Galois connections to adjoint functors we make a slight change of notation. The change is only cosmetic but it is very important for our intuition.

Definition of Poset Adjunction. Let $(\mathcal{P}, \leq_\mathcal{P})$ and $(\mathcal{Q}, \leq_\mathcal{Q})$ be posets. A pair of functions $L : \mathcal{P} \rightleftarrows \mathcal{Q} : R$ is called an *adjunction* if for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ we have

$$p \leq_\mathcal{P} R(q) \quad \Longleftrightarrow \quad L(p) \leq_\mathcal{Q} q$$

In this case we write $L \dashv R$ and call this an *adjoint pair* of functions. The function L is the *left adjoint* and R is the *right adjoint.* ///

The only difference between Galois connections and poset adjunctions is that we have **reversed** the partial order on $\mathcal{Q}$. To be precise, we define the *opposite poset* $\mathcal{Q}^{\mathsf{op}}$ with the same underlying set $\mathcal{Q}$, such that for all $q_1, q_2 \in \mathcal{Q}$ we have

$$q_1 \leq_{\mathcal{Q}^{\mathsf{op}}} q_2 \quad \Longleftrightarrow \quad q_2 \leq_\mathcal{Q} q_1.$$

Then an adjunction $\mathcal{P} \rightleftarrows \mathcal{Q}$ is just the same thing as a Galois connection $\mathcal{P} \rightleftarrows \mathcal{Q}^{\mathsf{op}}$.

However, this difference is important because it **breaks the symmetry**. It also prepares us for the notation of an adjunction between categories, where it is more common to use an "asymmetric pair of **covariant** functors" as opposed to a "symmetric pair of **contravariant** functors." But more on that later.

Now I will state the two most important properties of adjunctions and prove them for adjunctions between posets. The statements and proofs of these theorems for general categories are more involved, but will follow the same pattern.

Uniqueness of Adjoints for Posets. Let $\mathcal{P}$ and $\mathcal{Q}$ be posets and let $L : \mathcal{P} \rightleftarrows \mathcal{Q} : R$ be an adjunction. Then each of the adjoint functions $L \dashv R$ uniquely determines the other. ///

Proof: To prove that R determines L, suppose that $L' : \mathcal{P} \rightleftarrows \mathcal{Q} : R$ is another adjunction. Then by definition of adjunction we have for all $q \in \mathcal{Q}$ that

$$L(p) \leq_\mathcal{Q} q \quad \Longleftrightarrow \quad p \leq_\mathcal{P} R(q) \quad \Longleftrightarrow \quad L'(p) \leq_\mathcal{Q} q.$$

In particular, setting $q = L(p)$ gives

$$L(p) \leq_\mathcal{Q} L(p) \quad \Longrightarrow \quad L'(p) \leq_\mathcal{Q} L(p)$$

and setting $q = L'(p)$ gives

$$L'(p) \leq_\mathcal{Q} L'(p) \quad \Longrightarrow \quad L(p) \leq_\mathcal{Q} L'(p).$$

Then by the antisymmetry of $\leq_\mathcal{Q}$ we have $L(p) = L'(p)$. Since this holds for all $p \in \mathcal{P}$ we conclude that $L = L'$, as desired. □

[Remark: The general version uses something called "Yoneda's Lemma" to conclude that L and L' are merely "isomorphic in some natural way."]

RAPL Theorem for Posets. Let $L : \mathcal{P} \rightleftarrows \mathcal{Q} : R$ be an adjunction of posets. Then for all subsets $S \subseteq \mathcal{P}$ and $T \subseteq \mathcal{Q}$ we have

$$L(\vee_{\mathcal{P}} S) = \vee_{\mathcal{Q}} L(S) \quad \text{and} \quad R(\wedge_{\mathcal{Q}} T) = \wedge_{\mathcal{P}} R(T).$$

In words, we say that

"*left adjoints preserve join*" and "*right adjoints preserve meet.*"

///

Proof: You already proved this in a previous exercise when you showed that Galois connections send joins to meets. Now it's my turn.

So let $L : \mathcal{P} \rightleftarrows \mathcal{Q} : R$ be an adjunction of posets and consider any $S \subseteq \mathcal{P}$. If the set $S \subseteq \mathcal{P}$ has a least upper bound $\vee_{\mathcal{P}} S \in P$ then I claim that $L(\vee_{\mathcal{P}} S) \in Q$ is a least upper bound of the set $L(\vee_{\mathcal{P}} S)$. Indeed, for any $q \in \mathcal{Q}$ the definitions of "adjunction" and "join" tell us that

$$\begin{aligned} L(\vee_{\mathcal{P}} S) \leq_{\mathcal{Q}} q &\iff \vee_{\mathcal{P}} S \leq_{\mathcal{P}} R(q) \\ &\iff \forall s \in S, s \leq_{\mathcal{P}} R(q) \\ &\iff \forall s \in S, L(s) \leq_{\mathcal{Q}} q. \end{aligned}$$

Putting $q = L(\vee_P S)$ gives

$$L(\vee_{\mathcal{P}} S) \leq_{\mathcal{Q}} L(\vee_P S) \implies \forall s \in S, L(s) \leq_{\mathcal{Q}} L(\vee_P S),$$

which tells us that $L(\vee_{\mathcal{P}} S)$ is an upper bound of the set $L(S)$. Finally, if $q \in \mathcal{Q}$ is **any** upper bound of $L(S)$ then we obtain

$$\forall s \in S, L(s) \leq_{\mathcal{Q}} q \implies L(\vee_P S) \leq_{\mathcal{Q}} q,$$

which says that $L(\vee_P S)$ is a **least** upper bound. □

[Remark: The general version states that "left adjoints preserve colimits" and "right adjoints preserve limits," hence the name RAPL.]

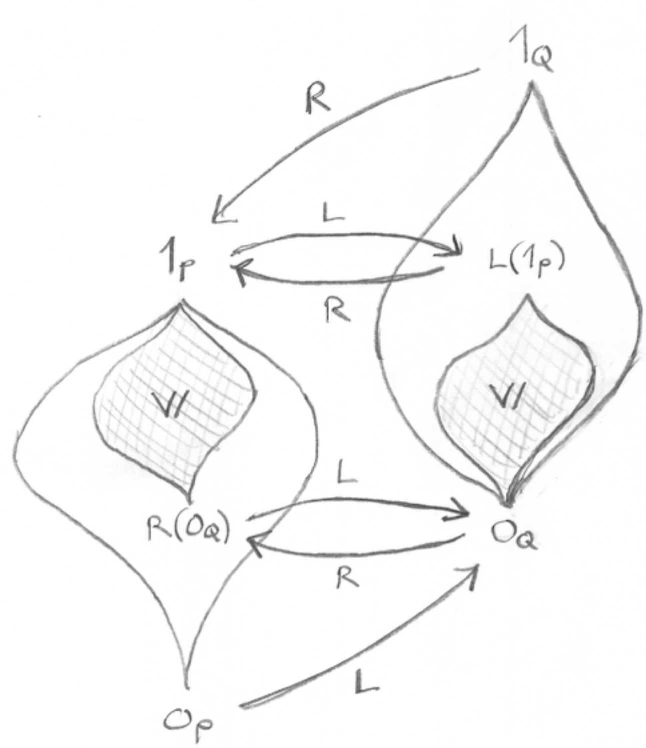
Exercise: In this proof we saw that the existence of $\vee_{\mathcal{P}} S$ implies the existence of $\vee_{\mathcal{Q}} L(S)$. [Dually, the existence of $\wedge_{\mathcal{Q}} T$ implies the existence of $\wedge_{\mathcal{P}} R(T)$.] But what about the other way around? That is, does the existence of $\vee_{\mathcal{Q}} L(S)$ imply the existence of $\vee_{\mathcal{P}} S$?[4] ///

Exercise: If a function of posets $L : \mathcal{P} \to \mathcal{Q}$ has a right adjoint $\mathcal{P} \leftarrow \mathcal{Q} : R$ then we have see that L necessarily (1) preserves order and (2) preserves any joins that exist in $\mathcal{P}$. In the

[4] That is, in addition do *preserving* joins, does a left adjoint also *create* joins?

case that **all** joins exist in $\mathcal{P}$ (i.e., if $\mathcal{P}$ is *co-complete*), prove that these conditions are also sufficient. That is, if $L : \mathcal{P} \to \mathcal{Q}$ is a function that (1) preserves order and (2) preserves joins, prove that L has a (necessarily unique) right adjoint $\mathcal{P} \leftarrow \mathcal{Q} : R$.[5] ///

It seems worthwhile to emphasize the new terminology with a picture. Suppose that the posets $\mathcal{P}$ and $\mathcal{Q}$ have top and bottom elements: $1_\mathcal{P}, 0_\mathcal{P} \in \mathcal{P}$ and $1_\mathcal{Q}, 0_\mathcal{Q} \in \mathcal{Q}$. Then a poset adjunction $L : \mathcal{P} \rightleftarrows \mathcal{Q} : R$ looks like this:



In this case $RL : \mathcal{P} \to \mathcal{P}$ is a closure operator as before, but now $LR : \mathcal{Q} \to \mathcal{Q}$ is called an *interior operator*. From the case of Galois connections we also know that $LRL = L$ and $RLR = R$. Since bottom elements are colmits and top elements are limits, the identities $L(0_\mathcal{P}) = 0_\mathcal{Q}$ and $R(1_\mathcal{Q}) = 1_\mathcal{P}$ are special cases of the RAPL Theorem.

///

Just as with Galois connections, adjunctions between the Boolean lattices 2^U and 2^V are in bijection with relations $\sim \subseteq U \times V$, but this time we will view the relation $\sim$ as a function $f^\sim : U \to 2^V$ that sends each $u \in U$ to the set $f^\sim(u) := \{v \in V : u \sim v\}$. We can also think of $f^\sim$ as a "multi-valued function" from U to V.

[5]This exercise is a very special case of Freyd's *adjoint functor theorem.*

Adjunctions of Boolean Lattices. Let U, V be sets and consider an arbitrary function $f : U \to 2^V$. Then for all subsets $S \in 2^U$ and $T \in 2^V$ we define

$$L_f(S) := \cup_{s \in S} f(s) \in 2^V,$$
$$R_f(T) := \{u \in U : f(u) \subseteq T\} \in 2^U.$$

I claim that the pair of functions $L_f : 2^U \rightleftarrows 2^V : R_f$ is an adjunction of Boolean lattices. To see this, note that for all $S \in 2^U$ and $T \in 2^V$ we have

$$\begin{aligned} S \subseteq R_f(T) &\iff \forall s \in S, s \in R_f(T) \\ &\iff \forall s \in S, f(s) \subseteq T \\ &\iff \cup_{s \in S} f(s) \subseteq T \\ &\iff L_f(S) \subseteq T. \end{aligned}$$

Moreover, one can prove that **any** adjunction between 2^U and 2^V arises in this way from a unique function $f : U \to 2^V$. ///

Exercise: Prove this.

The case of actual (i.e., single-valued) funtions is particularly interesting.

Example (Functions). Let $f : U \to V$ be any function. We can extend this to a function $f : U \to 2^V$ by defining $f(u) := \{f(u)\}$ for all $u \in U$. In this case we denote the corresponding left and right adjoint functions by $f_* := L_f : 2^U \to 2^V$ and $f^{-1} := R_f : 2^V \to 2^U$, so that for all $S \in 2^U$ and $T \in 2^V$ we have

$$f_*(S) = \{f(s) : s \in S\},$$
$$f^{-1}(T) = \{u \in U : f(s) \in T\}.$$

The resulting adjunction $f_* : 2^U \rightleftarrows 2^V : f^{-1}$ is called the *image* and *preimage* of the function. It follows from RAPL that image preserves unions and preimage preserves intersections.

But now something **surprising** happens. We can restrict the preimage $f^{-1} : 2^V \to 2^U$ to a function $f^{-1} : V \to 2^U$ by defining $f^{-1}(v) := f^{-1}(\{v\})$ for each $v \in V$. Then since $f^{-1} = L_{f^{-1}}$ we obtain **another adjunction**

$$f^{-1} : 2^V \rightleftarrows 2^U : R_{f^{-1}},$$

where this time f^{-1} is the **left adjoint**. The new right adjoint is defined for each $S \in 2^U$ by

$$R_{f^{-1}}(S) = \{v \in V : f^{-1}(v) \subseteq S\}.$$

There seems to be no standard notation for this function, but I've seen people call it $f_! := R_{f^{-1}}$ (the "!" is pronounced "shriek"). In summary, each function $f : U \to V$ determines a triple of adjoints

$$f_* \dashv f^{-1} \dashv f_!$$

where f_* preserves unions, $f_!$ preserves intersections, and f^{-1} **preserves both unions and intersections**. Logicians will tell you that the functions f_* and $f_!$ are closely related to the *existential* ($\exists$) and *universal* ($\forall$) *quantifiers*, in the sense that for all $S \in 2^U$ we have

$$f_*(S) = \{v \in V : \exists u \in f^{-1}(v), u \in S\},$$
$$f_!(S) = \{v \in V : \forall u \in f^{-1}(v), u \in S\}.$$

///

Exercise: Prove that in general the function $f_* : 2^U \to 2^V$ does **not** preserve intersections and the function $f_! : 2^V \to 2^U$ does **not** preserve unions. Hence the string of adjunctions $f_* \dashv f^{-1} \dashv f_!$ can not be extended.

This fundamental example can be dressed up in many ways. Here's one way.

Example (Group Homomorphisms). Given a group G we let $(\mathscr{L}(G), \subseteq)$ denote its poset of subgroups. Since the intersection of subgroups is again a subgroup, we have $\wedge = \cap$. Then since $\mathscr{L}(G)$ has arbitrary meets it also has arbitrary joins. In particular, the join of two subgroups $A, B \in \mathscr{L}(G)$ is given by

$$A \vee B = \bigcap \{C \in \mathscr{L}(G) : A \subseteq C \text{ and } B \subseteq C\},$$

which is the **smallest subgroup containing the union** $A \cup B$. Thus $\mathscr{L}(G)$ is a lattice, but since $A \vee B \neq A \cup B$ (in general) it is not a **sublattice** of 2^G.

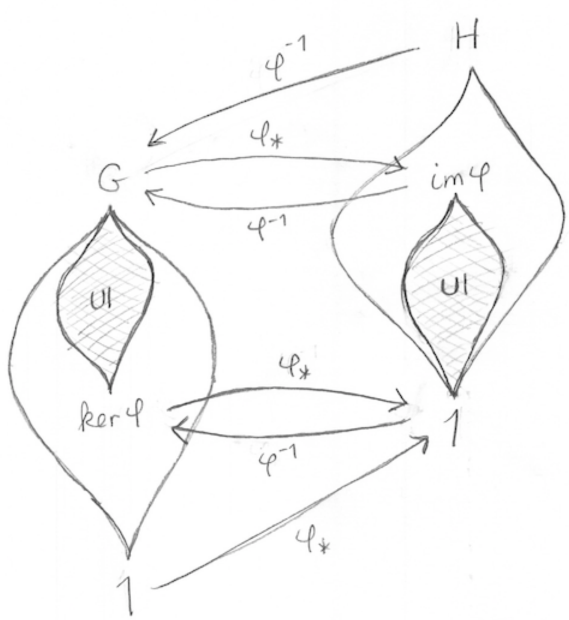
Now let $\varphi : G \to H$ be an arbitrary group homomorphism. One can check that the image and preimage $\varphi_* : 2^G \rightleftarrows 2^H : \varphi^{-1}$ send subgroups to subgroups, hence they restrict to an adjunction between subgroup lattices:

$$\varphi_* : \mathscr{L}(G) \rightleftarrows \mathscr{L}(H) : \varphi^{-1}.$$

The function $\varphi_! : 2^G \to 2^H$ from the previous example does **not** send subgroups to subgroups, and in general the function $\varphi^{-1} : \mathscr{L}(H) \to \mathscr{L}(G)$ does **not** have a right adjoint. For all subgroups $A \in \mathscr{L}(G)$ and $B \in \mathscr{L}(H)$ one can check that

$$\varphi^{-1}\varphi_*(A) = A \vee \ker\varphi \quad \text{and} \quad \varphi_*\varphi^{-1}(B) = B \wedge \operatorname{im}\varphi.$$

Thus the $\varphi^{-1}\varphi_*$-fixed subgroups of G are precisely those that **contain the kernel** and the $\varphi_*\varphi^{-1}$-fixed subgroups of H are precisely those **contained in the image**. Finally, the Fundamental Theorem gives us an order-preserving bijection as in the following picture:



All of Noether's "Isomorphism Theorems" are built on top of this picture. ///

1.4 The Definition of Adjoint Functors

We have called $L : \mathcal{P} \rightleftarrows \mathcal{Q} : R$ an adjoint pair of **functions**, but of course they more than just functions. If $L \dashv R$ is an adjunction, then property (2) of Galois connections says that for all $p_1, p_2 \in \mathcal{P}$ and $q_1, q_2 \in \mathcal{Q}$ we have

$$p_1 \leq_{\mathcal{P}} p_2 \Longrightarrow L(p_1) \leq_{\mathcal{Q}} L(p_2) \quad \text{and} \quad q_1 \leq_{\mathcal{Q}} q_2 \Longrightarrow R(q_1) \leq_{\mathcal{P}} R(q_2).$$

That is, the functions $L : \mathcal{P} \to \mathcal{Q}$ and $R : \mathcal{Q} \to \mathcal{P}$ are actually *homomorphisms of posets.*

To define adjunctions in general, we must first define "homomorphisms of categories." I already implicity used this concept when we discussed "diagrams."

Definition of Functor. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists a family of functions:

- A function on objects $F : \mathsf{Obj}(\mathcal{C}) \to \mathsf{Obj}(\mathcal{D})$,
- For each pair of objects $c_1, c_2 \in \mathcal{C}$ a function on hom sets:

$$F : \mathsf{Hom}_{\mathcal{C}}(c_1, c_2) \to \mathsf{Hom}_{\mathcal{D}}(F(c_1), F(c_2)).$$

These functions must preserve the category structure:

(i) Identity: For all objects $c \in \mathcal{C}$ we have $F(\mathsf{id}_c) = \mathsf{id}_{F(c)}$.

(ii) Composition: For all arrows $\alpha, \beta \in \mathcal{C}$ such that $\beta \circ \alpha$ is defined, we have

$$F(\beta \circ \alpha) = F(\beta) \circ F(\alpha).$$

Functors compose in an associative way, and for each category $\mathcal{C}$ there is a distinguished *identity functor* $\mathsf{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. In other words, the collection of all categories with functors between them forms a (very big) category, which we denote by Cat. ///

This definition is not surprising. It basically says that a functor $F : \mathcal{C} \to \mathcal{D}$ sends commutative diagrams in $\mathcal{C}$ to commutative diagrams in $\mathcal{D}$. That is, for each diagram $D : \mathcal{I} \to \mathcal{C}$ in $\mathcal{C}$ we have a diagram $F^{\mathcal{I}}(D) : \mathcal{I} \to \mathcal{D}$ in $\mathcal{D}$ (defined by $F^{\mathcal{I}}(D) := F \circ D$), which is commutative if and only if D is.

Exercise: Let $F : \mathcal{C} \to \mathcal{D}$ and suppose that $\alpha : x \leftrightarrow y : \beta$ is an isomorphism in $\mathcal{C}$. Apply F to the commutative diagram

$$\begin{tikzcd} x \arrow[loop left, "\mathsf{id}_x"] \arrow[r, bend left, "\alpha"] & y \arrow[l, bend left, "\beta"] \arrow[loop right, "\mathsf{id}_y"] \end{tikzcd}$$

to prove that the objects $F(x)$ and $F(y)$ are isomorphic in $\mathcal{D}$.

Now let's try to **guess** the definition of an "adjunction of categories." Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be any two functors. When $\mathcal{C}$ is a poset, recall that for all $x, y \in \mathcal{C}$ we have $|\mathsf{Hom}_{\mathcal{C}}(x, y)| \in \{0, 1\}$ and we use the notations

$$\begin{aligned} \text{“}x \leq y\text{”} &\iff |\mathsf{Hom}_{\mathcal{C}}(x, y)| = 1, \\ \text{“}x \nleq y\text{”} &\iff |\mathsf{Hom}_{\mathcal{C}}(x, y)| = 0. \end{aligned}$$

Thus if $\mathcal{C}$ and $\mathcal{D}$ are posets, we can rephrase the definition of a *poset adjunction* $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ by stating that for all objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ there exists a **bijection of hom sets**:

$$\mathsf{Hom}_{\mathcal{C}}(c, R(d)) \longleftrightarrow \mathsf{Hom}_{\mathcal{D}}(L(d), c).$$

In this form the definition now applies to any pair of functors between categories.

