Algebra Qualifying Exam **University of Miami**
Thurs, May 26, 2016 **Drew Armstrong**

There are 4 problems and 9 pages. You have 3 hours to write the exam.

1. Galois Connections. Let $(P, \le)$ and $(Q, \le)$ be partially ordered sets. We say that a pair of functions $* : P \rightleftarrows Q : *$ is a Galois connection if for all $p \in P$ and $q \in Q$ we have

$$p \le q^* \iff q \le p^*.$$

Since this relation is symmetric in P and Q, you need only prove half of parts (a)-(d) below.

(a) Prove that for all $p \in P$ and $q \in Q$ we have

$$p \le p^{**} \quad \text{and} \quad q \le q^{**}.$$

Proof. For all $p \in P$ we have $p^* \le p^*$ from the reflexivity of partial order. Then putting $q = p^*$ in the definition of Galois connection gives $(p^*) \le (p)^* \implies (p) \le (p^*)^* = p^{**}$. □

(b) Prove that for all $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ we have

$$p_1 \le p_2 \implies p_2^* \le p_1^* \quad \text{and} \quad q_1 \le q_2 \implies q_2^* \le q_1^*.$$

Proof. Consider $p_1, p_2 \in P$ such that $p_1 \le p_2$. By part (a) and the transitivity of partial order we have $p_1 \le p_2 \le p_2^{**}$ and then from the definition of Galois connection we have $(p_1) \le (p_2^*)^* \implies (p_2^*) \le (p_1)^*$. □

(c) Prove that for all $p \in P$ and $q \in Q$ we have

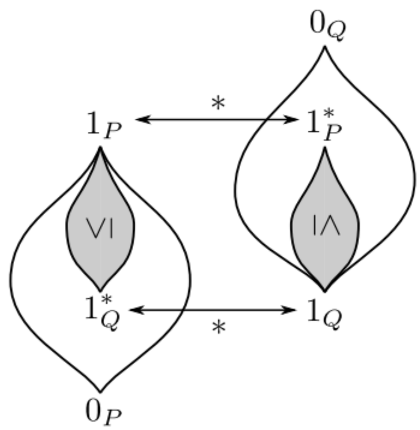
$$p^{***} = p^* \quad \text{and} \quad q^{***} = q^*.$$

Proof. Consider any $p \in P$. On the one hand, part (a) tells us that $(p^*) \le (p^*)^{**}$. On the other hand, part (a) says that $p \le p^{**}$ and then part (b) gives $(p) \le (p^{**}) \implies (p^{**})^* \le (p)^*$. Finally, the antisymmetry of partial order gives $p^{***} = p^*$. □

(d) We say that an element $p \in P$ (resp. $q \in Q$) is $**$**-closed** if $p^{**} = p$ (resp. $q^{**} = q$). Prove that the Galois connection $* : P \rightleftarrows Q : *$ restricts to an order-reversing **bijection** between $**$-closed elements.

Proof. Let $P^* \subseteq P$ and $Q^* \subseteq Q$ denote the images of the functions $* : Q \to P$ and $* : P \to Q$, respectively. I claim that $P^* \subseteq P$ is precisely the subset of $**$-closed elements. Indeed, if $p^{**} = p$ then $p = (p^*)^*$ is the image of p^*. Conversely, if $p = q^*$ for some $q \in Q$ then by part (c) we have $p^{**} = (q^*)^{**} = (q^*) = p$. Similarly, we can show that $Q^* \subseteq Q$ is the subset of $**$-closed elements in Q. It follows immediately that the functions $* : Q^* \rightleftarrows P^* : *$ are inverse to each other, hence they are bijections. [The fact that they reverse order follows from (b).] □

(e) Finally, suppose that P and Q have bottom and top elements $0_P, 1_P \in P$ and $0_Q, 1_Q \in Q$. In this case **draw a picture** of the bijection from part (d).



2. Image and Preimage. Let R be a ring and let $\varphi : M \to N$ be a homomorphism of (left) R-modules with kernel $\ker\varphi \subseteq M$ and image $\operatorname{im}\varphi \subseteq N$. For any (left) R-modules $Q \subseteq P$ let $\mathscr{L}(P, Q)$ be the lattice of submodules of P that contain Q, and let $\mathscr{L}(P) := \mathscr{L}(P, 0)$.

(a) For every submodule $A \subseteq M$ prove that the image $\varphi(A) := \{n \in N : \exists\, a \in A,\, \varphi(a) = n\}$ is a submodule of N.