However, if we want to preserve the important theorems (uniqueness of adjoints and RAPL) then we need to impose some "naturality" condition on the family of bijections between hom sets. This condition is automatic for posets, so we will have to look elsewhere for motivation. First I'll give an ugly definition that is completely explicit. Then we'll search for a more abstract definition that explains what's really going on.

Explicit Definition of Adjoint Functors. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that a pair of functors $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ is an *adjunction* if for all objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ there exists a bijection of hom sets

$$\mathsf{Hom}_{\mathcal{C}}(c, R(d)) \longleftrightarrow \mathsf{Hom}_{\mathcal{D}}(L(c), d).$$

Furthermore, we require that these bijections fit together in the following "natural" way. For each arrow $\gamma : c_2 \to c_1$ in $\mathcal{C}$ (not a typo) and each arrow $\delta : d_1 \to d_2$ in $\mathcal{D}$ we require that the following cube of functions commutes:

$$\begin{tikzcd}
 & \mathsf{Hom}_{\mathcal{C}}(c_2, R(d_2)) \arrow[rr, leftrightarrow] & & \mathsf{Hom}_{\mathcal{D}}(L(c_2), d_2) \\
\mathsf{Hom}_{\mathcal{C}}(c_2, R(d_1)) \arrow[ur, "R(\delta)\circ(-)"] \arrow[rr, leftrightarrow] & & \mathsf{Hom}_{\mathcal{D}}(L(c_2), d_1) \arrow[ur, "\delta\circ(-)"] & \\
 & \mathsf{Hom}_{\mathcal{C}}(c_1, R(d_2)) \arrow[uu, dotted, "(-)\circ\gamma"'] \arrow[rr, dotted, leftrightarrow] & & \mathsf{Hom}_{\mathcal{D}}(L(c_1), d_2) \arrow[uu, "(-)\circ L(\gamma)"'] \\
\mathsf{Hom}_{\mathcal{C}}(c_1, R(d_1)) \arrow[uu, "(-)\circ\gamma"] \arrow[ur, dotted, "R(\delta)\circ(-)"'] \arrow[rr, leftrightarrow] & & \mathsf{Hom}_{\mathcal{D}}(L(c_1), d_1) \arrow[uu, "(-)\circ L(\gamma)"] \arrow[ur, "\delta\circ(-)"'] &
\end{tikzcd}$$

It is convenient to denote the horizontal bijections of the cube by $\varphi \mapsto \overline{\varphi}$ in either direction (so that $\overline{\overline{\varphi}} = \varphi$). Then the adjunction can be summarized with two equations:

$$\overline{\varphi \circ L(\gamma)} = \overline{\varphi} \circ \gamma,$$
$$\overline{R(\delta) \circ \psi} = \delta \circ \overline{\psi}.$$

The first equation says that the front/back of the cube commutes, while the second equation says that the top/bottom of the cube commutes. (The sides of the cube automatically commute.)

If the categories $\mathcal{C}, \mathcal{D}$ are understood then we will denote the adjunction by $L \dashv R$. We say that L is the *left adjoint* and R is the *right adjoint* functor. ///

This is the definition we will use to verify whether a given pair of functors is an adjunction. However, this definition seems unmotivated. Next we will develop a higher level of abstraction to make the concept of "naturality" seem more natural. Maybe it will seem too abstract to you at first, but I will work hard to convince you that this is the correct level of abstraction.

Definition of Natural Transformation. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and consider two parallel functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\Phi : F \Rightarrow G$ consists of a family of arrows $\Phi_c : F(c) \to R(c)$, one for each object $c \in \mathcal{C}$, such that for each arrow $\gamma : c_1 \to c_2$ in $\mathcal{C}$ the following square commutes:

$$\begin{tikzcd}
F_1(c_2) \arrow[r, "\Phi_{c_2}"] & F_2(c_2) \\
F_1(c_1) \arrow[u, "F_1(\gamma)"] \arrow[r, "\Phi_{c_1}"'] & F_2(c_1) \arrow[u, "F_2(\gamma)"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{D} \\
\mathcal{C} \arrow[u, bend left=60, "F_1"] \arrow[u, bend right=60, "F_2"'] \arrow[u, phantom, "\overset{\Phi}{\Rightarrow}"]
\end{tikzcd}$$

The picture on the right is called a "2-cell diagram." It hints at the close relationship between category theory and topology.

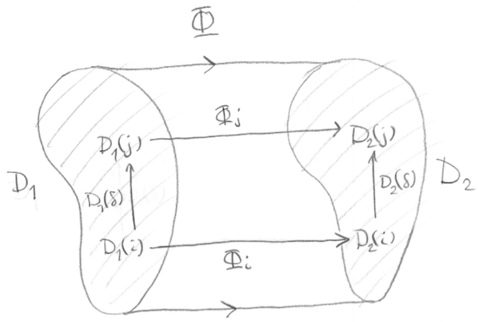
Let $\mathcal{D}^{\mathcal{C}}$ denote the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations between them. One can check that this forms a category (called a *functor category*).[6] Given $F_1, F_2 \in \mathcal{D}^{\mathcal{C}}$ we say that F_1 and F_2 are *naturally isomorphic* if they are isomorphic in $\mathcal{D}^{\mathcal{C}}$, i.e., if there exists a pair of natural transformations $\Phi : F_1 \Rightarrow F_2$ and $\Psi : F_2 \Rightarrow F_1$ such that $\Psi \circ \Phi = \mathsf{id}_{F_1}$ and $\Psi \circ \Phi = \mathsf{id}_{F_2}$ are the *identity natural transformations*. In this case we will write $F_1 \cong F_2$ and we will say that Φ and $\Phi^{-1} := \Psi$ are *natural isomorphisms*. ///

Exercise: Explicitly define the composition of natural transformations mentioned above, and check that $\mathcal{D}^{\mathcal{C}}$ is a category. Prove that a natural isomorphism $\Phi : F_1 \overset{\sim}{\Rightarrow} F_2$ is the same thing as a natural transformation $\Phi : F_1 \Rightarrow F_2$ in which each arrow $\Phi_c : F_1(c) \to F_2(c)$ is invertible. That is, prove that there exists a natural transformation $\Phi^{-1} : F_2 \Rightarrow F_1$ satisfying $(\Phi^{-1})_c = (\Phi_c)^{-1}$ for all $c \in \mathcal{C}$.

[Remark: The concept of "natural transformation" was invented by Eilenberg and Mac Lane in 1945. They viewed this as the central concept of their theory, with "functors" and "categories" playing only a secondary role.]

To develop some intuition for this definition, let $\mathcal{I}$ be a small category and let $\mathcal{C}$ be any category. We have previously referred to functors $D : \mathcal{I} \to \mathcal{C}$ as "diagrams of shape $\mathcal{I}$ in $\mathcal{C}$." Now we can think of $\mathcal{C}^{\mathcal{I}}$ as a *category of diagrams*. Given two such diagrams $D_1, D_2 \in \mathcal{C}^{\mathcal{I}}$, we visualize a natural transformation $\Phi : D_1 \Rightarrow D_2$ as a "cylinder":

[6] Warning: If $\mathcal{C}$ is not "small" then the "hom sets" in the category $\mathcal{D}^{\mathcal{C}}$ are too big to be sets, thus $\mathcal{D}^{\mathcal{C}}$ is not "locally small." But it's still perfectly nice.



The diagrams D_1 and D_2 need not be commutative, but if they are then the whole cylinder is commutative.

You probably recognize this picture because we already saw a version of it when we discussed limits and colimits. To make this precise, we define for each object $c \in \mathcal{C}$ the *constant diagram* $c^{\mathcal{I}} : \mathcal{I} \to \mathcal{C}$ that sends each object of $\mathcal{I}$ to c and each arrow of $\mathcal{I}$ to id_c.

Exercise: Let $c^{\mathcal{I}} : \mathcal{I} \to \mathcal{C}$ be a constant diagram and let $D : \mathcal{I} \to \mathcal{C}$ be any diagram. We previously defined a "cone under D" as a pair $(c, (\varphi_i)_{i \in \mathcal{I}})$ such that for all $\delta : i \to j$ in $\mathcal{I}$ we have $\varphi_j = D(\delta) \circ \varphi_i$. Verify that this concept is equivalent to a natural transformation $\varphi : c^{\mathcal{I}} \Rightarrow D$. Dually, a "cone over D" is the same as a natural transformation $\varphi : D \Rightarrow c^{\mathcal{I}}$.

Thus we can rephrase the definition of limits/colimits as follows.

Fancy Definition of Limit/Colimit. Consider a diagram $D : \mathcal{I} \to \mathcal{C}$. The *limit of* D, if it exists, consists of an object $\ell \in \mathcal{C}$ and a canonical natural transformation $\Lambda : \ell^{\mathcal{I}} \Rightarrow D$ such that for each object $c \in \mathcal{C}$ and natural transformation $\Phi : c^{\mathcal{I}} \Rightarrow D$ **there exists a unique natural transformation** $\varphi^{\mathcal{I}} : c^{\mathcal{I}} \Rightarrow \ell^{\mathcal{I}}$ (i.e., a unique arrow $\varphi : c \to \ell$) making the following diagram in $\mathcal{C}^{\mathcal{I}}$ commute:

$$\begin{tikzcd} & D & \\ c^{\mathcal{I}} \arrow[ur, Rightarrow, "\Phi"] \arrow[rr, Rightarrow, "\exists!\, \varphi^{\mathcal{I}}"'] & & \ell^{\mathcal{I}} \arrow[ul, Rightarrow, "\Lambda"'] \end{tikzcd}$$

The fancy definition of colimits is similar. ///

Exercise: Verify that this is equivalent to the original definition of limits/colimits.

Now we need just one more concept before I can state the fancy definition of adjoint functors.

Definition of Hom Functors. Let $\mathcal{C}$ be a category.[7] For each object $c \in \mathcal{C}$ the mapping $d \mapsto \mathsf{Hom}_{\mathcal{C}}(c, d)$ defines a functor from $\mathcal{C}$ to the category of sets Set. We denote it by

$$H^c := \mathsf{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathsf{Set}.$$

To define the action of H^c on arrows, consider any $\delta : d_1 \to d_2$ in $\mathcal{C}$. Then we must have a function $H^c(\delta) : H^c(d_1) \to H^c(d_2)$, i.e., a function $H^c(\delta) : \mathsf{Hom}_{\mathcal{C}}(c, d_1) \to \mathsf{Hom}_{\mathcal{C}}(c, d_2)$. There is only one way to define this:

$$H^c(\delta)(\varphi) := \delta \circ \varphi.$$

Similarly, for each arrow $\delta : c_1 \to c_2$ we can define a function $H_c(\delta) : \mathsf{Hom}_{\mathcal{C}}(d_2, c) \to \mathsf{Hom}_{\mathcal{C}}(d_1, c)$ by $H_c(\delta)(\varphi) := \varphi \circ \delta$. This again defines a functor into sets, but this time it is from the *opposite category* $\mathcal{C}^{\mathsf{op}}$ (which is defined by reversing all arrows in $\mathcal{C}$):

$$H_c := \mathsf{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}.$$

Finally, we can put these two functors together to obtain the *hom bifunctor*

$$\mathsf{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set},$$

which sends each pair of arrows $(\gamma : c_2 \to c_1, \delta : d_1 \to d_2)$ to the function

$$\mathsf{Hom}_{\mathcal{C}}(\gamma, \delta) : \mathsf{Hom}_{\mathcal{C}}(c_1, d_1) \to \mathsf{Hom}_{\mathcal{C}}(c_2, d_2)$$

defined by $\varphi \mapsto \delta \circ \varphi \circ \gamma$. The *product category* $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ is defined in the most obvious way. ///

Exercise: I skipped over several details in this definition. Check that everything makes sense. In particular, you should work out the definitions of "product categories" and "bifunctors." The only non-obvious part of the definition is the fact that a bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ should satisfy a commutative square of the form

$$\begin{tikzcd}
F(c_2, d_1) \arrow[r, "F(c_2,\delta)"] & F(c_2, d_2) \\
F(c_1, d_1) \arrow[u, "F(\gamma,d_1)"] \arrow[r, "F(c_1,\delta)"'] & F(c_1, d_2) \arrow[u, "F(\gamma,d_2)"']
\end{tikzcd}$$

for each pair of arrows $\gamma : c_1 \to c_2$ and $\delta : d_1 \to d_2$. ///

Now we are ready.

[7] locally small, I guess

Fancy Definition of Adjoint Functors. Let $\mathcal{C},\mathcal{D}$ be categories[8] and consider a pair of functors $L:\mathcal{C}\rightleftarrows\mathcal{D}:R$. By composing these with the hom bifunctors $\mathsf{Hom}_{\mathcal{C}}(-,-)$ and $\mathsf{Hom}_{\mathcal{D}}(-,-)$ we obtain two parallel bifunctors:

$$\mathsf{Hom}_{\mathcal{C}}(-,R(-)):\mathcal{C}^{\mathsf{op}}\times\mathcal{D}\to\mathsf{Set}\quad\text{and}\quad\mathsf{Hom}_{\mathcal{D}}(L(-),-):\mathcal{C}^{\mathsf{op}}\times\mathcal{D}\to\mathsf{Set}.$$

We say that $L:\mathcal{C}\rightleftarrows\mathcal{D}:R$ is an *adjunction* if the two bifunctors are naturally isomorphic:

$$\mathsf{Hom}_{\mathcal{C}}(-,R(-))\cong\mathsf{Hom}_{D}(L(-),-).$$

///

Exercise: Verify that the fancy definition is equivalent to the explicit definition.

I think you will agree that the definition looks good now. More importantly, this definition will lead us to proofs for the uniqueness of adjoints and the RAPL theorem.

1.5 From Vector Spaces to Categories

We began by thinking of categories as "posets with extra arrows." This analogy gives excellent intuition for the general facts about adjoint functors. However, our intuition from posets is insufficient to actually **prove** anything about adjoint functors.

To complete the proofs we will switch to a new analogy between **categories** and **vector spaces**. Let V be a vector space over a field K and let V^* be the *dual space* consisting of K-linear functions $V\to K$. Now consider any K-bilinear function $\langle -,-\rangle:V\times V\to K$. We say that the function $\langle -,-\rangle$ is *non-degenerate* in both coordinates if we have

$$\begin{aligned}\langle u_1,v\rangle=\langle u_2,v\rangle\text{ for all }v\in V&\quad\Longrightarrow\quad u_1=u_2,\\ \langle u,v_1\rangle=\langle u,v_2\rangle\text{ for all }u\in V&\quad\Longrightarrow\quad v_1=v_2.\end{aligned}$$

We say that two K-linear operators $L:V\rightleftarrows V:R$ define an *adjunction* with respect to $\langle -,-\rangle$ if, for all vectors $u,v\in V$, we have

$$\langle u,R(v)\rangle=\langle L(u),v\rangle.$$

Uniqueness of Adjoint Operators. Let $L\dashv R$ be an adjoint pair of operators with respect to a non-degenerate bilinear function $\langle -,-\rangle:V\times V\to K$. Then each of L and R determines the other uniquely. ///

[8] locally small

Proof: To show that R determines L, suppose that $L' \dashv R$ is another adjoint pair. Thus, for all vectors $u, v \in V$ we have

$$\langle L(u), v\rangle = \langle u, R(v)\rangle = \langle L'(u), v\rangle.$$

Now consider any vector $u \in V$. The non-degeneracy of $\langle -,-\rangle$ tells us that

$$\langle L(u), v\rangle = \langle L'(u), v\rangle \text{ for all } v \in V \quad \Longrightarrow \quad L(u) = L'(u),$$

and since this is true for all $u \in V$ we conclude that $L = L'$. □

The next theorem tells us that under certain conditions adjoint operators are continuous.

RAPL for Operators. Suppose that the function $\langle -,-\rangle : V \times V \to K$ is non-degenerate and continuous (in some appropriate sense). Now let $T : V \to V$ be any linear operator. If T has a left or a right adjoint, then T is continuous (in some appropriate sense). ///

Proof: Suppose that $T : V \to V$ has a left adjoint $L \dashv T$, and suppose that the sequence of vectors $v_i \in V$ has a limit $\mathsf{lim}_i v_i \in V$. Furthermore, suppose that the limit $\mathsf{lim}_i T(v_i) \in V$ exists. Then for each $u \in V$, the continuity of $\langle -,-\rangle$ in the second coordinate tells us that

$$\begin{aligned}
\langle u, T(\mathsf{lim}_i v_i)\rangle &= \langle L(u), \mathsf{lim}_i v_i\rangle \\
&= \mathsf{lim}_i \langle L(u), v_i\rangle \\
&= \mathsf{lim}_i \langle u, T(v_i)\rangle \\
&= \langle u, \mathsf{lim}_i T(v_i)\rangle.
\end{aligned}$$

Since this is true for all $u \in V$, non-degeneracy gives $T(\mathsf{lim}_i v_i) = \mathsf{lim}_i T(v_i)$. □

[Remark: Don't take this too literally; it's just for motivation. The theorem can be made rigorous if we work with topological vector spaces. If $(V, \|-\|)$ is a normed (real or complex) vector space, then an operator $T : V \to V$ is bounded if and only if it is continuous. Furthermore, if $(V, \langle -,-\rangle)$ is a Hilbert space then an operator $T : V \to V$ having an adjoint is necessarily bounded, hence continuous. Many theorems of category theory have direct analogues in functional analysis. After all, Grothendieck began as a functional analyst.]

We can summarize these two results as follows. Let $\langle -,-\rangle : V \times V \to K$ be a K-bilinear function. Then for each vector $v \in V$ we have two elements of the dual space $H^v, H_v \in V^*$ defined by

$$\begin{aligned}
H^v &:= \langle v, -\rangle : V \to K, \\
H_v &:= \langle -, v\rangle : V \to K.
\end{aligned}$$

The mappings $v \mapsto H^v$ and $v \mapsto H_v$ thus define two K-linear functions from V to V^*:

$$H^{(-)} : V \to V^* \quad \text{and} \quad H_{(-)} : V \to V^*.$$

Furthermore, if the function $\langle -, - \rangle$ is non-degenerate and continuous then the functions $H^{(-)}, H_{(-)} : V \to V^*$ are both **injective** and **continuous**. Now here is a dictionary connecting all of this back to categories:

vector space V over a field K	=	category $\mathcal{C}$
the field K	=	the category Set
K-linear function	=	functor
the dual space V^*	=	the category $\mathsf{Set}^{\mathcal{C}}$
bilinear pairing $V \times V \to K$	=	hom bifunctor $\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$
adjoint linear functions	=	adjoint functors
non-dengeneracy and continuity	=	?

In the next two sections we will flesh out the details of this analogy in order to prove the uniqueness of adjoints and the RAPL theorem for categories. There are quite a few details involved, but it is fair to summarize the whole story by saying that the hom bifunctor

$$\mathsf{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$$

behaves like a "non-degenerate and continuous bilinear function."

We will work at a level of generality that is high enough to make the proofs conceptual (and hence memorable), but low enough that a careful reader can actually follow the details and make sure that everything works. Non-careful readers can just skim the proofs to get a sense of their complexity. (I assure you that I'm not hiding anything.)

1.6 Uniqueness of Adjoints

In this section we will prove that adjoint functors determine each other up to isomorphism. The key tool is the concept of an "embedding of categories." In particular, the hom bifunctor $\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$ induces two "Yoneda embeddings"

$$H^{(-)} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}^{\mathcal{C}} \quad \text{and} \quad H_{(-)} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}.$$

These are analogous to the two embeddings of a vector space V into its dual space that are induced by a non-degenerate bilinear function $\langle -, - \rangle : V \times V \to K$.

Definition of Embedding of Categories. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ consists of:

- An object function $F : \mathsf{Obj}(\mathcal{C}) \to \mathsf{Obj}(\mathcal{D})$,

- For each pair of objects $c_1, c_2 \in \mathcal{C}$, a hom set function:

$$F : \mathsf{Hom}_{\mathcal{C}}(c_1, c_2) \to \mathsf{Hom}_{\mathcal{D}}(F(c_1), F(c_2)).$$

We say that F is a *full functor* when the hom set functions are **surjective**, and we say that F is a *faithful functor* when the hom set functions are **injective**. If the hom set functions are **bijective** then we say that F is a *fully faithful functor*, or an *embedding of categories.* ///

An embedding is in some sense the correct notion of an "injective functor." If $F : \mathcal{C} \to \mathcal{D}$ is an embedding, then the object function $F : \mathsf{Obj}(\mathcal{C}) \to \mathsf{Obj}(\mathcal{D})$ is **not necessarily injective**, but it is "injective up to isomorphism." This agrees with the general philosophy of category theory, i.e., that we should only care about objects up to isomorphism.