Proof. Consider any elements $n_1, n_2 \in \varphi(A)$ and $r \in R$. Since $n_1, n_2 \in \varphi(A)$ there exist $a_1, a_2 \in A$ such that $n_1 = \varphi(a_1)$ and $n_2 = \varphi(a_2)$. Then since φ is a homomorphism of R-modules we have

$$\varphi(a_1 + ra_2) = \varphi(a_1) + r\varphi(a_2) = n_1 + rn_2.$$

Finally, since $A \subseteq M$ is a submodule we have $a_1 + ra_2 \in A$, and it follows that $n_1 + rn_2 \in \varphi(A)$ as desired. □

(b) For every submodule $B \subseteq N$ prove that the preimage $\varphi^{-1}(B) := \{m \in M : \exists\, b \in B,\, \varphi(m) = b\}$ is a submodule of M.

Proof. Consider any elements $m_1, m_2 \in \varphi^{-1}(B)$ and $r \in R$. Since $m_1, m_2 \in \varphi^{-1}(B)$ there exist $b_1, b_2 \in B$ such that $\varphi(m_1) = b_1$ and $\varphi(m_2) = b_2$. Then since φ is a homomorphism of R-modules we have

$$\varphi(m_1 + rm_2) = \varphi(m_1) + r\varphi(m_2) = b_1 + rb_2.$$

Finally, since $B \subseteq N$ is a submodule we have $b_1 + rb_2 \in B$, and it follows that $m_1 + rm_2 \in \varphi^{-1}(B)$ as desired. □

(c) For all submodules $A \subseteq M$ and $B \subseteq N$ prove that we have

$$\varphi(A) \subseteq B \quad\Longleftrightarrow\quad A \subseteq \varphi^{-1}(B).$$

Proof. By definition we have

$$\begin{aligned} \varphi(A) \subseteq B &\Longleftrightarrow \forall\, a \in A,\, \varphi(a) \in B \\ &\Longleftrightarrow \forall\, a \in A,\, \exists\, b \in B,\, \varphi(a) = b \\ &\Longleftrightarrow \forall\, a \in A,\, a \in \varphi^{-1}(B) \\ &\Longleftrightarrow A \subseteq \varphi^{-1}(B). \end{aligned}$$

□

(d) For all submodules $A \subseteq M$ and $B \subseteq N$ you may assume without proof that

$$\varphi^{-1}(\varphi(A)) = A \vee \ker\varphi \quad \text{and} \quad \varphi(\varphi^{-1}(B)) = B \wedge \operatorname{im}\varphi.$$

Quote from Problem 1 to obtain a poset isomorphism $\mathscr{L}(M, \ker\varphi) \cong \mathscr{L}(\operatorname{im}\varphi)$.

Proof. From part (c) we see that $\varphi : \mathscr{L}(M)^{\mathrm{op}} \rightleftarrows \mathscr{L}(N) : \varphi^{-1}$ is a Galois connection in the sense of Problem 1, thus from Problem 1(d) we obtain an order-reversing bijection between the subposets of "closed submodules" in $\mathscr{L}(M)^{\mathrm{op}}$ and $\mathscr{L}(N)$. Equivalently, we obtain an **order-preserving** bijection (i.e. a poset isomorphism) between closed submodules in $\mathscr{L}(M)$ and $\mathscr{L}(N)$.

It remains only to determine the closed submodules. By assumption $A \subseteq M$ is $\varphi^{-1}\varphi$-closed if and only if $A = A \vee \ker\varphi$, and from the universal property of $\vee$ this is equivalent to saying that $\ker\varphi \subseteq A$. Similarly, a submodule $B \subseteq N$ is $\varphi\varphi^{-1}$-closed if and only if $B = B \wedge \operatorname{im}\varphi$, which is equivalent to $B \subseteq \operatorname{im}\varphi$. $\square$

(e) Prove that we have an isomorphism of (left) R-modules $M/\ker\varphi \cong \operatorname{im}\varphi$. [Hint: Show that the surjective homomorphism $(m + \ker\varphi) \mapsto \varphi(m)$ is well-defined and injective.]

Proof. For all elements $m_1, m_2 \in M$ we have

$$\begin{aligned}(m_1 + \ker\varphi) = (m_2 + \ker\varphi) &\iff (m_1 - m_2) \in \ker\varphi \\ &\iff \varphi(m_1 - m_2) = 0 \\ &\iff \varphi(m_1) = \varphi(m_2).\end{aligned}$$

The $\Rightarrow$ direction proves that the map is well-defined and the $\Leftarrow$ direction proves that it is injective. [This result is often called the 1st Isomorphism Theorem.] $\square$

(f) Conclude that we have an isomorphism of posets $\mathscr{L}(M, \ker\varphi) \cong \mathscr{L}(M/\ker\varphi)$.

Proof. The R-module isomorphism $\operatorname{im}\varphi \cong M/\ker\varphi$ from part (e) induces a poset isomorphism $\mathscr{L}(\operatorname{im}\varphi) \cong \mathscr{L}(M/\ker\varphi)$. Then combining this with part (d) gives

$$\mathscr{L}(M, \ker\varphi) \cong \mathscr{L}(\operatorname{im}\varphi) \cong \mathscr{L}(M/\ker\varphi).$$

[This result is often called the Correspondence Theorem. See the picture from 1(e).] $\square$

3. Direct Product of Rings. Let CRng be the category of commutative rings and consider $R, S \in \mathsf{CRng}$. We define the **direct product ring** $R \times S$ as the Cartesian product set with componentwise addition and multiplication. Note that it has a unit: $(1_R, 1_S) \in R \times S$.

(a) Prove that $R \times S$ is the categorical product in CRng. [Hint: You can assume that the Cartesian product is the categorical product in Set.]

Proof. The definition of categorical product is given by the following diagram:

$$\begin{tikzcd} & & R \\ T \arrow[rru, bend left, "\varphi_R"] \arrow[rrd, bend right, "\varphi_S"'] \arrow[r, dashed, "\varphi_R \times \varphi_S"] & R \times S \arrow[ru, "\pi_R"'] \arrow[rd, "\pi_S"] & \\ & & S \end{tikzcd}$$

By assumption we know that there exist set functions $\pi_R : R \times S \to R$ and $\pi_S : R \times S \to S$ such that for all set functions $\varphi_R : T \to R$ and $\varphi_S : T \to S$ there exists a unique set function $\varphi_R \times \varphi_S : T \to R \times S$ making the above diagram commute. Explicitly, these functions are given by

$$\pi_R(r,s) = r, \quad \pi_S(r,s) = s, \qquad \text{and} \qquad (\varphi_R \times \varphi_S)(t) = (\varphi_R(t), \varphi_S(t)).$$

To lift this diagram to the category CRng, first note that π_r and π_S are clearly ring homomorphisms. If $T \in \mathsf{CRng}$ and if φ_R and φ_S are ring homomorphisms then it is also clear that the function $\varphi_S \times \varphi_R$ defined by $(\varphi_R \times \varphi_S)(t) = (\varphi_R(t), \varphi_S(t))$ is a ring homomorphism. Finally, the uniqueness of the ring homomorphism $\varphi_R \times \varphi_S$ follows from the uniqueness of the underlying set function. $\square$

(b) If $R \cong S \times T$ for some $R, S, T \in \mathsf{CRng}$ where neither of S or T is the zero ring, prove that R contains a **nontrivial idempotent**, i.e., an element $e \in R$ such that $e^2 = e$ and $e \notin \{0_R, 1_R\}$.

Proof. Since S and T are both nonzero rings we have $0_S \neq 1_S$ and $0_T \neq 1_T$. Now I claim that $e := (1_S, 0_T) \in R \times S$ is a nontrivial idempotent. Indeed, since $0_S \neq 1_S$ and $0_T \neq 1_T$ we see that $e \neq (0_S, 0_T) = 0_R$ and $e \neq (1_S, 1_T) = 1_R$, hence e is nontrivial. And e is idempotent because

$$e^2 = (1_S, 0_T)(1_S, 0_T) = (1_S 1_S, 0_T 0_T) = (1_S, 0_T) = e.$$

[Note that $f = (0_S, 1_T) = 1_R - e$ is another perfectly good choice.] $\square$

(c) Given any ring $R \in \mathsf{CRng}$ and an element $e \in R$, prove that

$$e \text{ is a nontrivial idempotent} \quad \Longleftrightarrow \quad 1_R - e \text{ is a nontrivial idempotent.}$$

Proof. First note that $e \notin \{0_R, 1_R\}$ if and only if $(1_R - e) \notin \{0_R, 1_R\}$. Now assume that $e^2 = e$. From this it follows that

$$(1_R - e)^2 = (1_R)^2 - e - e + e^2 = 1_R - e - \cancel{e} + \cancel{e} = (1_R - e).$$

Finally, if $f := (1_R - e)$ satisfies $f^2 = f$ then the same computation shows that $e = (1_R - f)$ satisfies $e^2 = e$. $\square$

(d) If $e \in R$ is idempotent, prove that $eR := \{er : r \in R\}$ is a commutative ring with unit element $e \in eR$. But note that $eR \subseteq R$ is (probably) **not a subring**.