Embedding Lemma. Let $F : \mathcal{C} \to \mathcal{D}$ be an embedding of categories. Then F is *essentially injective* in the sense that for all objects $c_1, c_2 \in \mathcal{C}$ we have

$$c_1 \cong c_2 \text{ in } \mathcal{C} \quad \Longleftrightarrow \quad F(c_1) \cong F(c_2) \text{ in } \mathcal{D}.$$

Furthermore, F is *essentially monic*[9] in the sense that for all functors $G_1, G_2 : \mathcal{B} \to \mathcal{C}$ we have

$$G_1 \cong G_2 \text{ in } \mathcal{C}^{\mathcal{B}} \quad \Longleftrightarrow \quad F \circ G_1 \cong F \circ G_2 \text{ in } \mathcal{D}^{\mathcal{B}}.$$

///

Proof: Let $F : \mathcal{C} \to \mathcal{D}$ be full and faithful, i.e., bijective on hom sets.

To prove that F is essentially injective, suppose that $\alpha : c_1 \leftrightarrow c_2 : \beta$ is an isomorphism in $\mathcal{C}$ and apply F to obtain arrows $F(\alpha) : F(c_1) \rightleftarrows F(c_2) : F(\beta)$ in $\mathcal{D}$. Then by the functoriality of F we have

$$\begin{aligned} F(\alpha) \circ F(\beta) = F(\alpha \circ \beta) = F(\mathsf{id}_{c_2}) = \mathsf{id}_{F(c_2)}, \\ F(\beta) \circ F(\alpha) = F(\beta \circ \alpha) = F(\mathsf{id}_{c_1}) = \mathsf{id}_{F(c_1)}, \end{aligned}$$

which implies that $F(\alpha) : F(c_1) \leftrightarrow F(c_2) : F(\beta)$ is an isomorphism in $\mathcal{D}$. Conversely, suppose that $\alpha' : F(c_1) \leftrightarrow F(c_2) : \beta'$ is an isomorphism in $\mathcal{D}$. By the **fullness** of F there exist arrows $\alpha : c_1 \rightleftarrows c_2 : \beta$ such that $F(\alpha) = \alpha'$ and $F(\beta) = \beta'$, and by the functoriality of F we have

$$\begin{aligned} F(\alpha \circ \beta) = F(\alpha) \circ F(\beta) = \alpha' \circ \beta' = \mathsf{id}_{F(c_2)} = F(\mathsf{id}_{c_2}), \\ F(\beta \circ \alpha) = F(\beta) \circ F(\alpha) = \beta' \circ \alpha' = \mathsf{id}_{F(c_1)} = F(\mathsf{id}_{c_1}). \end{aligned}$$

Then by the **faithfulness** of F we have $\alpha \circ \beta = \mathsf{id}_{c_2}$ and $\beta \circ \alpha = \mathsf{id}_{c_1}$, which implies that $\alpha : c_1 \leftrightarrow c_2 : \beta$ is an isomorphism in $\mathcal{C}$.

[9] The word "monic" will be explained in the next chapter.

To prove that F is essentially monic, let $G_1, G_2 : \mathcal{B} \to \mathcal{C}$ be any functors and suppose that we have a natural isomorphism $\Phi : G_1 \overset{\sim}{\Rightarrow} G_2$. This means that for each object $b \in \mathcal{B}$ we have an isomorphism $\Phi_b : G_1(b) \overset{\sim}{\to} G_2(b)$ in $\mathcal{C}$ and for each arrow $\beta : b_1 \to b_2$ in $\mathcal{B}$ we have a commutative square:

$$\begin{tikzcd}
G_1(b_2) \arrow[r, "\Phi_{b_2}", "\sim"'] & G_2(b_2) \\
G_1(b_1) \arrow[u, "G_1(\beta)"] \arrow[r, "\sim", "\Phi_{b_1}"'] & G_2(b_1) \arrow[u, "G_2(\beta)"']
\end{tikzcd}$$

Recall from the previous argument that **any** functor sends isomorphisms to isomorphisms, thus by the functoriality of F we obtain another commutative square

$$\begin{tikzcd}
F(G_1(b_2)) \arrow[r, "F(\Phi_{b_2})", "\sim"'] & F(G_2(b_2)) \\
F(G_1(b_1)) \arrow[u, "F(G_1(\beta))"] \arrow[r, "\sim", "F(\Phi_{b_1})"'] & F(G_2(b_1)) \arrow[u, "F(G_2(\beta))"']
\end{tikzcd}$$

in which the horizontal arrows are isomorphisms in $\mathcal{D}$. In other words, the assignment $F(\Phi)_b := F(\Phi_b)$ defines a natural isomorphism $F(\Phi) : F \circ G_1 \overset{\sim}{\Rightarrow} F \circ G_2$.

Conversely, suppose that we have a natural isomorphism $\Phi' : F \circ G_1 \overset{\sim}{\Rightarrow} F \circ G_2$, meaning that for each object $b \in \mathcal{B}$ we have an isomorphism $\Phi'_b : F(G_1(b)) \overset{\sim}{\to} F(G_2(b))$ in $\mathcal{C}$, and for each arrow $\beta : b_1 \to b_2$ in $\mathcal{B}$ we have a commutative square:

$$\begin{tikzcd}
F(G_1(b_2)) \arrow[r, "\Phi'_{b_2}", "\sim"'] & F(G_2(b_2)) \\
F(G_1(b_1)) \arrow[u, "F(G_1(\beta))"] \arrow[r, "\sim", "\Phi'_{b_1}"'] & F(G_2(b_1)) \arrow[u, "F(G_2(\beta))"']
\end{tikzcd}$$

Since F is **fully faithful**, we know from the previous result that for each $b \in \mathcal{B}$ there exists an isomorphism $\Phi_b : G_1(b) \overset{\sim}{\to} G_2(b)$ in $\mathcal{C}$ with the property $\Phi_b = F(\Phi'_b)$. Then by the functoriality of F and the commutativity of the above square we have

$$\begin{aligned}
F(\Phi_{b_2} \circ G_1(\beta)) &= F(\Phi_{b_2}) \circ F(G_1(\beta)) \\
&= \Phi'_{b_2} \circ F(G_1(\beta)) \\
&= F(G_2(\beta)) \circ \Phi'_{b_1} \\
&= F(G_2(\beta)) \circ F(\Phi_{b_1}) \\
&= F(G_2(\beta) \circ \Phi'_{b_1}),
\end{aligned}$$

and by the **faithfulness** of F it follows that $\Phi_{b_2} \circ G_1(\beta) = G_2(\beta) = \Phi_{b_1}$. We conclude that the following square commutes:

$$\begin{tikzcd} G_1(b_2) \arrow[r, "\Phi_{b_2}", "\sim"'] & G_2(b_2) \\ G_1(b_1) \arrow[u, "G_1(\beta)"] \arrow[r, "\sim", "\Phi_{b_1}"'] & G_2(b_1) \arrow[u, "G_2(\beta)"'] \end{tikzcd}$$

In other words, the arrows Φ_b assemble into a natural isomorphism $\Phi : G_1 \overset{\sim}{\Rightarrow} G_2$. □

So embeddings of categories are quite general and have nice properties. Now we will define the specific embeddings that we need.

Lemma (The Yoneda Embeddings). Let $\mathcal{C}$ be a category and recall that for each object $c \in \mathcal{C}$ we have two hom functors

$$H^c = \mathsf{Hom}_{\mathcal{C}}(c,-) : \mathcal{C} \to \mathsf{Set} \quad \text{and} \quad H_c : \mathsf{Hom}_{\mathcal{C}}(-,c) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}.$$

I claim that the mappings $c \mapsto H^c$ and $c \mapsto H_c$ define two embeddings of categories:

$$H^{(-)} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}^{\mathcal{C}} \quad \text{and} \quad H_{(-)} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}.$$

///

We will prove that $H^{(-)}$ is an embedding. Then the fact that $H_{(-)}$ is an embedding follows by substituting $\mathcal{C}^{\mathsf{op}}$ in place of $\mathcal{C}$.

Proof: The proof has three steps.

(1) $H^{(-)}$ **is a Functor.** For each arrow $\gamma : c_1 \to c_2$ in $\mathcal{C}^{\mathsf{op}}$ (i.e., for each arrow $\gamma : c_2 \to c_1$ in $\mathcal{C}$) we must define a natural transformation $H^{(-)}(\gamma) : H^{(-)}(c_1) \Rightarrow H^{(-)}(c_2)$, i.e., a natural transformation $H^\gamma : H^{c_1} \Rightarrow H^{c_2}$. And this means that for each object $d \in \mathcal{C}$ we must define an arrow $(H^\gamma)_d : H^{c_1}(d) \to H^{c_2}(d)$, i.e., a function $(H^\gamma)_d : \mathsf{Hom}_{\mathcal{C}}(c_1,d) \to \mathsf{Hom}_{\mathcal{C}}(c_2,d)$. Note that the **only possible choice** is to send each arrow $\alpha : c_1 \to d$ to the arrow $\alpha \circ \gamma : c_2 \to d$. In other words, for all $d \in \mathcal{C}$ we define

$$(H^\gamma)_d := (-) \circ \gamma.$$

To check that this is indeed a natural transformation $H^\gamma : H^{c_1} \Rightarrow H^{c_2}$, consider any arrow $\delta : d_1 \to d_2$ in $\mathcal{C}$ and observe that the following diagram commutes:

$$\begin{tikzcd} H^{c_1}(d_2) \arrow[r, "(H^\gamma)_{d_2}"] & H^{c_2}(d_2) \\ H^{c_1}(d_1) \arrow[u, "H^{c_1}(\delta)"] \arrow[r, "(H^\gamma)_{d_1}"'] & H^{c_2}(d_1) \arrow[u, "H^{c_2}(\delta)"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Hom}_{\mathcal{C}}(c_1,d_2) \arrow[r, "(-)\circ\gamma"] & \mathsf{Hom}_{\mathcal{C}}(c_2,d_2) \\ \mathsf{Hom}_{\mathcal{C}}(c_1,d_1) \arrow[u, "\delta\circ(-)"] \arrow[r, "(-)\circ\gamma"'] & \mathsf{Hom}_{\mathcal{C}}(c_2,d_1) \arrow[u, "\delta\circ(-)"'] \end{tikzcd}$$

Indeed, the commutativity of this square is just the associative axiom for composition of arrows. Thus we have defined the action of $H^{(-)}$ on arrows in $\mathcal{C}^{\mathsf{op}}$. To see that this defines a **functor** $\mathcal{C}^{\mathsf{op}} \to \mathsf{Set}^{\mathcal{C}}$, we need to show that for any composible arrows $\gamma_1, \gamma_2 \in \mathsf{Arr}(\mathcal{C})$ we have $H^{\gamma_1 \circ \gamma_2} = H^{\gamma_2} \circ H^{\gamma_1}$. So consider any arrows $\gamma_1 : c_2 \to c_1$ and $\gamma_2 : c_3 \to c_2$. Then for all objects $d \in \mathcal{C}$ and for all arrows $\delta : c_1 \to d$ we have

$$\begin{aligned}
[H^{\gamma_2} \circ H^{\gamma_1}]_d(\delta) &= [(H^{\gamma_2})_d \circ (H^{\gamma_1})_d](\delta) \\
&= (H^{\gamma_2})_d[(H^{\gamma_1})_d(\delta)] \\
&= (H^{\gamma_2})_d(\delta \circ \gamma_1) \\
&= (\delta \circ \gamma_1) \circ \gamma_2 \\
&= \delta \circ (\gamma_1 \circ \gamma_2) \\
&= (H^{\gamma_1 \circ \gamma_2})_d(\delta).
\end{aligned}$$

Since this holds for all $\delta \in H^{c_1}(d)$ we have $[H^{\gamma_2} \circ H^{\gamma_1}]_d = (H^{\gamma_1 \circ \gamma_2})_d$, and then since this holds for all $d \in \mathcal{C}$ we conclude that $H^{\gamma_1 \circ \gamma_2} = H^{\gamma_2} \circ H^{\gamma_1}$ as desired.

(2) $H^{(-)}$ **is Faithful.** For each pair of objects $c_1, c_2 \in \mathcal{C}$ we want to show that the function

$$H^{(-)} : \mathsf{Hom}_{\mathcal{C}^{\mathsf{op}}}(c_1, c_2) \to \mathsf{Hom}_{\mathsf{Set}^{\mathcal{C}}}(H^{c_1}, H^{c_2})$$

defined in part (1) is injective. So consider any two arrows $\alpha, \beta : c_2 \to c_1$ in $\mathcal{C}$ and suppose that we have $H^\alpha = H^\beta$ as natural transformations. In this case we want to show that $\alpha = \beta$.

Recall that for all objects $d \in \mathcal{C}$ and all arrows $\delta \in H^{c_1}(d)$ we have defined $(H^\alpha)_d(\delta) = \delta \circ \alpha$. Since $H^\alpha = H^\beta$, this means that

$$\delta \circ \alpha = (H^\alpha)_d(\delta) = (H^\beta)_d(\delta) = \delta \circ \beta.$$

Now we just take $d = c_1$ and $\delta = \mathsf{id}_{c_1}$ to obtain

$$\alpha = (\mathsf{id}_{c_1} \circ \alpha) = (\mathsf{id}_{c_1} \circ \beta) = \beta,$$

as desired.

(3) $H^{(-)}$ **is Full.** For each pair of objects $c_1, c_2 \in \mathcal{C}$ we want to show that the function

$$H^{(-)} : \mathsf{Hom}_{\mathcal{C}^{\mathsf{op}}}(c_1, c_2) \to \mathsf{Hom}_{\mathsf{Set}^{\mathcal{C}}}(H^{c_1}, H^{c_2})$$

is surjective. So consider any natural transformation $\Phi : H^{c_1} \Rightarrow H^{c_2}$. In this case we want to find an arrow $\varphi : c_2 \to c_1$ with the property $H^\varphi = \Phi$. Where can we find such an arrow?

By definition of "natural transformation" we have a function $\Phi_d : H^{c_1}(d) \to H^{c_2}(d)$ for each object $d \in \mathcal{C}$, and for each arrow $\delta : d_1 \to d_2$ we know that the following square commutes:

$$\begin{tikzcd}
H^{c_1}(d_2) \arrow[r, "\Phi_{d_2}"] & H^{c_2}(d_2) \\
H^{c_1}(d_1) \arrow[u, "H^{c_1}(\delta)"] \arrow[r, "\Phi_{d_1}"'] & H^{c_2}(d_1) \arrow[u, "H^{c_2}(\delta)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c_1, d_2) \arrow[r, "\Phi_{d_2}"] & \mathsf{Hom}_{\mathcal{C}}(c_2, d_2) \\
\mathsf{Hom}_{\mathcal{C}}(c_1, d_1) \arrow[u, "\delta \circ (-)"] \arrow[r, "\Phi_{d_1}"'] & \mathsf{Hom}_{\mathcal{C}}(c_2, d_1) \arrow[u, "\delta \circ (-)"']
\end{tikzcd}$$

Note that the category $\mathcal{C}$ might have very few arrows. (Indeed, $\mathcal{C}$ might be a discrete category, i.e., with only the identity arrows.) This suggests that our **only possible choice** is to evaluate the function $\Phi_{c_1} : H^{c_1}(c_1) \to H^{c_2}(c_1)$ at the identity arrow to obtain an arrow $\varphi := \Phi_{c_1}(\mathsf{id}_{c_1}) \in H^{c_2}(c_1)$. Now **hopefully** we have $H^\varphi = \Phi$ (otherwise the theorem is not true). To check this, consider any element $d \in \mathcal{C}$ and any arrow $\delta : c_1 \to d$. Substituting this δ into the above diagram gives a commutative square:

$$\begin{tikzcd}
H^{c_1}(d) \arrow[r, "\Phi_d"] & H^{c_2}(d) \\
H^{c_1}(c_1) \arrow[u, "H^{c_1}(\delta)"] \arrow[r, "\Phi_{c_1}"'] & H^{c_2}(c_1) \arrow[u, "H^{c_2}(\delta)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c_1, d) \arrow[r, "\Phi_d"] & \mathsf{Hom}_{\mathcal{C}}(c_2, d) \\
\mathsf{Hom}_{\mathcal{C}}(c_1, c_1) \arrow[u, "\delta\circ(-)"] \arrow[r, "\Phi_{c_1}"'] & \mathsf{Hom}_{\mathcal{C}}(c_2, c_1) \arrow[u, "\delta\circ(-)"']
\end{tikzcd}$$

Then by following the arrow $\mathsf{id}_{c_1} \in H^{c_1}(c_1)$ around the square in two different ways, and by using the definition $(H^\varphi)_d(\delta) := \delta \circ \varphi$ from part (1), we obtain

$$\begin{aligned}
\Phi_d(\delta \circ \mathsf{id}_{c_1}) &= \delta \circ \Phi_{c_1}(\mathsf{id}_{c_1}) \\
\Phi_d(\delta) &= \delta \circ \varphi \\
\Phi_d(\delta) &= (H^\varphi)_d(\delta).
\end{aligned}$$

Since this holds for all arrows $\delta \in H^{c_1}(d)$ we have $\Phi_d = (H^\varphi)_d$, and then since this holds for all objects $d \in \mathcal{C}$ we conclude that $\Phi = H^\varphi$ as desired. □

Let's pause to apply the Embedding Lemma to the Yoneda embedding $H^{(-)} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}^{\mathcal{C}}$. The fact that $H^{(-)}$ is "essentially injective" means that for all objects $c_1, c_2 \in \mathcal{C}$ we have

$$c_1 \cong c_2, \text{ in } \mathcal{C} \quad \Longleftrightarrow \quad H^{c_1} \cong H^{c_2} \text{ in } \mathsf{Set}^{\mathcal{C}}.$$

[Note that $c_1 \cong c_2$ in $\mathcal{C}$ if and only if $c_1 \cong c_2$ in $\mathcal{C}^{\mathsf{op}}$.] This useful fact is the starting point for many areas of modern mathematics. It tells us that if we know all the information about arrows pointing to (or from) an object $c \in \mathcal{C}$, then we know the object up to isomorphism. In some sense this is a justification for the philosophy of category theory.

The Embedding Lemma also implies that the Yoneda embedding is "essentially monic," i.e., "left-cancellable up to natural isomorphism." We will use this fact to prove the uniqueness of adjoints.

Theorem (Uniqueness of Adjoints). Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction of categories. Then each of L and R determines the other up to natural isomorphism. ///

Proof: We will prove that R determines L. The other direction is similar.

So suppose that $L' : \mathcal{C} \rightleftarrows \mathcal{D} : R$ is another adjunction. Then we have two bijections

$$\mathsf{Hom}_{\mathcal{D}}(L(c), d) \cong \mathsf{Hom}_{\mathcal{C}}(c, R(d)) \cong \mathsf{Hom}_{\mathcal{D}}(L'(c), d)$$

that are natural in $(c,d)\in\mathcal{C}^{\mathsf{op}}\times\mathcal{D}$, and by composing them we obtain a bijection

$$\mathsf{Hom}_{\mathcal{D}}(L(c),d)\cong\mathsf{Hom}_{\mathcal{D}}(L'(c),d)$$

that is natural in $(c,d)\in\mathcal{C}^{\mathsf{op}}\times\mathcal{D}$. [Exercise: Why does this work?] Naturality in $d\in\mathcal{D}$ means that for each $c\in\mathcal{C}^{\mathsf{op}}$ we have a natural isomorphism of functors $\mathsf{Hom}_{\mathcal{D}}(L(c),-)\cong\mathsf{Hom}_{\mathcal{D}}(L'(c),-)$ in the category $\mathsf{Set}^{\mathcal{D}}$.

Now let us compose the functor $L:\mathcal{C}^{\mathsf{op}}\to\mathcal{D}^{\mathsf{op}}$ (this is just a trick) with the Yoneda embedding $H^{(-)}:\mathcal{D}^{\mathsf{op}}\to\mathsf{Set}^{\mathcal{D}}$ to obtain a functor $(H^{(-)}\circ L):\mathcal{C}^{\mathsf{op}}\to\mathsf{Set}^{\mathcal{D}}$. Observe that if we apply the functor $H^{(-)}\circ L$ to an object $c\in\mathcal{C}^{\mathsf{op}}$ then we obtain the functor

$$(H^{(-)}\circ L)(c)=\mathsf{Hom}_{\mathcal{D}}(L(c),-)\in\mathsf{Set}^{\mathcal{D}}.$$

Thus, naturality in $c\in\mathcal{C}^{\mathsf{op}}$ means exactly that we have a natural isomorphism of functors $(H^{(-)}\circ L)\cong(H^{(-)}\circ L')$ in the category $(\mathsf{Set}^{\mathcal{D}})^{\mathcal{C}^{\mathsf{op}}}$. Finally, since the "Yoneda embedding" $H^{(-)}$ is an embedding of categories, the Embedding Lemma tells us that we can cancel $H^{(-)}$ on the left to obtain a natural isomorphism:

$$(H^{(-)}\circ L)\cong(H^{(-)}\circ L')\text{ in }(\mathsf{Set}^{\mathcal{D}})^{\mathcal{C}^{\mathsf{op}}}\quad\Longrightarrow\quad L\cong L'\text{ in }(\mathcal{D}^{\mathsf{op}})^{\mathcal{C}^{\mathsf{op}}}.$$

In other words, we have $L\cong L'$ in $\mathcal{D}^{\mathcal{C}}$. □

This theorem allows us to work with adjoint functors **as if they exist**, even when we don't know how to define them. I call it the "I don't care if the tensor product really exists" theorem. (See the final section of this chapter.)

1.7 RAPL

To prove the RAPL theorem it is helpful to reformulate the definition of limits/colimits in a language that is compatible with the definition of adjoint functors. Recall that a diagram is a diagram is a functor $D:\mathcal{I}\to\mathcal{C}$ is a functor from a small (index) category. If $\mathcal{C}$ is locally small then we have a locally small category $\mathcal{C}^{\mathcal{I}}$ whose objects are diagrams of shape $\mathcal{I}$ and whose arrows are natural transformations. For each object $c\in\mathcal{C}$ we also have the constant diagram $c^{\mathcal{I}}:\mathcal{I}\to\mathcal{C}$ that sends each object $i\in\mathsf{Obj}(\mathcal{I})$ to c and each arrow $\delta\in\mathsf{Arr}(\mathcal{I})$ to id_c.

Exercise: For any category, show that the object map $c\mapsto c^{\mathcal{I}}$ extends to a fully faithful functor $(-)^{\mathcal{I}}:\mathcal{C}\to\mathcal{C}^{\mathcal{I}}$, which we call the *diagonal embedding.*

Recall, further, that a limit of the diagram $D:\mathcal{I}\to\mathcal{C}$ consists of an object $\ell\in\mathcal{C}$ and a natural transformation $\Lambda:\ell^{\mathcal{I}}\Rightarrow D$, together called a "cone under D," such that for any cone $\Phi:c^{\mathcal{I}}\Rightarrow D$ there exists a unique arrow $\varphi:c\to\ell$ making the following triangle commute:

$$\begin{tikzcd} & D & \\ c^{\mathcal{I}} \arrow[ur, Rightarrow, "\Phi"] \arrow[rr, Rightarrow, "\varphi^{\mathcal{I}}"'] & & \ell^{\mathcal{I}} \arrow[ul, Rightarrow, "\Lambda"'] \end{tikzcd}$$

The following lemma reformulates this definition in terms of adjoint functors.

Theorem (Colimit $\dashv$ Diagonal $\dashv$ Limit). Fix a small category $\mathcal{I}$ and a diagram $D:\mathcal{I}\to\mathcal{C}$ of shape $\mathcal{I}$. Then the limit of D, if it exists, consists of an object $\ell\in\mathcal{C}$ and family of bijections

$$\Upsilon_c : \mathsf{Hom}_{\mathcal{C}}(c,\ell) \xrightarrow{\sim} \mathsf{Nat}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, D)$$

that is natural in $c\in\mathcal{C}^{\mathsf{op}}$. In other words, a limit for D is the same thing as an isomorphism

$$\Upsilon : \mathsf{Hom}_{\mathcal{C}}(-,\ell) \overset{\sim}{\Longrightarrow} \mathsf{Nat}_{\mathcal{C}^{\mathcal{I}}}((-)^{\mathcal{I}}, D)$$

in the category $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$. In other, other words, a limit of the diagram D is precisely a "representation of the pre-sheaf of cones under D":

$$H_\ell \cong \mathsf{Nat}((-)^{\mathcal{I}}, D).$$

///

Proof: Suppose we are given such a family of bijections Υ_c. Then the limit object is $\ell\in\mathcal{C}$, but where is the cone $\ell^{\mathcal{I}}\Rightarrow D$? This cone must be an element of the set $\mathsf{Nat}(\ell^{\mathcal{I}}, D)$, which means that there is only one possible choice:

$$\Upsilon_\ell(\mathsf{id}_\ell) : \ell^{\mathcal{I}} \Rightarrow D.$$

To show that this cone has the desired universal property, let $\Phi : c^\ell \Rightarrow D$ be any other cone under D. In this case we want to show that **there exists a unique arrow** $\varphi : c\to\ell$ making the following triangle commute:

$$\begin{tikzcd} & D & \\ c^{\mathcal{I}} \arrow[ur, Rightarrow, "\Phi"] \arrow[rr, Rightarrow, "\varphi^{\mathcal{I}}"'] & & \ell^{\mathcal{I}} \arrow[ul, Rightarrow, "\Upsilon_\ell(\mathsf{id}_\ell)"'] \end{tikzcd}$$

Suppose, hypothetically, that there exists such an arrow $\varphi : c\to\ell$ satisfying $\Phi = \Upsilon_\ell(\mathsf{id}_\ell)\circ\varphi^{\mathcal{I}}$. Then applying the natural transformation Υ to φ gives the following commutative square:

$$\begin{tikzcd} \mathsf{Hom}(c,\ell) \arrow[r, "\Upsilon_c", "\sim"'] & \mathsf{Nat}(c^{\mathcal{I}}, D) \\ \mathsf{Hom}(\ell,\ell) \arrow[u, "(-)\circ\varphi"] \arrow[r, "\sim", "\Upsilon_\ell"'] & \mathsf{Nat}(\ell^{\mathcal{I}}, D) \arrow[u, "(-)\circ\varphi^{\mathcal{I}}"'] \end{tikzcd}$$

And following the element id_ℓ from the bottom left to the top right in two ways gives

$$\begin{aligned} \Upsilon_c(\mathsf{id}_\ell\circ\varphi) &= \Upsilon_\ell(\mathsf{id}_\ell)\circ\varphi^{\mathcal{I}} \\ \Upsilon_c(\varphi) &= \Phi \\ \varphi &= \Upsilon_c^{-1}(\Phi). \end{aligned}$$

We conclude that there exists **at most one such arrow** φ. Finally, to see that the arrow $\varphi := \Upsilon_c^{-1}(\Phi)$ actually does satisfy $\Phi = \Upsilon_\ell(\mathsf{id}_\ell) \circ \varphi^{\mathcal{I}}$, we apply Υ to φ once again to obtain

$$\begin{aligned}
\Upsilon_c(\mathsf{id}_\ell \circ \varphi) &= \Upsilon_\ell(\mathsf{id}_\ell) \circ \varphi^{\mathcal{I}} \\
\Upsilon_c(\varphi) &= \Upsilon_\ell(\mathsf{id}_\ell) \circ \varphi^{\mathcal{I}} \\
\Upsilon_c(\Upsilon_c^{-1}(\Phi)) &= \Upsilon_\ell(\mathsf{id}_\ell) \circ \Upsilon_c^{-1}(\Phi)^{\mathcal{I}} \\
\Phi &= \Upsilon_\ell(\mathsf{id}_\ell) \circ \Upsilon_c^{-1}(\Phi)^{\mathcal{I}},
\end{aligned}$$

as desired. □

Exercise: Prove the other direction of the theorem, i.e., that every ///

1.8 RAPL (Old Version)

To prove the RAPL theorem we must first translate the definition of limit/colimit into a language that is compatible with the definition of adjoint functors.

Recall that a diagram is a functor $D : \mathcal{I} \to \mathcal{C}$ from a small category $\mathcal{I}$. If $\mathcal{C}$ is locally small then we have a locally small category $\mathcal{C}^{\mathcal{I}}$ consisting of diagrams and natural transformations between them. For each object $c \in \mathcal{C}$ we also have the constant diagram $c^{\mathcal{I}} : \mathcal{I} \to \mathcal{C}$ that sends each object $i \in \mathsf{Obj}(\mathcal{I})$ to $c^{\mathcal{I}}(i) := c$ and each arrow $\delta \in \mathsf{Arr}(\mathcal{I})$ to $c^{\mathcal{I}}(\delta) := \mathsf{id}_c$.

It is a general phenomenon that many categorical properties of $\mathcal{C}^{\mathcal{I}}$ are inherited from $\mathcal{C}$. The next lemma collects a few of these properties that we will need later.

Diagram Lemma. Fix a small category $\mathcal{I}$ and locally small categories $\mathcal{C}, \mathcal{D}$. Then:

(i) For any category $\mathcal{C}$, the mapping $c \mapsto c^{\mathcal{I}}$ defines a fully faithful functor $(-)^{\mathcal{I}} : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ which we call the *diagonal embedding*.

(ii) For any functor $F : \mathcal{C} \to \mathcal{D}$ the mapping $F^I(D) := F \circ D$ defines a functor $F^{\mathcal{I}} : \mathcal{C}^{\mathcal{I}} \to \mathcal{D}^{\mathcal{I}}$ with the property that $F(-)^{\mathcal{I}} = F^{\mathcal{I}}((-)^{\mathcal{I}})$.

(iii) Any adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ induces an adjunction $L^{\mathcal{I}} : \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{D}^{\mathcal{I}} : R^{\mathcal{I}}$. That is, we have a natural isomorphism of bifunctors

$$\mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(-, R^{\mathcal{I}}(-)) \cong \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L^{\mathcal{I}}(-), -)$$

from $(\mathcal{C}^{\mathcal{I}})^{\mathsf{op}} \times \mathcal{D}^{\mathcal{I}}$ to Set.

(iv) In particular, naturality in $\mathcal{D}^{\mathcal{I}}$ tells us that for all objects $\ell \in \mathcal{C}$ and all natural transformations $\Lambda : \ell^{\mathcal{I}} \Rightarrow D$ we have a commutative square:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(R(\ell)^{\mathcal{I}}, R^{\mathcal{I}}(D)) \arrow[r, leftrightarrow] & \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(LR(\ell)^{\mathcal{I}}, D) \\
\mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(R(\ell)^{\mathcal{I}}, R(\ell)^{\mathcal{I}}) \arrow[r, leftrightarrow] \arrow[u, "R^{\mathcal{I}}(\Lambda)\circ(-)"] & \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(LR(\ell)^{\mathcal{I}}, \ell^{\mathcal{I}}) \arrow[u, "\Lambda\circ(-)"']
\end{tikzcd}$$

///

Proving this is tedious but we will do it anyway. The proof becomes "obvious" if we are allowed to hide it inside the concept of "2-categories" (in particular, using the fact that $(-)^{\mathcal{I}} : \mathsf{Cat} \to \mathsf{Cat}$ is a "2-functor") but it would ultimately take **much longer** to develop those ideas.

Proof: (i): For any arrow $\alpha : c_1 \to c_2$ in $\mathcal{C}$ we want to define a natural transformation of diagrams $\alpha^{\mathcal{I}} : c_1^{\mathcal{I}} \Rightarrow c_2^{\mathcal{I}}$, and there is only one way to do this. Since $(c_1^{\mathcal{I}})_i = c_1$ and $(c_2^{\mathcal{I}})_i = c_2$ for all $i \in \mathcal{I}$, the arrow $(\alpha^{\mathcal{I}})_i := (c_1^{\mathcal{I}})_i \to (c_2^{\mathcal{I}})_i$ must be defined by $(\alpha^{\mathcal{I}})_i := \alpha$. Then for any arrow $\delta : i \to j$ in $\mathcal{I}$ we have $c_1^{\mathcal{I}}(\delta) = \mathsf{id}_{c_1}$ and $c_2^{\mathcal{I}}(\delta) = \mathsf{id}_{c_2}$, so that

$$(\alpha^{\mathcal{I}})_i \circ (c_1^{\mathcal{I}})(\delta) = (\alpha \circ \mathsf{id}_{c_1}) = (\mathsf{id}_{c_2} \circ \alpha) = (c_2^{\mathcal{I}})_i(\delta) \circ (\alpha^{\mathcal{I}})_i,$$

and hence we obtain a natural transformation $\alpha^{\mathcal{I}} : c_1^{\mathcal{I}} \Rightarrow c_2^{\mathcal{I}}$. The assignment $\alpha \mapsto \alpha^{\mathcal{I}}$ is functorial since for all arrows α, β such that $\alpha \circ \beta$ exists and for all $i \in \mathcal{I}$ we have

$$(\alpha \circ \beta)_i^{\mathcal{I}} = \alpha \circ \beta = (\alpha^{\mathcal{I}})_i \circ (\beta^{\mathcal{I}})_i = (\alpha^{\mathcal{I}} \circ \beta^{\mathcal{I}})_i,$$

and hence $(\alpha \circ \beta)^{\mathcal{I}} = \alpha^{\mathcal{I}} \circ \beta^{\mathcal{I}}$. Finally, note that we have a bijection of hom sets

$$\mathsf{Hom}_{\mathcal{C}}(c_1, c_2) \leftrightarrow \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c_1^{\mathcal{I}}, c_2^{\mathcal{I}})$$

given by $\alpha \leftrightarrow \alpha^{\mathcal{I}}$, and hence the functor $(-)^{\mathcal{I}} : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ is fully faithful.

(ii): Let $F : \mathcal{C} \to \mathcal{D}$ be any functor. Then for any diagram $D : \mathcal{I} \to \mathcal{C}$ we obtain a diagram $F^{\mathcal{I}}(D) : \mathcal{I} \to \mathcal{D}$ by composition: $F^{\mathcal{I}}(D) := F \circ D$. I claim that this assignment is functorial in $D \in \mathcal{C}^I$. To see this, consider any natural transformation $\Phi : D_1 \Rightarrow D_2$ in the category $\mathcal{C}^{\mathcal{I}}$. Then for any arrow $\delta : i \to j$ in $\mathcal{I}$ we can apply F to the naturality square for Φ to obtain another commutative square:

$$\begin{tikzcd}
D_1(j) \arrow[r, "\Phi_j"] & D_2(j) \\
D_1(i) \arrow[r, "\Phi_i"'] \arrow[u, "D_1(\delta)"] & D_2(i) \arrow[u, "D_2(\delta)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(D_1(j)) \arrow[r, "F(\Phi_j)"] & F(D_2(j)) \\
F(D_1(i)) \arrow[r, "F(\Phi_i)"'] \arrow[u, "F(D_1(\delta))"] & F(D_2(j)) \arrow[u, "F(D_2(\delta))"']
\end{tikzcd}$$

If we define $F^{\mathcal{I}}(\Phi)_i := F(\Phi_i)$ for all $i \in \mathcal{I}$ then this second commutative square says that $F^{\mathcal{I}}(\Phi) : F^{\mathcal{I}}(D_1) \Rightarrow F^{\mathcal{I}}(D_2)$ is a natural transformation in $\mathcal{D}^{\mathcal{I}}$. If Φ and Ψ are two arrows (natural transformations) in $\mathcal{C}^{\mathcal{I}}$ such that $\Phi \circ \Psi$ is defined, then for all $i \in \mathcal{I}$ we have $F^{\mathcal{I}}(\Phi \circ \Psi)_i = F((\Phi \circ \Psi)_i) = F(\Phi_i \circ \Psi_i) = F(\Phi_i) \circ F(\Psi_i) = F^{\mathcal{I}}(\Phi)_i \circ F^{\mathcal{I}}(\Psi)_i = (F^{\mathcal{I}}(\Phi) \circ F^{\mathcal{I}}(\Psi))_i$ and hence $F^{\mathcal{I}}(\Phi \circ \Psi) = F^{\mathcal{I}}(\Phi) \circ F^{\mathcal{I}}(\Psi)$. Thus we have defined a functor $F^{\mathcal{I}} : \mathcal{C}^{\mathcal{I}} \to \mathcal{D}^{\mathcal{I}}$. Finally, note that for all $i \in \mathcal{I}$, $c \in \mathcal{C}$, and $\alpha \in \mathsf{Arr}(\mathcal{C})$ we have

$$\begin{aligned} F^{\mathcal{I}}(c^{\mathcal{I}})_i &= F((c^{\mathcal{I}})_i) = F(c) = ((F(c))^{\mathcal{I}})_i \\ F^{\mathcal{I}}(\alpha^{\mathcal{I}})_i &= F((\alpha^{\mathcal{I}})_i) = F(\alpha) = ((F(\alpha))^{\mathcal{I}})_i \end{aligned}$$

and hence we have an equality of functors $F^{\mathcal{I}}((-)^{\mathcal{I}}) = F(-)^{\mathcal{I}}$ from $\mathcal{C}$ to $\mathcal{D}^{\mathcal{I}}$.

(iii): Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be any adjunction. We will denote each bijection $\mathsf{Hom}_{\mathcal{C}}(-, R(-)) \leftrightarrow \mathsf{Hom}_{\mathcal{C}}(L(-), -)$ by $\varphi \mapsto \overline{\varphi}$, so that $\overline{\overline{\varphi}} = \varphi$. Now we want to define a natural family of bijections

$$\mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(-, R^{\mathcal{I}}(-)) \cong \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L^{\mathcal{I}}(-), -).$$

To do this, consider diagrams $C \in \mathcal{C}^{\mathcal{I}}$, $D \in \mathcal{D}^{\mathcal{I}}$, and a natural transformation $\Phi : C \Rightarrow R^{\mathcal{I}}(D)$. Then for each index $i \in \mathcal{I}$ we have an arrow $\Phi_i : C(i) \to R(D(i))$, which determines an arrow $\overline{\Phi_i} : L(C(i)) \to D(i)$ by adjunction. I claim that the arrows $\overline{\Phi_i}$ assemble into a natural transformation $\overline{\Phi} : L^{\mathcal{I}}(C) \Rightarrow D$. To see this, consider any arrow $\delta : i \in j$ in $\mathcal{I}$. Then from the naturality of Φ and the adjunction $L \dashv R$ we have

$$\begin{aligned} D(\delta) \circ \overline{\Phi_i} &= \overline{R(D(\delta)) \circ \Phi_i} && \text{naturality of } L \dashv R \\ &= \overline{\Phi_j \circ C(\delta)} && \text{naturality of } \Phi \\ &= \overline{\Phi_j} \circ L(C(\delta)), && \text{naturality of } L \dashv R \end{aligned}$$

as desired. In a similar way one can check that for each natural transformation $\Psi : L^{\mathcal{I}}(C) \Rightarrow D$, the arrows $\overline{\Psi_i} : C(i) \to R(D(i))$ assemble into a natural transformation $\overline{\Psi} : C \Rightarrow R^{\mathcal{I}}(D)$. Thus we have established the desired bijection of hom sets $\mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(C, R^{\mathcal{I}}(D)) \leftrightarrow \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L^{\mathcal{I}}(C), D)$.

To prove that this bijection is natural in $(C, D) \in (\mathcal{C}^{\mathcal{I}})^{\mathsf{op}} \times \mathcal{D}^{\mathcal{I}}$, consider any pair of natural transformations $\Gamma : C_2 \Rightarrow C_1$ in $\mathcal{C}^{\mathcal{I}}$ and $\Delta : D_1 \Rightarrow D_2$ in $\mathcal{D}^{\mathcal{I}}$. We need to show that a certain cube of functions commutes. Well, I'm not going to draw that cube, and I'm not going to prove that it's fully commutative. Instead, I'll just show that for a fixed diagram $C \in \mathcal{C}^{\mathcal{I}}$ the following square commutes:

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(C, R^{\mathcal{I}}(D_2)) \arrow[r, leftrightarrow] & \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L^{\mathcal{I}}(C), D_2) \\ \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(C, R^{\mathcal{I}}(D_1)) \arrow[r, leftrightarrow] \arrow[u, "R^{\mathcal{I}}(\Delta)\circ(-)"] & \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L^{\mathcal{I}}(C), D_1) \arrow[u, "\Delta\circ(-)"'] \end{tikzcd}$$

First, recall that the natural transformation $R^{\mathcal{I}}(\Delta) : R^{\mathcal{I}}(D_1) \Rightarrow R^{\mathcal{I}}(D_2)$ is defined pointwise by $R^{\mathcal{I}}(\Delta)_i := R(\Delta_i) : R(D_1(i)) \to R(D_2(i))$. Now consider any $\Phi : C \Rightarrow R^{\mathcal{I}}(D_1)$. The naturality of the original adjunction tells us that $\overline{R(\Delta_i)\circ\Phi_i} = \Delta_i \circ \overline{\Phi_i}$, and hence we have

$$\begin{aligned}\overline{(R^{\mathcal{I}}(\Delta)\circ\Phi)_i} &= \overline{R^{\mathcal{I}}(\Delta)_i\circ\Phi_i}\\ &= \overline{R(\Delta_i)\circ\Phi_i}\\ &= \Delta_i\circ\overline{\Phi_i}\\ &= (\Delta\circ\overline{\Phi})_i\end{aligned}$$

for all $i\in\mathcal{I}$. By definition this means that $\overline{R^{\mathcal{I}}(\Delta)\circ\Phi} = \Delta\circ\overline{\Phi}$, and hence the desired square commutes. It remains only to check that the cube is natural in $(\mathcal{C}^{\mathcal{I}})^{\mathsf{op}}$. This follows from a similar pointwise computation.

(iv): Now fix an element $\ell\in\mathcal{C}$, a diagram $D\in\mathcal{D}^{\mathcal{I}}$, and a natural transformation $\Lambda:\ell^{\mathcal{I}}\Rightarrow D$. By substituting $C = R(\ell)^{\mathcal{I}}$, $D_1 = \ell^{\mathcal{I}}$, $D_2 = D$, and $\Delta = \Lambda$ into the above commutative square and using part (ii), we obtain the commutative square from the statement of the lemma. In particular, following the identity arrow $\mathsf{id}^{\mathcal{I}}_{R(\ell)}$ around the square in two ways gives

$$\overline{R^{\mathcal{I}}(\Lambda)\circ\mathsf{id}^{\mathcal{I}}_{R(\ell)}} = \Lambda\circ\overline{\mathsf{id}^{\mathcal{I}}_{R(\ell)}}.$$

Finally, one can check pointwise that $\overline{\mathsf{id}^{\mathcal{I}}_{R(\ell)}} = \left(\overline{\mathsf{id}_{R(\ell)}}\right)^{\mathcal{I}}$, and hence we obtain the identity

$$\overline{R^{\mathcal{I}}(\Lambda)\circ\mathsf{id}^{\mathcal{I}}_{R(\ell)}} = \left(\overline{\mathsf{id}_{R(\ell)}}\right)^{\mathcal{I}}.$$

This identity is not very interesting but we will need it later. □

Now we will reformulate the definition of limit/colimit in terms of adjoint functors. If all limits/colimits of shape $\mathcal{I}$ **exist** in some category $\mathcal{C}$ then it turns out (surprisingly) that we can think of limits/colimits as right/left adjoints to the diagonal embedding $(-)^{\mathcal{I}}:\mathcal{C}\to\mathcal{C}^{\mathcal{I}}$:

$$\mathsf{colim}_{\mathcal{I}} \dashv (-)^{\mathcal{I}} \dashv \mathsf{lim}_{\mathcal{I}}.$$

In the following lemma we will prove something slightly more general. We will characterize a **specific** limit/colimit of shape $\mathcal{I}$, without assuming that **all** limits/colimits of shape $\mathcal{I}$ exist.

Lemma (Colimit ⊣ Diagonal ⊣ Limit). Fix a small category $\mathcal{I}$ and a diagram $D:\mathcal{I}\to\mathcal{C}$ of shape $\mathcal{I}$. Then the limit of D, if it exists, consists of an object $\ell\in\mathcal{C}$ and a natural isomorphism

$$\mathsf{Cone}:\mathsf{Hom}_{\mathcal{C}}(-,\ell)\cong\mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}((-)^{\mathcal{I}},D):\mathsf{Uni}$$

in the category $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$. ///

This is intuitively plausible if we recall the definition of limits. Recall that a cone under D consists of an object $\ell\in\mathcal{C}$ and a natural transformation $\Lambda:\ell^{\mathcal{I}}\Rightarrow D$. We say that the cone

(ℓ, Λ) is the the limit of D if, for any other cone $\Phi : c^{\mathcal{I}} \Rightarrow D$, there exists a unique arrow $\upsilon : c \to \ell$ making the following diagram in $\mathcal{C}^{\mathcal{I}}$ commute:

$$\begin{tikzcd} & D & \\ c^{\mathcal{I}} \arrow[ur, Rightarrow, "\Phi"] \arrow[rr, Rightarrow, "\upsilon^{\mathcal{I}}"'] & & \ell^{\mathcal{I}} \arrow[ul, Rightarrow, "\Lambda"'] \end{tikzcd}$$

The map sending the cone $\Phi : c^{\mathcal{I}} \Rightarrow D$ to the unique arrow $\upsilon : c \to \ell$ is the desired function $\mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, D) \to \mathsf{Hom}_{\mathcal{C}}(c, \ell)$. Furthermore, it's clear that this function is a bijection since we can pull back any arrow $\alpha : c \to \ell$ to the cone $\Lambda \circ \alpha^{\mathcal{I}} : c^{\mathcal{I}} \Rightarrow D$. The main difficulty is to show that the data of naturality for these bijections is equivalent to the data of the canonical cone $\Lambda : \ell^{\mathcal{I}} \Rightarrow D$.

Proof: First assume that the limit of D exists and is given by the cone $(\mathsf{lim}_{\mathcal{I}} D, \Lambda)$. In this case we want to define a family of bijections

$$\mathsf{Uni}_c : \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, D) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(c, \mathsf{lim}_{\mathcal{I}} D)$$

that is natural in $c \in \mathcal{C}^{\mathsf{op}}$. (Then the inverse $\mathsf{Cone}_c := \mathsf{Uni}_c^{-1}$ is automatically natural, as you proved in a previous exercise.) So consider any element $\Phi \in \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, D)$, i.e., any cone $\Phi : c^{\mathcal{I}} \Rightarrow D$. By the definition of limits we know that there exists a unique arrow $\upsilon : c \to \mathsf{lim}_{\mathcal{I}} D$ making the following diagram commute:

$$\begin{tikzcd} & D & \\ c^{\mathcal{I}} \arrow[ur, Rightarrow, "\Phi"] \arrow[rr, Rightarrow, "\upsilon^{\mathcal{I}}"'] & & (\mathsf{lim}_{\mathcal{I}} D)^{\mathcal{I}} \arrow[ul, Rightarrow, "\Lambda"'] \end{tikzcd}$$

Therefore the assignment $\mathsf{Uni}_c(\Phi) := \upsilon$ defines an **injective function** (recall that the functor $(-)^I$ is faithful, so that $\upsilon_1^{\mathcal{I}} = \upsilon_2^{\mathcal{I}}$ implies $\upsilon_1 = \upsilon_2$). To see that Uni_c is **surjective**, consider any arrow $\alpha : c \to \mathsf{lim}_{\mathcal{I}} D$ in $\mathcal{C}$. We want to define a cone $\Phi_\alpha : c^{\mathcal{I}} \Rightarrow D$ with the property that $\mathsf{Uni}_c(\Phi_\alpha) = \alpha$. By definition of Uni_c this means that we must have $\Phi_\alpha := \Lambda \circ \alpha^{\mathcal{I}}$ — in other words, we must have $(\Phi_\alpha)_i := \Lambda_i \circ \alpha$ for all indices $i \in \mathcal{I}$. And note that this **does** define a natural transformation $\Phi_\alpha : c^{\mathcal{I}} \Rightarrow D$ since for all arrows $\delta : i \in j$ in $\mathcal{I}$ we have

$$\begin{aligned} D(\delta) \circ (\Phi_\alpha)_i &= D(\delta) \circ (\Lambda_i \circ \alpha) \\ &= (D(\delta) \circ \Lambda_i) \circ \alpha \\ &= \Lambda_j \circ \alpha && \text{naturality of } \Lambda \\ &= (\Phi_\alpha)_j. \end{aligned}$$

We conclude that Uni_c is a bijection. To see that Uni_c is natural in $c \in \mathcal{C}^{\mathsf{op}}$, consider any arrow $\gamma : c_1 \to c_2$ in $\mathcal{C}^{\mathsf{op}}$ (i.e., any arrow $\gamma : c_2 \to c_1$ in $\mathcal{C}$). We want to show that the following diagram commutes:

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c_2^{\mathcal{I}}, D) \arrow[r, "\mathsf{Uni}_{c_2}", "\sim"'] & \mathsf{Hom}_{\mathcal{C}}(c_2, \mathsf{lim}_{\mathcal{I}} D) \\ \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c_1^{\mathcal{I}}, D) \arrow[r, "\sim", "\mathsf{Uni}_{c_1}"'] \arrow[u, "(-)\circ\gamma^{\mathcal{I}}"] & \mathsf{Hom}_{\mathcal{C}}(c_1, \mathsf{lim}_{\mathcal{I}} D) \arrow[u, "(-)\circ\gamma"'] \end{tikzcd}$$

And to see this, consider any cone $\Phi : c_1^{\mathcal{I}} \Rightarrow D$. By composing with the natural transformation $\gamma^{\mathcal{I}} : c_2^{\mathcal{I}} \Rightarrow c_1^{\mathcal{I}}$ we obtain the following commutative diagram in $\mathcal{C}^{\mathcal{I}}$:

$$\begin{tikzcd} & D & \\ c_2^{\mathcal{I}} \arrow[ur, Rightarrow, "\Phi\circ\gamma^{\mathcal{I}}"] \arrow[r, Rightarrow, "\gamma^{\mathcal{I}}"'] \arrow[rr, Rightarrow, bend right, "(\mathsf{Uni}_{c_1}(\Phi))^{\mathcal{I}}\circ\gamma^{\mathcal{I}}"'] & c_1^{\mathcal{I}} \arrow[u, Rightarrow, "\Phi"'] \arrow[r, Rightarrow, "(\mathsf{Uni}_{c_1}(\Phi))^{\mathcal{I}}"'] & (\mathsf{lim}_{\mathcal{I}} D)^{\mathcal{I}} \arrow[ul, Rightarrow, "\Lambda"'] \end{tikzcd}$$

Since the diagonal embedding $(-)^{\mathcal{I}} : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ is a functor, the bottom arrow is given by

$$(\mathsf{Uni}_{c_1}(\Phi))^{\mathcal{I}} \circ \gamma^{\mathcal{I}} = (\mathsf{Uni}_{c_1}(\Phi) \circ \gamma)^{\mathcal{I}}.$$

But by the definition of the function Uni_{c_2} this arrow also equals

$$(\mathsf{Uni}_{c_2}(\Phi \circ \gamma^{\mathcal{I}}))^{\mathcal{I}}.$$

Then since $(-)^{\mathcal{I}}$ is a faithful functor we conclude that

$$\mathsf{Uni}_{c_2}(\Phi \circ \gamma^{\mathcal{I}}) = \mathsf{Uni}_{c_1}(\Phi) \circ \gamma,$$

and hence the desired square commutes.

Conversely, consider an object $\ell \in \mathcal{C}$ and suppose that we have a bijection

$$\mathsf{Cone}_c : \mathsf{Hom}_{\mathcal{C}}(c, \ell) \longleftrightarrow \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, D) : \mathsf{Uni}_c$$

that is natural in $c \in \mathcal{C}^{\mathsf{op}}$. In other words, suppose that for each arrow $\gamma : c_1 \to c_2$ in $\mathcal{C}^{\mathsf{op}}$ (i.e., for each arrow $\gamma : c_2 \to c_1$ in $\mathcal{C}$) we have a commutative square:

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{C}}(c_2, \ell) \arrow[r, bend left, "\mathsf{Cone}_{c_2}"] & \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c_2^{\mathcal{I}}, D) \arrow[l, bend left, "\mathsf{Uni}_{c_2}"] \\ \mathsf{Hom}_{\mathcal{C}}(c_1, \ell) \arrow[u, "(-)\circ\gamma"] \arrow[r, bend left, "\mathsf{Cone}_{c_1}"] & \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c_1^{\mathcal{I}}, D) \arrow[u, "(-)\circ\gamma^{\mathcal{I}}"'] \arrow[l, bend left, "\mathsf{Uni}_{c_1}"] \end{tikzcd}$$

We want to show that this determines a unique cone $\Lambda : \ell^{\mathcal{I}} \Rightarrow D$ such that (ℓ, Λ) is the limit of D. **The only possible choice** is to define $\Lambda := \mathsf{Cone}_\ell(\mathsf{id}_\ell)$. Now given **any** cone $\Phi : c^{\mathcal{I}} \Rightarrow D$ we want to show that there exists a unique arrow $\upsilon : c \to \ell$ with the property $\Lambda \circ \upsilon^{\mathcal{I}} = \Phi$.

So suppose that there exists **some** arrow $\upsilon : c \to \ell$ with the property $\Lambda \circ \upsilon^{\mathcal{I}} = \Phi$. By substituting $\gamma := \upsilon$ into the above diagram we obtain a commutative square:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c,\ell) \arrow[r, bend left, "\mathsf{Cone}_c"] & \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, D) \arrow[l, bend left, "\mathsf{Uni}_c"] \\
\mathsf{Hom}_{\mathcal{C}}(\ell,\ell) \arrow[u, "(-)\circ\upsilon"] \arrow[r, bend left, "\mathsf{Cone}_\ell"] & \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(\ell^{\mathcal{I}}, D) \arrow[l, bend left, "\mathsf{Uni}_\ell"] \arrow[u, "(-)\circ\upsilon^{\mathcal{I}}"']
\end{tikzcd}$$

Then following the arrow $\mathsf{id}_\ell \in \mathsf{Hom}_{\mathcal{C}}(\ell,\ell)$ around the square in two different ways gives

$$\begin{aligned}
\mathsf{id}_\ell \circ \upsilon &= \mathsf{Uni}_c(\mathsf{Cone}_\ell(\mathsf{id}_\ell) \circ \upsilon^{\mathcal{I}}) \\
\upsilon &= \mathsf{Uni}_c(\Lambda \circ \upsilon^{\mathcal{I}}) \\
\upsilon &= \mathsf{Uni}_c(\Phi).
\end{aligned}$$

Thus there exists **at most one** such arrow υ. To show that there exists **at least one** such arrow, we must check that the arrow $\mathsf{Uni}_c(\Phi)$ actually **does** satisfy $\Lambda \circ (\mathsf{Uni}_c(\Phi))^{\mathcal{I}} = \Phi$. Indeed, by substituting $\upsilon := \mathsf{Uni}_c(\Phi)$ into the above diagram we obtain a commutative square:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(c,\ell) \arrow[r, bend left, "\mathsf{Cone}_c"] & \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, D) \arrow[l, bend left, "\mathsf{Uni}_c"] \\
\mathsf{Hom}_{\mathcal{C}}(\ell,\ell) \arrow[u, "(-)\circ\mathsf{Uni}_c(\Phi)"] \arrow[r, bend left, "\mathsf{Cone}_\ell"] & \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(\ell^{\mathcal{I}}, D) \arrow[l, bend left, "\mathsf{Uni}_\ell"] \arrow[u, "(-)\circ(\mathsf{Uni}_c(\Phi))^{\mathcal{I}}"']
\end{tikzcd}$$

Then following the arrow $\mathsf{id}_\ell \in \mathsf{Hom}_{\mathcal{C}}(\ell,\ell)$ around the square in two ways gives

$$\begin{aligned}
\mathsf{Cone}_\ell(\mathsf{id}_\ell) \circ (\mathsf{Uni}_c(\Phi))^{\mathcal{I}}) &= \mathsf{Cone}_c(\mathsf{id}_\ell \circ \mathsf{Uni}_c(\Phi)) \\
\Lambda \circ (\mathsf{Uni}_c(\Phi))^{\mathcal{I}} &= \mathsf{Cone}_c(\mathsf{Uni}_c(\Phi)) \\
\Lambda \circ (\mathsf{Uni}_c(\Phi))^{\mathcal{I}} &= \Phi,
\end{aligned}$$

as desired. □

[Remark: We have proved that the limit of a diagram $D : \mathcal{I} \to \mathcal{C}$, if it exists, consists of an object $\mathsf{lim}_{\mathcal{I}} D \in \mathcal{C}$ and a natural isomorphism

$$\mathsf{Hom}_{\mathcal{C}}(-, \mathsf{lim}_{\mathcal{I}} D) \cong \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}((-)^{\mathcal{I}}, D)$$

of functors $\mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$. It turns out that if **all** limits of shpae $\mathcal{I}$ exist in $\mathcal{C}$ then there is a unique way to extend this to a natural isomorphism

$$\mathsf{Hom}_{\mathcal{C}}(-, \mathsf{lim}_{\mathcal{I}} -) \cong \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}((-)^{\mathcal{I}}, -)$$

of functors $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}^{\mathcal{I}} \to \mathsf{Set}$, and hence that we have an adjunction $(-)^{\mathcal{I}} : \mathcal{C} \rightleftarrows \mathcal{C}^{\mathcal{I}} : \mathsf{lim}_{\mathcal{I}}$. However, we don't need this result right now so we won't prove it. Dually, the colimit of D, if it exists, consists of an object $\mathsf{colim}_{\mathcal{I}} D \in \mathcal{C}$ and a natural isomorphism

$$\mathsf{Hom}_{\mathcal{C}}(\mathsf{colim}_{\mathcal{I}} D, -) \cong \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(D, (-)^{\mathcal{I}})$$

of functors $\mathcal{C} \to \mathsf{Set}$. If **all** colimits of shape $\mathcal{I}$ exist in $\mathcal{C}$ then this extends uniquely to an adjunction $\mathsf{colim}_{\mathcal{I}} : \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{C} : (-)^{\mathcal{I}}$. This explains the title of the previous lemma.]

Finally, here it is.

Theorem (RAPL). Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction and consider a diagram $D : \mathcal{I} \to \mathcal{D}$ of shape $\mathcal{I}$ in $\mathcal{D}$. If the diagram $D : \mathcal{I} \to \mathcal{D}$ has a limit cone $\Lambda : \ell^{\mathcal{I}} \Rightarrow D$ then the composite diagram $R^{\mathcal{I}}(D) : \mathcal{I} \to \mathcal{C}$ also has a limit cone, which is given by $R^{\mathcal{I}}(\Lambda) : R(\ell)^{\mathcal{I}} \Rightarrow R^{\mathcal{I}}(D)$.

I will present two proofs: first, a simplified proof that conforms to our intuition about adjoint operators between vector spaces; then the full proof, which is a bit more complicated. The result of the simplified proof is sufficient for most applications.

Simplified Proof: In this proof we will write $\mathsf{lim}_{\mathcal{I}} D := \ell \in \mathcal{D}$, and we will just **assume** that the limit object $\mathsf{lim}_I R^{\mathcal{I}}(D) \in \mathcal{C}$ **exists**. Now we want to show that the following objects are isomorphic in $\mathcal{C}$: $R(\mathsf{lim}_{\mathcal{I}} D) \cong \mathsf{lim}_I R^{\mathcal{I}}(D)$. (We will ignore the data of the limit cone.)

So assume that $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ is an adjunction. Then we have the following sequence of bijections, each of which is natural in $c \in \mathcal{C}^{\mathsf{op}}$:

$$\begin{aligned}
\mathsf{Hom}_{\mathcal{C}}(c, R(\mathsf{lim}_{\mathcal{I}} D)) &\xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(L(c), \mathsf{lim}_I D) && L \dashv R \\
&\xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L(c)^{\mathcal{I}}, D) && \text{Diagonal} \dashv \text{Limit} \\
&= \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L^{\mathcal{I}}(c^{\mathcal{I}}), D) && \text{Diagram Lemma (ii)} \\
&\xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, R^{\mathcal{I}}(D)) && \text{Diagram Lemma (iii)} \\
&\xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(c, \mathsf{lim}_{\mathcal{I}} R^{\mathcal{I}}(D)). && \text{Diagonal} \dashv \text{Limit}
\end{aligned}$$

By composing these we obtain a family of bijections

$$\mathsf{Hom}_{\mathcal{C}}(c, R(\mathsf{lim}_{\mathcal{I}} D)) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(c, \mathsf{lim}_{\mathcal{I}} R^{\mathcal{I}}(D))$$

that is natural in $c \in \mathcal{C}^{\mathsf{op}}$. In other words, we obtain an isomorphism of hom functors $H_{R(\lim_{\mathcal{I}}(D))} \cong H_{\lim_{\mathcal{I}} R^{\mathcal{I}}(D)}$ in the category $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$. Then since the Yoneda embedding $H_{(-)} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ is essentially injective (from the Embedding Lemma), we obtain an isomorphism of objects $R(\lim_{\mathcal{I}} D) \cong \lim_I R^{\mathcal{I}}(D)$ in the category $\mathcal{C}$. □

Full Proof: Now we will prove that the limit of the diagram $R^{\mathcal{I}}(D) : \mathcal{I} \to \mathcal{D}$ **actually exists** and is given by the correct limit cone. Then the uniqueness of the limit object up to isomorphism follows from the universal property of limit cones.

So Assume that $\Lambda : \ell^{\mathcal{I}} \Rightarrow D$ is the limit of the diagram $D : \mathcal{I} \to \mathcal{D}$ and that $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ is an adjunction. To show that the cone $R^{\mathcal{I}}(\Lambda) : R(\ell)^{\mathcal{I}} \Rightarrow R^{\mathcal{I}}(D)$ is the limit of the diagram $R^{\mathcal{I}}(D) : \mathcal{I} \to \mathcal{C}$ we must construct a family of bijections

$$\mathsf{Cone}_c : \mathsf{Hom}_{\mathcal{C}}(c, R(\ell)) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, R^{\mathcal{I}}(D))$$

that is natural in $c \in \mathcal{C}^{\mathsf{op}}$, and then we must verify that $\mathsf{Cone}_{R(\ell)}(\mathsf{id}_{R(\ell)}) = R^{\mathcal{I}}(\Lambda)$.

To construct the bijection Cone_c note that we have a sequence of bijections, each of which is natural in $c \in \mathcal{C}^{\mathsf{op}}$:

$$\begin{aligned}
\mathsf{Hom}_{\mathcal{C}}(c, R(\ell)) &\xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}}(L(c), \ell) && L \dashv R \\
&\xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L(c)^{\mathcal{I}}, D) && \text{Diagonal} \dashv \text{Limit} \\
&= \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(L^{\mathcal{I}}(c^{\mathcal{I}}), D) && \text{Diagram Lemma (ii)} \\
&\xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(c^{\mathcal{I}}, R^{\mathcal{I}}(D)). && \text{Diagram Lemma (iii)}
\end{aligned}$$

Composing these gives the desired natural isomorphism:

$$\mathsf{Cone} : \mathsf{Hom}_{\mathcal{C}}(-, R(\ell)) \overset{\sim}{\Rightarrow} \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}((-)^{\mathcal{I}}, R^{\mathcal{I}}(D)).$$

Then to compute $\mathsf{Cone}_{R(\ell)}(\mathsf{id}_{R(\ell)})$ we follow the arrow $\mathsf{id}_{R(\ell)} \in \mathsf{Hom}_{\mathcal{C}}(R(\ell), R(\ell))$ through the sequence of bijections above to get

$$\begin{aligned}
\mathsf{id}_{R(\ell)} &\mapsto \overline{\mathsf{id}_{R(\ell)}} && \in \mathsf{Hom}_{\mathcal{D}}(LR(\ell), \ell) \\
&\mapsto \Lambda \circ \left(\overline{\mathsf{id}_{R(\ell)}}\right)^{\mathcal{I}} && \in \mathsf{Hom}_{\mathcal{D}^{\mathcal{I}}}(LR(\ell)^{\mathcal{I}}, D) \\
&= \overline{R^{\mathcal{I}}(\Lambda) \circ \mathsf{id}_{R(\ell)}^{\mathcal{I}}} && \text{Diagram Lemma (iv)} \\
&= \overline{R^{\mathcal{I}}(\Lambda)} && \text{definition of } \mathsf{id}_{R(\ell)} \text{ :)} \\
&\mapsto R^{\mathcal{I}}(\Lambda), && \in \mathsf{Hom}_{\mathcal{C}^{\mathcal{I}}}(R(\ell)^{\mathcal{I}}, R(D))
\end{aligned}$$

as desired. □

[Remark: I get angry when books give a false sense of the difficulty of a result. Therefore, it was my intention not to skip any details in this proof. Now we can see how long the proof actually is.]

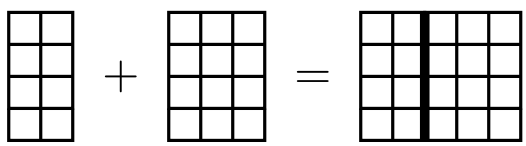
Let's get to the applications.

1.9 Three Fundamental Examples

We saw earlier that adjoint functors between posets (i.e., Galois connections) are already quite interesting. In this section I will investigate three fundamental adjunctions that don't come from posets. Hopefully these examples will demonstrate that the concept of adjoint functors is one of the key ideas in mathematics.

1.9.1 Multiplication $\dashv$ Exponentiation

Why does multiplication of natural numbers distribute over addition? Here's one possible explanation:



But let me suggest an alternative. Given a set $Z \in \mathsf{Set}$ we can define two functors by "multiplication" $(-) \times Z : \mathsf{Set} \to \mathsf{Set}$ and by "exponentiation" $(-)^Z : \mathsf{Set} \to \mathsf{Set}$. The functor $(-) \times Z$ sends each $X \in \mathsf{Set}$ to the Cartesian product set $X \times Z$ and the functor $(-)^Z$ sends each $X \in \mathsf{Set}$ to the power set $X^Z := \mathsf{Hom}_{\mathsf{Set}}(Z, X)$. (Note that $(-)^Z$ is just a different name for the hom functor H^Z.) For each function $f : X_1 \to X_2$ we define a function $f \times Z : X_1 \times Z \to X_2 \times Z$ by setting $(f \times Z)(x, z) := (f(x), z)$ for all $(x, z) \in X_1 \times Z$, and we define $f^Z : X_1^Z \to X_2^Z$ by setting $f^Z(g) := f \circ g$ for all $g \in X_1^Z$.

Exercise: Check that $(f_1 \circ f_2) \times Z = (f_1 \times Z) \circ (f_2 \times Z)$ and $(f_1 \circ f_2)^Z = f_1^Z \circ f_2^Z$.

Now I claim that for each set $S \in \mathsf{Set}$ we have an adjunction:

$$\begin{tikzcd} \mathsf{Set} \arrow[r, bend left, "(-)\times Z"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, bend left, "(-)^Z"] \end{tikzcd}$$

That is, I claim that we have a family of bijections

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Set}}(X \times Z, Y) & \longleftrightarrow & \mathsf{Hom}_{\mathsf{Set}}(X, \mathsf{Hom}_{\mathsf{Set}}(Z, Y)) \\ \mathsf{Hom}_{\mathsf{Set}}(X \times Z, Y) & \longleftrightarrow & \mathsf{Hom}_{\mathsf{Set}}(X, Y^Z) \\ Y^{(X \times Z)} & \longleftrightarrow & (Y^Z)^X \end{array}$$

that is natural in $(X, Y) \in \mathsf{Set}^{\mathsf{op}} \times \mathsf{Set}$. [Remark: Maybe this would look nicer written as $Y^{(X \times Z)} \leftrightarrow (Y^X)^Z$ but I prefer the convention of "moving Z across the comma." This convention also generalizes better.] Here's the idea: Given a function $f : X \times Z \to Y$ and an element $x \in X$ we define a function $f^x : Z \to Y$ by

$$f^x(z) := f(x,z).$$

Then the mapping $x \mapsto f^x$ defines a function $f^{(-)} : X \to Y^Z$ and the mapping $f \mapsto f^{(-)}$ defines a function $\mathsf{Hom}_{\mathsf{Set}}(X \times Z, Y) \to \mathsf{Hom}_{\mathsf{Set}}(X, Y^Z)$. This function is traditionally called "currying," after the American logician Haskell Curry:

$$\boxed{\begin{array}{rcl} \mathsf{Curry}_{X,Y} : \mathsf{Hom}_{\mathsf{Set}}(X \times Z, Y) & \to & \mathsf{Hom}_{\mathsf{Set}}(X, \mathsf{Hom}_{\mathsf{Set}}(Z, Y)) \\ f & \mapsto & f^{(-)} \end{array}}$$

Proof: We want to show that Curry is a natural isomorphism. In other words, we want to show that $\mathsf{Curry}_{X,Y}$ is a family of bijections that is natural in $(X,Y) \in \mathsf{Set}^{\mathsf{op}} \times \mathsf{Set}$.

First we will show that $\mathsf{Curry}_{X,Y}$ is a bijection. To see that $\mathsf{Curry}_{X,Y}$ is surjective, consider any function $F : X \to Y^Z$ so that for each $x \in X$ we have a function $F(x) : Z \to Y$. Now define a function $f : X \times Z \to Y$ by setting

$$f(x,z) := F(x)(z)$$

for all $(x,z) \in X \times Z$. Then by definition we have $f^x = F(x)$ for all $x \in X$, and hence $f^{(-)} = F$. To see that $\mathsf{Curry}_{X,Y}$ is injective, consider any two functions $f, g : X \times Z \to Y$ and assume that we have $f^{(-)} = g^{(-)}$ as functions $X \to Y^Z$. By definition this means that for all $x \in X$ we have $f^x = g^x$ as functions $Z \to Y$, which by definition this means that for all $z \in Z$ we have

$$f(x,z) = f^x(z) = g^x(z) = g(x,z).$$

Since this holds for all $(x,z) \in X \times Z$ we have $f = g$, as desired.

Now we will show that $\mathsf{Curry}_{X,Y}$ is natural in $(X,Y) \in \mathsf{Set}^{\mathsf{op}} \times \mathsf{Set}$. So consider any pair of functions $\gamma : X_2 \to X_1$ and $\delta : Y_1 \to Y_2$. We want to show that the following cube commutes:

$$\begin{tikzcd}
 & Y_2^{(X_2 \times Z)} \arrow[rr, leftrightarrow] & & (Y_2^Z)^{X_2} \\
Y_1^{(X_2 \times Z)} \arrow[ur, "\delta\circ(-)"] \arrow[rr, leftrightarrow] & & (Y_1^Z)^{X_2} \arrow[ur, "\delta\circ(-)"] & \\
 & Y_2^{(X_1 \times Z)} \arrow[uu, dotted, "(-)\circ(\gamma\times\mathsf{id}_Z)"] \arrow[rr, dotted, leftrightarrow] & & (Y_2^Z)^{X_1} \arrow[uu, "(-)\circ\gamma"'] \\
Y_1^{(X_1 \times Z)} \arrow[uu, "(-)\circ(\gamma\times\mathsf{id}_Z)"] \arrow[ur, dotted, "\delta\circ(-)"'] \arrow[rr, leftrightarrow] & & (Y_1^Z)^{X_1} \arrow[uu, "(-)\circ\gamma"] \arrow[ur, "\delta\circ(-)"'] &
\end{tikzcd}$$

To show that the front/back of the cube commutes (i.e., that $\mathsf{Curry}_{X,Y}$ is natural in $X \in \mathsf{Set}^{\mathsf{op}}$), consider any function $f : X_1 \times Z \to Y_1$. Going around the bottom of the square gives the function $f^{(-)} \circ \gamma : X_2 \to Y_1^Z$ and going around the top of the square gives the function $(f \circ \gamma)^{(-)} : X_2 \to Y_1^Z$. Are these the same function? To check, consider any elements $x \in X_2$ and $z \in Z$. Then we have

$$[(f^{(-)} \circ \gamma)(x)](z) = [f^{(-)}(\gamma(x))](z) = f^{\gamma(x)}(z) = f(\gamma(x), z)$$

and

$$[(f \circ (\gamma \times \mathsf{id}_Z))^{(-)}(x)](z) = [f(\gamma(-), -)^{(-)}(x)](z) = [f(\gamma(x), -)](z) = f(\gamma(x), z),$$

which are equal as desired. To show that the top/bottom of the cube commutes (i.e., that $\mathsf{Curry}_{X,Y}$ is natural in $Y \in \mathsf{Set}$), consider any function $g : X_1 \times Z \to Y_1$. Going around the front of the square gives the function $\delta \circ g^{(-)} : X_1 \to Y_2^Z$ and going around the back of the square gives the function $(\delta \circ g)^{(-)} : X_1 \to Y_2^Z$. Are these the same function? To check, consider any elements $x \in X_1$ and $z \in Z$. Then we have

$$[(\delta \circ g^{(-)})(x)](z) = [\delta \circ g^x](z) = \delta(g^x(z)) = \delta(g(x, z))$$

and

$$[(\delta \circ g)^{(-)}(x)](z) = [\delta(g(-,-))^{(-)}(x)](z) = [\delta(g(x,-))](z) = \delta(g(x,z)),$$

which are equal as desired. □

[Remark: I did my best with the notation here but I realize that it's still confusing.]

Now let's apply the RAPL theorem to the Curry adjunction. Recall that the product and coproduct in Set are the Cartesian product $\times$ and the disjoint union $\coprod$, respectively. Consider any three sets $X, Y, Z \in \mathsf{Set}$ with cardinalities $x, y, z \in \mathbb{N}$. Then since the functor $(-) \times Z$ is **left adjoint** and the functor $(-)^Z$ is **right adjoint**, we obtain two bijections:

$$(X \coprod Y) \times Z \leftrightarrow (X \times Z) \coprod (Y \times Z) \qquad \text{and} \qquad (X \times Y)^Z \leftrightarrow (X^Z) \times (Y^Z).$$

Finally, applying cardinality to both bijections gives

$$(x + y)z = xz + yz \qquad \text{and} \qquad (xy)^z = x^z y^z.$$

1.9.2 Tensor ⊣ Hom

You may think that this was an unnecessarily complicated way to explain the properties of addition/multiplication/exponentiation for natural numbers. However, it's very important to understand arithmetic in this way if we want to generalize it to other categories.

In some naive sense, the notions of addition/multiplication/exponentiation of natural numbers should generalize to the notions of categorical coproduct / categorical product / hom sets, respectively. But we quickly find that these notions don't behave properly in most categories. [Remark: They only behave properly in caterogies (called "toposes") that are sufficiently like the category of sets.]

For example, consider the category Ab of abelian groups. Here are three important facts about this category:[10]

- The trivial group $0 \in \mathsf{Ab}$ is both the initial and the final object, called the **zero object**.
- The product and coproduct of $A, B \in \mathsf{Ab}$ coincide, and this unique group is called the **direct sum** $A \oplus B \in \mathsf{Ab}$.
- For any abelian groups $A, B \in \mathsf{Ab}$, the hom set $B^A := \mathsf{Hom}_{\mathsf{Ab}}(A, B)$ naturally carries the structure of an **abelian group**, defined by adding homomorphisms "pointwise."

Thus we might hope that the Curry adjunction in Set lifts to an adjunction in Ab:

$$\mathsf{Hom}_{\mathsf{Ab}}(A \oplus C, B) \overset{?}{\cong} \mathsf{Hom}_{\mathsf{Ab}}(A, B^C).$$

If this were true, then $(-) \oplus C$ would be a left adjoint functor and hence (by RAPL) it would preserve colimits. Then since the initial object $0 \in \mathsf{Ab}$ is an example of a colimit we would have an isomorphism of groups:

$$0 \oplus C \cong 0.$$

But this is certainly not true! It follows that the functor $(-) \oplus C$ **has no right adjoint**, and hence the Curry adjunction fails in Ab. [Remark: Ab is not a topos.]

The problem here is that the product operation in Ab collapsed into the sum operation and stopped behaving like "multiplication." Similarly, the final object $* \in \mathsf{Set}$ (which satisfies $* \times S \leftrightarrow S$ for all $S \in \mathsf{Set}$) collapsed into the initial object in Ab and stopped behaving like "the number 1." Where we had both "addition" and "multiplication" operations in Set, we now have only "addition" in Ab.

This leads to an important question:

Does there exist a binary operation in Ab *that behaves like multiplication?*

One can prove that the hom functor $(-)^C : \mathsf{Ab} \to \mathsf{Ab}$ for any $C \in \mathsf{Ab}$ preserves limits (in fact, the hom functors $H^c : \mathcal{C} \to \mathsf{Set}$ preserve limits in any[11] category $\mathcal{C}$). In particular, one can check using the universal property of $A \oplus B$ (thought of as the categorical **product**) that we have an isomorphism:

$$(A \oplus B)^C \cong A^C \oplus B^C.$$

Exercise: Check this.

[10] We will examine these facts in great detail in the next chapter.

[11] locally small

This suggests that the functor $(-)^C$ could possibly have a left adjoint. Let's assume for the sake of argument that such a left adjoint does exist, and let's call it $(-)\otimes C$. That is, let's assume that we have a family of bijections

$$\boxed{\mathsf{Hom}_{\mathsf{Ab}}(A\otimes C, B) \leftrightarrow \mathsf{Hom}_{\mathsf{Ab}}(A, B^C)}$$

that is natural in $(A,B)\in\mathsf{Ab}^{\mathsf{op}}\times\mathsf{Ab}$. Note that we have chosen the symbol "$\otimes$" because this functor **necessarily behaves like multiplication**. For example, since the coproduct $A\oplus B$ and the initial object $0\in\mathsf{Ab}$ are examples of colimits, the RAPL theorem gives us isomorphisms

$$(A\oplus B)\otimes C\cong (A\otimes C)\oplus(B\otimes C) \qquad \text{and} \qquad 0\otimes C\cong 0.$$

Furthermore, note that the abelian group $(\mathbb{Z},+)$ has the special property that for each abelian group $B\in\mathsf{Ab}$ there is a natural isomorphism $B^{\mathbb{Z}}\cong B$, defined by sending each homomorphism $\varphi:\mathbb{Z}\to B$ to the element $\varphi(1_{\mathbb{Z}})\in B$.

Exercise: Check this.

Thus, for each $A\in\mathsf{Ab}$, if we compose the two natural bijections

$$\mathsf{Hom}_{\mathsf{Ab}}(A\otimes\mathbb{Z}, B)\leftrightarrow\mathsf{Hom}_{\mathsf{Ab}}(A, B^{\mathbb{Z}})\leftrightarrow\mathsf{Hom}_{\mathsf{Ab}}(A,B)$$

then we obtain a bijection $\mathsf{Hom}_{\mathsf{Ab}}(A\otimes\mathbb{Z}, B)\leftrightarrow\mathsf{Hom}_{\mathsf{Ab}}(A,B)$ that is natural in $B\in\mathsf{Ab}$. In other words, we have a natural isomorphism of hom functors $H^{A\otimes\mathbb{Z}}\cong H^A$. Finally, by applying the fact that the Yoneda embedding $H^{(-)}$ is essentially injective, we obtain an isomorphism of groups:

$$A\otimes\mathbb{Z}\cong A.$$

In summary, if we assume that each hom functor $(-)^C:\mathsf{Ab}\to\mathsf{Set}$ has a left adjoint (which is a reasonable assumption, because hom functors preserve limits), then we obtain an operation $\otimes:\mathsf{Obj}(\mathsf{Ab})\times\mathsf{Obj}(\mathsf{Ab})\to\mathsf{Obj}(\mathsf{Ab})$ that behaves very much like "multiplication," and which has a "unit object" $\mathbb{Z}\in\mathsf{Ab}$. With a little more work we could also show that this operation is associative, and hence that $(\mathsf{Obj}(\mathsf{Ab}),\otimes,\mathbb{Z})$ is a monoid.[12]

So, does the operation $\otimes$ really exist? There are two possible responses:

- Yes, it exists. I'll sketch a proof below in Example (3) by constructing it out of things you believe in.
- Who cares if it really exists? The uniqueness of adjoints tells us that the operation $\otimes$ is uniquely determined by its universal property, and this is actually the best way to work with it. What more do you want?

[12] Well, it would be if $\mathsf{Obj}(\mathsf{Ab})$ were a set.

Let me elaborate on this second response. We already know that the family of hom functors $(-)^C = \mathsf{Hom}_{\mathsf{Ab}}(C, -) : \mathsf{Ab} \to \mathsf{Ab}$ assembles into the hom bifunctor:

$$\mathsf{Hom}_{\mathsf{Ab}}(-,-) : \mathsf{Ab}^{\mathsf{op}} \times \mathsf{Ab} \to \mathsf{Ab}.$$

If each functor $(-)^C$ has a (unique) left adjoint, called $(-) \otimes C$, then I claim that these left adjoints also assemble into a (unique) bifunctor:

$$(-) \otimes (-) : \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}.$$

In fact, I will prove something more general.

Theorem (Two Variable Adjunction). Consider any bifunctor[13] $R : \mathcal{D}^{\mathsf{op}} \times \mathcal{E} \to \mathcal{C}$ and suppose that for each object $d \in \mathcal{D}$ the induced functor $R(d, -) : \mathcal{E} \to \mathcal{C}$ has a left adjoint $L_d : \mathcal{C} \to \mathcal{E}$. That is, suppose that we have a family of bijections

$$\Phi_{c,d,e} : \mathsf{Hom}_{\mathcal{E}}(L_d(c), e) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(c, R(d, e))$$

that is natural in $(c, e) \in \mathcal{C}^{\mathsf{op}} \times \mathcal{E}$. In this case, I claim that there exists a unique bifunctor $L : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ with the property $L(c, d) = L_d(c)$ and such that the family of bijections $\Phi_{c,d,e}$ **is also natural in** $d \in \mathcal{D}^{\mathsf{op}}$. In other words, we obtain a natural isomorphism

$$\mathsf{Hom}_{\mathcal{E}}(L(-,-),-) \cong \mathsf{Hom}_{\mathcal{C}}(-, R(-,-))$$

of functors $\mathcal{C}^{\mathsf{op}} \times \mathcal{D}^{\mathsf{op}} \times \mathcal{E} \to \mathsf{Set}$. ///

Assuming this, our desired result follows by substituting

$$\mathcal{C} = \mathcal{D} = \mathcal{E} = \mathsf{Ab}, \quad R(-,-) = \mathsf{Hom}_{\mathsf{Ab}}(-,-), \quad \text{and} \quad L_C(-) = (-) \otimes C \text{ for each } C \in \mathsf{Ab}.$$

Proof: We must construct a bifunctor $L : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ with the desired properties. So consider any arrow $\delta : d_2 \to d_1$ and then define the function $\Lambda_\delta : \mathsf{Hom}_{\mathcal{E}}(L_{d_1}(c), e) \to \mathsf{Hom}_{\mathcal{E}}(L_{d_2}(c), e)$ so that the following square commutes:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{E}}(L_{d_2}(c), e) & \mathsf{Hom}_{\mathcal{C}}(c, R(d_2, e)) \arrow[l, "\Phi^{-1}_{c,d_2,e}"'] \\
\mathsf{Hom}_{\mathcal{E}}(L_{d_1}(c), e) \arrow[u, "\Lambda_\delta"] \arrow[r, "\Phi_{c,d_1,e}"'] & \mathsf{Hom}_{\mathcal{C}}(c, R(d_1, e)) \arrow[u, "R(\delta,e)\circ(-)"']
\end{tikzcd}$$

Our goal is to show that **there exists a unique arrow** $L_\delta(c) : L_{d_2}(c) \to L_{d_1}(c)$ **with the property** $\Lambda_\delta(-) = (-) \circ L_\delta(c)$. If this is true, then by composing two such squares vertically (coming, say, from two arrows $\delta, \varepsilon \in \mathsf{Arr}(\mathcal{D})$) then we obtain the equation

$$(-) \circ L_{\delta\circ\varepsilon}(c) = \Lambda_{\delta\circ\varepsilon} = (-) \circ (L_\delta(c) \circ L_\varepsilon(c)),$$

[13] of locally small categories

and from uniqueness it will follow that $L_{\delta\circ\varepsilon}(c) = L_\delta(c) \circ L_\varepsilon(c)$. Hence $L(c,d) \coloneqq L_d(c)$ will be the desired (unique) bifunctor.

Now, if **some** arrow $L_\delta(c) : L_{d_2}(c) \to L_{d_1}(c)$ exists with the property $\Lambda_\delta(-) = (-) \circ L_\delta(c)$ then it **must** be defined by

$$L_\delta(c) = \mathsf{id}_{L_{d_1}(c)} \circ L_\delta(c) = \Lambda_\delta(\mathsf{id}_{L_{d_1}(c)}),$$

and it remains only to check that the equality $\Lambda_\delta(-) = (-) \circ \Lambda_\delta(\mathsf{id}_{L_{d_1}(c)})$ is **actually true**. In other words, for any fixed arrow $\varphi : L_{d_1}(c) \to e$ we must show that $\Lambda_\delta(\varphi) = \varphi \circ \Lambda_\delta(\mathsf{id}_{L_{d_1}(c)})$.

We will prove this by drawing **two commutative diagrams**, both of which will use the assumption that $\Phi_{c,d,e}$ is natural in $e \in \mathcal{E}$. The first diagram is obtained by composing the original commutative square with a naturality square for the arrow $\varphi \in \mathsf{Arr}(\mathcal{E})$:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{E}}(L_{d_2}(c), e) \arrow[r, "\Phi_{c,d_2,e}"] & \mathsf{Hom}_{\mathcal{C}}(c, R(d_2, e)) \\
\mathsf{Hom}_{\mathcal{E}}(L_{d_1}(c), e) \arrow[u, "\Lambda_\delta"] \arrow[r, "\Phi_{c,d_1,e}"] & \mathsf{Hom}_{\mathcal{C}}(c, R(d_1, e)) \arrow[u, "R(\delta,e)\circ(-)"'] \\
\mathsf{Hom}_{\mathcal{E}}(L_{d_1}(c), L_{d_1}(c)) \arrow[u, "\varphi\circ(-)"] \arrow[r, "\Phi_{c,d_1,L_{d_1}(c)}"] & \mathsf{Hom}_{\mathcal{C}}(c, R(d_1, L_{d_1}(c))) \arrow[u, "R(d_1,\varphi)\circ(-)"']
\end{tikzcd}$$

Following $\mathsf{id}_{L_{d_1}(c)} \in \mathsf{Hom}_{\mathcal{E}}(L_{d_1}(c), L_{d_1}(c))$ from the bottom left to the top right gives

$$\begin{aligned}
R(\delta, e) \circ R(d_1, \varphi) \circ \Phi_{c,d_1,L_{d_1}(c)}(\mathsf{id}_{L_{d_1}(c)}) &= \Phi_{c,d_2,e}(\Lambda_\delta(\varphi \circ \mathsf{id}_{L_{d_1}(c)})) \\
&= \Phi_{c,d_2,e}(\Lambda_\delta(\varphi)).
\end{aligned}$$

The second diagram is obtained by substituting $e = L_{d_1}(c)$ into the original commutative square and then by composing this with a different naturality square for $\varphi \in \mathsf{Arr}(\mathcal{E})$:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{E}}(L_{d_2}(c), e) \arrow[r, "\Phi_{c,d_2,e}"] & \mathsf{Hom}_{\mathcal{C}}(c, R(d_2, e))) \\
\mathsf{Hom}_{\mathcal{E}}(L_{d_2}(c), L_{d_1}(c)) \arrow[u, "\varphi\circ(-)"] \arrow[r, "\Phi_{c,d_2,L_{d_1}(c)}"] & \mathsf{Hom}_{\mathcal{C}}(c, R(d_2, L_{d_1}(c))) \arrow[u, "R(d_2,\varphi)\circ(-)"'] \\
\mathsf{Hom}_{\mathcal{E}}(L_{d_1}(c), L_{d_1}(c)) \arrow[u, "\Lambda_\delta"] \arrow[r, "\Phi_{c,d_1,L_{d_1}(c)}"] & \mathsf{Hom}_{\mathcal{C}}(c, R(d_1, L_{d_1}(c))) \arrow[u, "R(\delta,L_{d_1}(c))\circ(-)"']
\end{tikzcd}$$

Following $\mathsf{id}_{L_{d_1}(c)} \in \mathsf{Hom}_{\mathcal{E}}(L_{d_1}(c), L_{d_1}(c))$ from the bottom left to the top right gives

$$R(d_2, \varphi) \circ R(\delta, L_{d_1}(c)) \circ \Phi_{c,d_1,L_{d_1}(c)}(\mathsf{id}_{L_{d_1}(c)}) = \Phi_{c,d_2,e}(\varphi \circ \Lambda_\delta(\mathsf{id}_{L_{d_1}(c)})).$$

Finally, recall that $R : \mathcal{D}^{\mathsf{op}} \times \mathcal{E} \to \mathcal{C}$ was assumed to be a "bifunctor." By definition this implies that $R(\delta, e) \circ R(d_1, \varphi) = R(d_2, \varphi) \circ R(\delta, L_{d_1}(c))$. Then putting the two previous identities together gives

$$\begin{aligned}\Phi_{c,d_2,e}(\Lambda_\delta(\varphi)) &= R(\delta, e) \circ R(d_1, \varphi) \circ \Phi_{c,d_1,L_{d_1}(c)}(\mathsf{id}_{L_{d_1}(c)})\\ &= R(d_2, \varphi) \circ R(\delta, L_{d_1}(c)) \circ \Phi_{c,d_1,L_{d_1}(c)}(\mathsf{id}_{L_{d_1}(c)})\\ &= \Phi_{c,d_2,e}(\varphi \circ \Lambda_\delta(\mathsf{id}_{L_{d_1}(c)})),\end{aligned}$$

and applying $\Phi^{-1}_{c,d_2,e}$ to both sides gives $\Lambda_\delta(\varphi) = \varphi \circ \Lambda_\delta(\mathsf{id}_{L_{d_1}(c)})$, as desired.

In fact, we could have drawn the two diagrams together as a **cube** (it would have looked terrible), in which five of the six sides commute by assumption. The argument just presented says that the sixth side of the cube necessarily commutes. □

[Remark: This also completes the proof from the previous section that if all limits of shape $\mathcal{I}$ exist in a category $\mathcal{C}$ then we have an adjunction $(-)^{\mathcal{I}} : \mathcal{C} \rightleftarrows \mathcal{C}^{\mathcal{I}} : \lim_I$. I didn't prove it at the time because we didn't need it. But now we get it for free.]

Thus the hom bifunctor $\mathsf{Hom}_{\mathsf{Ab}}(-,-) : \mathsf{Ab}^{\mathsf{op}} \times \mathsf{Ab} \to \mathsf{Ab}$ (contravariant in the first coordinate) uniquely determines the bifunctor $(-) \otimes (-) : \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$ (covariant in both coordinates); that is, if such a functor really exists. It is amusing to note that a dual version of the same argument shows that the hom bifunctor is uniquely determined by the tensor product. That could be useful on some planet.

1.9.3 Free ⊣ Forget

People usually prove the existence of the tensor product by constructing it as a quotient of a "free abelian group." What does this mean?

I previously defined a "subcategory" $\mathcal{C} \subseteq \mathcal{D}$ by saying that $\mathsf{Obj}(\mathcal{C}) \subseteq \mathsf{Obj}(\mathcal{D})$ is a subcollection, and for each pair of objects $c_1, c_2 \in \mathcal{C}$ we have a subcollection of arrows $\mathsf{Hom}_{\mathcal{C}}(c_1, c_2) \subseteq \mathsf{Hom}_{\mathcal{D}}(c_1, c_2)$. But that's not a very categorical definition. In fact, it seems that the notion of "subcategory" is very difficult to define in categorical terms.[14] So instead of **subcategories** and **full subcategories** $\mathcal{C} \subseteq \mathcal{D}$ we prefer to talk about **faithful** and **fully faithful** functors $U : \mathcal{C} \to \mathcal{D}$, respectively.

Definition of Concrete Category. A *concrete category* is a pair $(\mathcal{C}, U)$ where $\mathcal{C}$ is a category and U is a **faithful** functor to the category of sets: $U : \mathcal{C} \to \mathsf{Set}$. In this case, every object $c \in \mathcal{C}$ has an *underlying set* $U(c) \in \mathsf{Set}$ and every arrow $\varphi : c_1 \to c_2$ in $\mathcal{C}$ has an *underlying function* $U(\varphi) : U(c_1) \to U(c_2)$. This explains our use of the letter "U". ///

[14] Some people say the notion of subcategory is "evil."

Any category of "sets with structure" is concrete. As an illustration let's consider the concrete category of abelian groups:

$$U : \mathsf{Ab} \to \mathsf{Set}.$$

One can show that this functor **preserves limits**. For example, it sends the product group $A \oplus B$ to the Cartesian product of sets $U(A \oplus B) = U(A) \times U(B)$ and it sends the zero group $0 \in \mathsf{Ab}$ (thought of as the final object in Ab) to the one point set $U(0) = * \in \mathsf{Set}$, which is the final object in Set.

Exercise: Show that $U : \mathsf{Ab} \to \mathsf{Set}$ does **not** preserve colimits.

This suggests that the functor $U : \mathsf{Ab} \to \mathsf{Set}$ could possibly have a left adjoint. Let's assume for the sake of argument that such a left adjoint does exist, and let's call it $F : \mathsf{Set} \to \mathsf{Ab}$. That is, let's assume that we have a family of bijections

$$\boxed{\mathsf{Hom}_{\mathsf{Set}}(S, U(A)) \leftrightarrow \mathsf{Hom}_{\mathsf{Ab}}(F(S), A)}$$

that is natural in $(S, A) \in \mathsf{Set}^{\mathsf{op}} \times \mathsf{Ab}$. What are the properties of this hypothetical functor?

First of all, let S be **any set** and consider the identity homomorphism $\mathsf{id}_{F(S)} : F(S) \to F(S)$. Applying the adjunction map $\mathsf{Hom}_{\mathsf{Ab}}(F(S), F(S)) \to \mathsf{Hom}_{\mathsf{Set}}(S, U(F(S)))$ to this homomorphism yields a function from S to the underlying set of the group $F(S)$:

$$\overline{\mathsf{id}_{F(S)}} : S \to U(F(S)).$$

Now consider **any abelian group** $A \in \mathsf{Ab}$ and suppose that we have a function from S into the underlying set of A:

$$\varphi : S \to U(A).$$

In this case, I claim **that there exists a unique group homomorphism** $\upsilon : F(S) \to A$ making the following diagram commute:

$$\begin{tikzcd}
U(F(S)) \arrow[rr, "U(\upsilon)"] & & U(A) \\
& S \arrow[ul, "\overline{\mathsf{id}_{F(S)}}"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Indeed, if **some** homomorphism υ exists with the property $\varphi = U(\upsilon) \circ \overline{\mathsf{id}_{F(S)}}$, then from the naturality of the adjunction we **must** have

$$\begin{aligned}
\upsilon &= \upsilon \circ \mathsf{id}_{F(S)} \\
&= \upsilon \circ \overline{\overline{\mathsf{id}_{F(S)}}} \\
&= \overline{U(\upsilon) \circ \overline{\mathsf{id}_{F(S)}}} && \text{naturality of } F \dashv U \\
&= \overline{\varphi}.
\end{aligned}$$

It remains only to show that the equation $\varphi = U(\overline{\varphi}) \circ \overline{\mathsf{id}_{F(S)}}$ is **actually true**. This also follows from the naturality of adjunction. Indeed, we have

$$\begin{aligned} \overline{\varphi} &= \overline{\varphi} \circ \mathsf{id}_{F(S)} \\ &= \overline{\varphi} \circ \overline{\overline{\mathsf{id}_{F(S)}}} \\ &= \overline{U(\overline{\varphi}) \circ \overline{\mathsf{id}_{F(S)}}}, && \text{naturality of } F \dashv U \end{aligned}$$

and then applying the adjunction to both sides gives the result.

We can summarize the situation as follows:

$F(S)$ is the free abelian group generated by the set S.

Let me try to make this clear. For each set $S \in \mathsf{Set}$ we have an abelian group $F(S)$ "freely generated by S" and a canonical function $\eta_S := \overline{\mathsf{id}_{F(S)}}$ that "inserts the generators" into the group, $\eta_S : S \to U(F(S))$. Furthermore, let $A \in \mathsf{Ab}$ be any abelian group and let $\varphi : S \to U(A)$ be any assignment of the generators to elements of A. The "freeness" of $F(S)$ asserts that the function φ **extends uniquely** to a homomorphism on the whole group $\overline{\varphi} : F(S) \to A$. Compare this to the situation in linear algebra, where a linear function is determined uniquely by its values on a basis. Thus we can think of the image $\eta_S(S) \subseteq U(F(S))$ as a "basis" for the group $F(S)$.

Definition of Free Functor. Let $(\mathcal{C}, U)$ be any concrete category and suppose that the faithful functor $U : \mathcal{C} \to \mathsf{Set}$ has a left adjoint $F : \mathsf{Set} \to \mathcal{C}$. Then we call F a *free functor* and for each set $S \in \mathsf{Set}$ we call $F(S) \in \mathcal{C}$ the *free object generated by* S. This explains our use of the letter "F". ///

The uniqueness of adjoints says that free abelian groups are determined up to isomorphism by the "underlying set" functor $U : \mathsf{Ab} \to \mathsf{Set}$, and the RAPL theorem tells us things like $F(A \coprod B) \cong F(A) \oplus F(B)$ and $F(\varnothing) \cong 0$. As with the tensor product $(-) \otimes (-) : \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$, our only question about the functor $F : \mathsf{Set} \to \mathsf{Ab}$ is whether it really exists.

The existence of F is easer to prove than the existence of $\otimes$. To gain a bit of intuition for the proof, let me sketch the computation of $F(S)$ for **finite** sets S. We begin by showing that $F(*) \cong \mathbb{Z}$. Recall that for any abelian group $A \in \mathsf{Ab}$ we have a bijection $\mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}, A) \leftrightarrow U(A)$ defined by sending each homomorphism $\varphi : \mathbb{Z} \to A$ to the element $\varphi(1_{\mathbb{Z}}) \in U(A)$. Furthermore, recall that the one point set $*$ is the final object in Set so we have a bijection $U(A) \leftrightarrow \mathsf{Hom}_{\mathsf{Set}}(*, U(A))$. Thus we have a chain of bijections

$$\mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}, A) \leftrightarrow U(A) \leftrightarrow \mathsf{Hom}_{\mathsf{Set}}(*, U(A)) \leftrightarrow \mathsf{Hom}_{\mathsf{Ab}}(F(*), A),$$

and by composing them we obtain a bijection $\mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}, A) \leftrightarrow \mathsf{Hom}_{\mathsf{Ab}}(F(*), A)$. If you're willing to believe that this bijection is natural in $A \in \mathsf{Ab}$ then by the essential injectivity of the Yoneda embedding we obtain an isomorphism of groups: $F(*) \cong \mathbb{Z}$. Now let $S \in \mathsf{Set}$ be any finite set of cardinality $s \in \mathbb{N}$ and note that we have a bijection between S and the disjoint union of s one point sets:

$$S \leftrightarrow \overbrace{* \coprod * \coprod \cdots \coprod *}^{s \text{ times}}.$$

Finally, the functor F sends this bijection to an isomorphism of groups:

$$\begin{aligned} F(S) &\cong F(* \coprod * \coprod \cdots \coprod *) \\ &\cong F(*) \oplus F(*) \oplus \cdots \oplus F(*) && \text{RAPL} \\ &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \\ &=: \mathbb{Z}^{\oplus s}. \end{aligned}$$

The following theorem just makes this calculation rigorous and extends it to infinite sets.

Theorem (Free Abelian Groups Exist). For any set S, we can add two functions $f_1, f_2 : S \to U(\mathbb{Z})$ by defining $(f_1 + f_2)(s) := f_1(s) + f_2(s)$ for all $s \in S$. Thus we obtain two abelian groups:

$$\begin{aligned} \mathbb{Z}^S &:= \{ \text{ functions } f : S \to U(\mathbb{Z}) \ \} \\ \mathbb{Z}^{\oplus S} &:= \{ \text{ functions } f : S \to U(\mathbb{Z}) : f(s) = 0_\mathbb{Z} \text{ for all but finitely many } s \in S \ \}. \end{aligned}$$

When S is **finite** we have $\mathbb{Z}^{\oplus S} = \mathbb{Z}^S$, but for **infinite sets** S we have $\mathbb{Z}^{\oplus S} \subsetneq \mathbb{Z}^S$. In general, I claim that $\mathbb{Z}^{\oplus S}$ is the free abelian group generated by S. ///

Proof: We need to construct a functor $F : \mathsf{Set} \to \mathsf{Ab}$ that is left adjoint to the forgetful functor $U : \mathsf{Ab} \to \mathsf{Set}$, and that is defined on objects by $F(S) := \mathbb{Z}^{\oplus S}$.

To define F on arrows we need to look more closely at the structure of the group $\mathbb{Z}^{\oplus S}$. First note that for each element $s \in S$ we have a "delta function" $\delta_s \in \mathbb{Z}^{\oplus S}$ defined by

$$\delta_s(t) := \begin{cases} 1_\mathbb{Z} & s = t \\ 0_\mathbb{Z} & s \neq t \end{cases}.$$

Then we can express any function of finite support $f \in \mathbb{Z}^{\oplus S}$ as a $\mathbb{Z}$-linear combination of delta functions: $f = \sum_{s \in S} f(s) \cdot \delta_s$. (Even though S may be an infinite set, we can regard this as a **finite sum** since all but finitely many of the summands $f(s) \cdot \delta_s$ are the zero function.) Note that this expression is unique since if $\sum_{s\in S} f(s) \cdot \delta_s = \sum_{s \in S} c_s \cdot \delta_s$ for some integers $c_s \in \mathbb{Z}$, then by applying these functions to the element $t \in S$ we obtain

$$c_t = \left(\sum_{s \in S} c_s \cdot \delta_s \right)(t) = \left(\sum_{s \in S} f(s) \cdot \delta_s \right)(t) = f(t).$$

In other words, the set of functions $\{\delta_s\}_{s\in S}$ is a "$\mathbb{Z}$-basis for the $\mathbb{Z}$-module $\mathbb{Z}^{\oplus S}$." Now given any function $g : S_1 \to S_2$ we define a mapping on basis elements by $\delta_s \mapsto \delta_{g(s)}$ and then we extend this to a group homomorphism $F(g) : \mathbb{Z}^{\oplus S_1} \to \mathbb{Z}^{\oplus S_2}$ by $\mathbb{Z}$-linearity:

$$F(g)\left(\sum_{s\in S_1} c_s \cdot \delta_s\right) := \sum_{s\in S_1} c_s \cdot \delta_{g(s)} \in \mathbb{Z}^{\oplus S_2}.$$

One can check that $F(g_1 \circ g_2) = F(g_1) \circ F(g_2)$, and hence we obtain a functor $F : \mathsf{Set} \to \mathsf{Ab}$.

To show that $F \dashv U$, we must construct a family of bijections

$$\mathsf{Hom}_{\mathsf{Set}}(S, U(A)) \leftrightarrow \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}^{\oplus S}, A)$$

that is natural in $(S, A) \in \mathsf{Set}^{\mathsf{op}} \times \mathsf{Ab}$. This family of bijections can be summarized as follows:

any homomorphism $\mathbb{Z}^{\oplus S} \to A$ is uniquely determined by its values on the basis δ_s.

To be precise, consider any group homomorphism $\varphi : \mathbb{Z}^{\oplus S} \to A$. Then we will define the function $\overline{\varphi} : S \to U(A)$ by setting $\overline{\varphi}(s) := \varphi(\delta_s)$ for all $s \in S$. To show that the mapping $\varphi \mapsto \overline{\varphi}$ is **injective**, suppose that we have $\overline{\varphi_1} = \overline{\varphi_2}$, so that $\varphi_1(\delta_s) = \varphi_2(\delta_s)$ for all $s \in S$. Then for any function of finite support $f \in \mathbb{Z}^{\oplus S}$ we have

$$\begin{aligned}
\varphi_1(f) &= \varphi_1\left(\sum_{s\in S} f(s)\cdot\delta_s\right)\\
&= \sum_{s\in S} f(s)\cdot\varphi_1(\delta_s)\\
&= \sum_{s\in S} f(s)\cdot\varphi_2(\delta_s)\\
&= \varphi_2\left(\sum_{s\in S} f(s)\cdot\delta_s\right)\\
&= \varphi_2(f),
\end{aligned}$$

and it follows that $\varphi_1 = \varphi_2$. To show that the mapping $\varphi \mapsto \overline{\varphi}$ is **surjective**, consider any function $b : S \to U(A)$. Then we obtain a group homomorphism $\varphi_b : \mathbb{Z}^{\oplus S} \to A$ by defining $\varphi_b(\delta_s) := b(s)$ for each $s \in S$ and extending by $\mathbb{Z}$-linearity. That is, for all $f \in \mathbb{Z}^{\oplus S}$ we define

$$\varphi_b(f) = \varphi_b\left(\sum_{s\in S} f(s)\cdot\delta_s\right) := \sum_{s\in S} f(s)\cdot b(s) \in A.$$

One can check that φ_b is indeed a homomorphism and that $\overline{\varphi_b} = b$, as desired.

Finally, we will show that the family of bijections

$$\begin{array}{ccc}
\mathsf{Hom}_{\mathsf{Set}}(S, U(A)) & \leftrightarrow & \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}^{\oplus S}, A)\\
\overline{\varphi} & \leftarrow\!\shortmid & \varphi
\end{array}$$

is natural in $(S, A) \in \mathsf{Set}^{\mathsf{op}} \times \mathsf{Ab}$. First consider any arrow $g : S_1 \to S_2$ in $\mathsf{Set}^{\mathsf{op}}$ (i.e., any function $g : S_2 \to S_1$). We want to show that the the following square commutes:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Set}}(S_2, U(A)) \arrow[r, leftrightarrow] & \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}^{\oplus S_2}, A) \\
\mathsf{Hom}_{\mathsf{Set}}(S_1, U(A)) \arrow[r, leftrightarrow] \arrow[u, "(-)\circ g"] & \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}^{\oplus S_1}, A) \arrow[u, "(-)\circ F(g)"']
\end{tikzcd}$$

So consider any group homomorphism $\varphi : \mathbb{Z}^{\oplus S_1} \to A$. Note that for all $s \in S_2$ the function $\overline{\varphi} \circ g : S_2 \to U(A)$ is defined by

$$(\overline{\varphi} \circ g)(s) = \overline{\varphi}(g(s)) = \varphi(\delta_{g(s)}),$$

and the function $\overline{\varphi \circ F(g)} : S_2 \to U(A)$ is defined by

$$\overline{\varphi \circ F(g)}(s) = (\varphi \circ F(g))(\delta_s) = \varphi(F(g)(\delta_s)) = \varphi(\delta_{g(s)}).$$

Hence we have $\overline{\varphi} \circ g = \overline{\varphi \circ F(g)}$ as desired.

Then for any homomorphism $\alpha : A_1 \to A_2$ we want to show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Set}}(S, U(A_2)) \arrow[r, leftrightarrow] & \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}^{\oplus S}, A_2) \\
\mathsf{Hom}_{\mathsf{Set}}(S, U(A_1)) \arrow[r, leftrightarrow] \arrow[u, "U(\alpha)\circ(-)"] & \mathsf{Hom}_{\mathsf{Ab}}(\mathbb{Z}^{\oplus S}, A_1) \arrow[u, "\alpha\circ(-)"']
\end{tikzcd}$$

So consider any group homomorphism $\varphi : \mathbb{Z}^{\oplus S} \to A_1$. Note that for all $s \in S$ the function $U(\alpha) \circ \overline{\varphi} : S \to U(A_2)$ is defined by

$$(U(\alpha) \circ \overline{\varphi})(s) = U(\alpha)(\overline{\varphi}(s)) = \alpha(\varphi(\delta_s))$$

and the function $\overline{\alpha \circ \varphi} : S \to U(A_2)$ is defined by

$$\overline{\alpha \circ \varphi}(s) = (\alpha \circ \varphi)(\delta_s) = \alpha(\varphi(\delta_s)).$$

Hence we have $U(\alpha) \circ \overline{\varphi} = \overline{\alpha \circ \varphi}$ as desired. □

Finally, we can construct the tensor product of abelian groups.

Theorem (Tensor Products Exist). Consider two abelian groups $A, B \in \mathsf{Ab}$. Now consider the free abelian group $F(S) \in \mathsf{Ab}$ generated by the following set of abstract symbols:

$$S := \{\text{“}a \otimes b\text{”} : a \in A, b \in B\}.$$

Furthermore, let $I \subseteq F(S)$ be the smallest subgroup of $F(S)$ containing the elements:

- “$a_1 \otimes b$” + “$a_2 \otimes b$” − “$(a_1 + a_2) \otimes b$” for all $a_1, a_2 \in A$ and $b \in B$,

- "$a \otimes b_1$" + "$a \otimes b_2$" − "$a \otimes (b_1 + b_2)$" for all $a \in A$ and $b_1, b_2 \in B$.

Then I claim that the quotient group $F(S)/I \in \mathsf{Ab}$ (which **exists**) satisfies the universal property of the tensor product $A \otimes B$. ///

This is not a real theorem so I'm not going to prove it. Instead I'll just enunciate the universal property that it is trying to model. Recall from the theorem on Two Variable Adjunctions that the bifunctor $(-) \otimes (-) : \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$, if it exists at all, is uniquely characterized by a family of bijections

$$\Phi_{A,B,C} : \mathsf{Hom}_{\mathsf{Ab}}(A \otimes B, C) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{Ab}}(A, \mathsf{Hom}_{\mathsf{Ab}}(B, C))$$

that is natural in $(A, B, C) \in \mathsf{Ab}^{\mathsf{op}} \times \mathsf{Ab}^{\mathsf{op}} \times \mathsf{Ab}$. If we want, we can think of the elements $\varphi \in \mathsf{Hom}_{\mathsf{Ab}}(A, \mathsf{Hom}_{\mathsf{Ab}}(B, C))$ as "bihomomorphisms" $A \times B \to C$. Indeed, for each element $a \in A$ we have a group homomorphism $\varphi(a) : B \to C$. Then by abuse of notation we can define a function $\varphi : A \times B \to C$ by $\varphi(a, b) := \varphi(a)(b)$ for all $(a, b) \in A \times B$. Observe from the definition that each of the component functions $\varphi(a)(-) : B \to C$ and $\varphi(-)(b) : A \to C$ is a group homomorphism, i.e., that φ is a *bihomomorphism*. This suggests that we should introduce the following notation:

$$\mathsf{Hom}_{\mathsf{Ab}}(A, B; C) := \mathsf{Hom}_{\mathsf{Ab}}(A, \mathsf{Hom}_{\mathsf{Ab}}(B, C)).$$

The adjunction can now be summarized as follows:

homomorphisms from $A \otimes B$ are the same as bihomomorphisms from $A \times B$.

Now let us substitute $C = A \otimes B$ and consider the family of bijections

$$\Phi_{A,B,A\otimes B} : \mathsf{Hom}_{\mathsf{Ab}}(A \otimes B, A \otimes B) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{Ab}}(A, B; A \otimes B),$$

which is natural in $(A, B) \in \mathsf{Ab}^{\mathsf{op}} \times \mathsf{Ab}^{\mathsf{op}}$. By applying this to the identity arrow $\mathsf{id}_{A\otimes B}$ we obtain a *canonical bihomomorphism*

$$\Phi_{A,B,A\otimes B}(\mathsf{id}_{A\otimes B}) : A \times B \to A \otimes B.$$

This notation is getting out of hand, so let's simplify things by writing

$$\tau := \Phi_{A,B,A\otimes B}(\mathsf{id}_{A\otimes B}).$$

Then for each pair of elements $(a, b) \in A \times B$ we will write "$a \otimes b$" $:= \tau(a, b) \in A \otimes B$.

Finally, consider **any group homomorphism** $\varphi : A \otimes B \to C$. By the naturality of the hom-tensor adjunction this φ induces a commutative square:

$$\begin{tikzcd}
\varphi \in & \mathsf{Hom}_{\mathsf{Ab}}(A \otimes B, C) \arrow[r, leftrightarrow] & \mathsf{Hom}_{\mathsf{Ab}}(A, B; C) & \ni \varphi \circ \tau \\
\mathsf{id}_{A\otimes B} \in & \mathsf{Hom}_{\mathsf{Ab}}(A \otimes B, A \otimes B) \arrow[r, leftrightarrow] \arrow[u, "\varphi\circ(-)"] & \mathsf{Hom}_{\mathsf{Ab}}(A, B; A \otimes B) \arrow[u, "\varphi\circ(-)"'] & \ni \tau
\end{tikzcd}$$

Now let's follow the canonical bihomomorphism $\tau \in \mathsf{Hom}_{\mathsf{Ab}}(A, B; A \otimes B)$ from the bottom right to the top left in two different ways. Going around the bottom/left of the square just gives φ. Then going around the right/top of the square tells us that φ is uniquely determined by the bihomomorphism $\varphi \circ \tau : A \times B \to C$. In other words, the homomorphism $\varphi : A \otimes B \to C$ is uniquely determined by the elements $\varphi(\tau(a, b)) = \varphi(\text{“}a \otimes b\text{”}) \in C$.

This suggests that $A \otimes B$ can be constructed as an abelian group **generated** by the abstract symbols “$a \otimes b$” and subject to some **relations** turning the insertion of generators function $(a, b) \mapsto$ “$a \otimes b$” into a bihomomorphism. The theorem just makes this idea explicit. Many books will give this construction as the **definition** of the tensor product, which I think is completely backwards.

Chapter 2

The Category of G-Sets

2.1 What is Representation Theory?

Abstraction turns a concept (such as symmetries of an object) into an axiomatic theory (such as the definition of a group). It is often helpful to ignore context and to develop group theory in the abstract. However, we should not forget why we care about groups. That is, given an abstract group G we should want to view (represent) G as symmetries of some object. More formally, given an abstract structure such as a group G we should consider the collection (hopefully a category) of all possible ways that G can "act" as symmetries on an object. The philosophy of representation theory is that we don't lose any information by doing this (see Tannaka Duality below), but in fact we gain more structure that can be used to solve problems.

2.2 Actions are Functors

A "representation" of an algebraic object A should be viewed as a functor $F : A \to \mathcal{C}$ from A into some suitable category $\mathcal{C}$. For this to work, we first need to view A itself as a category. To see how this works, we consider the most primitive example: a monoid acting on a set.

Monoid actions are functors. Looping/Delooping adjunction. The category of M-sets.

2.3 Tannaka Duality

Does the category of M-sets determine the monoid M? Not quite, but it we view M-Set as a "concrete category," i.e., if we consider M-Set together with the faithful "underlying set" functor $U : M\text{-}\mathsf{Set} \to \mathsf{Set}$ then the answer is yes.

Lemma. (Full Version of Yoneda) $\mathsf{Nat}(H^c, F) \cong F(c)$

Theorem (Tannaka Duality): Consider the underlying set functor $U : M\text{-}\mathsf{Set} \to \mathsf{Set}$ as an element of the functor category $[M\text{-}\mathsf{Set} \to \mathsf{Set}]$. Then we have a canonical isomorphism

$$\mathsf{End}_{[M\text{-}\mathsf{Set}\to\mathsf{Set}]}(U) \cong M.$$

(Does it remind us of Galois Theory?)

Proof: Recall that M-Set is by definition the functor category $[BM \to \mathsf{Set}]$. In this language, we observe that $U : M\text{-}\mathsf{Set} \to \mathsf{Set}$ is the same as the evaluation functor $\mathsf{Eval}(-, *) : [BM \to \mathsf{Set}] \to \mathsf{Set}$ at the unique object $* \in M\text{-}\mathsf{Set}$. Then Yoneda gives us an isomorphism

$$U = \mathsf{Eval}(-, *) \cong \mathsf{Nat}(H^*, -) = H^{H^*}.$$

Finally, we apply Yoneda twice to obtain a natural bijections

$$\begin{aligned}
\mathsf{Hom}(U, U) &\cong \mathsf{Hom}(H^{H^*}, H^{H^*}) && \text{in } [[BM \to \mathsf{Set}] \to \mathsf{Set}] \\
&\cong \mathsf{Hom}(H^*, H^*) && \text{in } [BM \to \mathsf{Set}] \\
&\cong \mathsf{Hom}(*, *) && \text{in } BM \\
&= M,
\end{aligned}$$

as desired. □

2.4 The Category of G-Sets

Indecomposable M-set implies irreducible? Only if $M = G$ is a group. Then we get a unique factorization into simples. Note that morphisms in G-Set are determined by morphisms between simples: $\mathsf{Hom}(G/H, G/K)$. What is this set?

2.5 The Burnside Ring (Grothendieck Ring of G-Set)

[Solomon 1967] The Burnside Algebra of a Finite Group.

[Greene 1973] On the Möbius Algebra of a Partially Ordered Set

An opportunity to discuss complete systems of orthogonal idempotents.

Chapter 3

Enriched Categories

The material in Chapter 2 is in some sense the prototype for all of representation theory. If G is any monoid or group, recall that we defined the category of G-sets as follows:

- Think of G as a category BG with one object.
- Consider the category of functors $F : BG \to \mathsf{Set}$.

In this chapter we will examine how to define "representations" of algebraic structures beyond just monoids and groups. Here is the motivating question:

Question: How can we view a ring as a category with one object?

3.1 Motivation: Categories Enriched Over $\mathbf{2} = (0 \to 1)$.

Categories enriched over $\mathbf{2} = (0 \to 1)$ are called posets. Poset version of Yoneda (Birkhoff transform). Dedekind-MacNeille completion.

Let P be a poset. Then we have two (enriched) Yoneda embeddings:

$$P \hookrightarrow \left(\mathbf{2}^P\right)^{\mathsf{op}},$$
$$P \hookrightarrow \mathbf{2}^{P^{\mathsf{op}}}.$$

By general properties of the Yoneda embedding we know that $\left(\mathbf{2}^P\right)^{\mathsf{op}}$ has all meets and that $\mathbf{2}^{P^{\mathsf{op}}}$ has all joins. Furthermore, since we are working with posets, the existence of all meets/joins implies the existence of all joins/meets, hence both posets are lattices.

Question: Thus we have two different ways to embed a general poset P into a lattice. Is it possible to find a **smallest** such embedding $P \hookrightarrow \bar{P}$? This emebeeding should factor through the Yoneda embeddings as follows:

$$\begin{tikzcd}
 & & (\mathbf{2}^P)^{\mathsf{op}} \\
P \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & \bar{P} \arrow[ur] \arrow[dr] & \\
 & & \mathbf{2}^{P^{\mathsf{op}}}
\end{tikzcd}$$

This problem was solved by Dedekind for $P = \mathbb{Q}$ and generalized to other posets by MacNeille. Here's the idea: We first have an adjunction between Boolean lattices

$$(-)^\vee : 2^P \rightleftarrows (2^P)^{\mathsf{op}} : (-)^\wedge,$$

where $A^\wedge$ and $A^\vee$ are the sets of lower and upper bounds of $A \subseteq P$, respectively. This is an adjunction because for all $A, B \subseteq P$ we have

$$\begin{aligned}
A \subseteq B^\wedge \quad &\Longleftrightarrow \quad \forall x \in A, x \in B^\wedge \\
&\Longleftrightarrow \quad \forall x \in A, \forall y \in B, x \leq y \\
&\Longleftrightarrow \quad \forall y \in B, \forall x \in A, x \leq y \\
&\Longleftrightarrow \quad B \subseteq A^\vee.
\end{aligned}$$

This restricts trivially to an adjunction between the down-closed and the up-closed sets:

$$(-)^\vee : \mathbf{2}^{P^{\mathsf{op}}} \rightleftarrows (\mathbf{2}^P)^{\mathsf{op}} : (-)^\wedge.$$

Now the fundamental theorem of Galois connections says that this adjunction restricts to an isomorphism between subposets of "closed" elements:

$$(-)^\vee : \left(\mathbf{2}^{P^{\mathsf{op}}}\right)^\vee \overset{\sim}{\longleftrightarrow} \left((\mathbf{2}^P)^{\mathsf{op}}\right)^\wedge : (-)^\wedge.$$

By abuse of terminology, let us denote both subposets by $\bar{P}$. Since $\mathbf{2}^{P^{\mathsf{op}}}$ has all colimits/joins, LAPC implies that $\bar{P}$ has all joins, and since $(\mathbf{2}^P)^{\mathsf{op}}$ has all limits/meets, RAPL implies that $\bar{P}$ has all meets. We conclude that $\bar{P}$ is both complete and co-complete, i.e., a lattice.

Finally, we note that P has two distinct embeddings into $\bar{P}$ via $(-)^\wedge$ and $(-)^\vee$. The following diagram summarizes the situation:

$$\begin{tikzcd}
P \arrow[dr] & \bar{P} \arrow[r] \arrow[d, leftrightarrow, "\sim"] & \mathbf{2}^{P^{\mathsf{op}}} \arrow[r] \arrow[d, bend right, "\vee"'] \arrow[d, phantom, "\dashv"] & 2^P \arrow[d, bend right, "\vee"'] \arrow[d, phantom, "\dashv"] \\
P \arrow[ur] & \bar{P} \arrow[r] & (\mathbf{2}^P)^{\mathsf{op}} \arrow[r] \arrow[u, bend right, "\wedge"'] & (2^P)^{\mathsf{op}} \arrow[u, bend right, "\wedge"']
\end{tikzcd}$$

Remark: It is a bit difficult to state the universal property of the Dedekind-MacNeille completion.

3.2 Motivation: Categories Enriched over Ab.

If G, H are groups then we can define a group structure on the set of functions $G \to H$. However, the product of homomorphisms may not be a homomorphism. If H is an **abelian** group then the set of homomorphisms $\mathsf{Hom}(G, H)$ is not only a group, but an abelian group. Furthermore, if A, B, C are abelian groups then the composition function

$$\mathsf{Hom}(B,C) \times \mathsf{Hom}(A,B) \to \mathsf{Hom}(A,C)$$

is a "bihomomorphism." Define Ab-category.

A ring is an Ab-category with one object.

An R-module is an "additive" functor $F : BR \to \mathsf{Ab}$.

R-modules form an (abelian) category under "additive" natural transformations.

3.3 Definition of Monoidal Category

Unify previous examples. Let $(\mathcal{V}, \otimes, 1)$ be a monoidal category. Then we define a category *enriched over* $\mathcal{V}$. We think of a small $\mathcal{V}$-category $\mathcal{C}$ as an "algebroid" and we think of a $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{V}$ as $\mathcal{V}$-representation of $\mathcal{C}$, or a $\mathcal{C}$-module.

Examples.

Change of base via Kan left and right Kan extensions of a $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$. General existence of Kan extensions when $\mathcal{V}$ is complete. Formula via (co)ends?

3.4 The Category of R-Modules is Sometimes Monoidal

Chapter 4

Morita Theory

We have seen Tannaka Duality for monoids acting on sets. More generally, we have a $\mathcal{V}$-enriched version of Tannaka Duality for any small $\mathcal{V}$-category. But sometimes this is overkill. What should we do in "partially enriched" situations?

4.1 Endomorphisms of the Identity Endofunctor