Proof. We would usually write eR as the principal ideal $(e) \subseteq R$. To see that this is indeed an ideal note that for all $er_1, er_2 \in eR$ and $r_3 \in R$ we have

$$er_1 - r_3(er_2) = e(r_1 - r_3 r_2) \in eR.$$

In particular we see that $(eR, +, 0_R)$ is an abelian group and that multiplication is a commutative and associative operation $eR \times eR \to eR$ that distributes over $+$. It remains only to show that $e \in eR$ is a unit element. To see this, observe that for all $er \in eR$ we have $e(er) = e^2 r = er$.

Finally, observe that $eR \subseteq R$ is a subring if and only if $e \in \{0_R, 1_R\}$. $\square$

(e) Finally, suppose that $R \in \mathsf{CRng}$ contains a nontrivial idempotent $e \in R$. In this case prove that R is isomorphic to a direct product of nonzero rings. This is the converse of part (b). [Hint: Use parts (c) and (d).]

Proof. Let $e \in R$ be a nontrivlal idempotent, so that $1_R - e \in R$ is also a nontrivial idempotent by part (c). From part (d) we know that eR and $(1_R - e)R$ are nonzero commutative rings. We will prove that there is a ring isomorphism $R \cong eR \times (1_R - e)R$.

Indeed, consider the obvious ring homomorphisms $R \to eR$ and $R \to (1_R - e)R$ defined by $r \mapsto er$ and $r \mapsto (1_R - e)r$, respectively. From part (a) these define a canonical product homomorphism $\varphi(r) := (er, (1_R - e)r)$. To prove that φ is injective we will show $\ker \varphi = \{0_R\}$. Indeed, if $\varphi(r) = (er, (1_R - e)r) = (0_R, 0_R)$ then we have

$$r = 1_R r = (e + (1_R - e))r = er + (1_R - e)r = 0_R + 0_R = 0_R$$

as desired. To prove that φ is surjective, consider a general element $(er_1, (1_R - e)r_2)$ of the ring $eR \times (1_R - e)R$. Then since $e(1_R - e) = (1_R - e)e = 0_R$ we have

$$\begin{aligned}
\varphi(er_1 + (1_R - e)r_2) &= (e(er_1 + (1_R - e)r_2), (1_R - e)(er_1 + (1_R - e)r_2)) \\
&= (e^2 r_1 + \cancel{e(1_R - e)}r_2, \cancel{(1_R - e)e}r_1 + (1_R - e)^2 r_2) \\
&= (er_1 + 0_R r_2, 0_R r_1 + (1_R - e)r_2) \\
&= (er_1, (1_R - e)r_2).
\end{aligned}$$

□

4. Companion Matrices. Let K be a field.

(a) Use the fact that $K[x]$ is a Euclidean domain to prove that $K[x]$ is a PID. [Hint: Consider a nonzero ideal $0 \subsetneq I \subseteq K[x]$ and let $m(x) \in I$ be a monic polynomial of minimal degree.]

Proof. Suppose that $I \subseteq K[x]$ is a nonzero ideal and let $m(x) \in I$ be a monic polynomial of minimal degree. Note that $(m(x)) \subseteq I$. Now consider any polynomial $f(x) \in I$. Since $m(x)$ is monic we can use long division to obtain polynomials $q(x), r(x) \in K[x]$ such that $f(x) = q(x)m(x) + r(x)$ and such that $r(x) = 0$ or $\deg(r) < \deg(m)$. Since I is an ideal we have $r(x) = q(x)m(x) - f(x) \in I$. Then if $r(x) \neq 0$ we find that $\deg(r) < \deg(m)$, which contradicts the minimality of $\deg(m)$. It follows that $r(x) = 0$ and hence $f(x) = q(x)m(x) \subseteq (m(x))$. Since this is true for all $f(x)$ we have $I \subseteq (m(x))$, and hence $I = (m(x))$. □

(b) Given a monic polynomial $m(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in K[x]$, prove that (the images of) $1, x, x^2, \ldots, x^{n-1}$ are a basis for the K-vector space $K[x]/(m(x))$.

Proof. A general K-linear combination of the elements $1, x, \ldots, x^{n-1} \in K[x]/(m(x))$ is just a coset $r(x) + (m(x))$ where $r(x) \in K[x]$ satisfies $r(x) = 0$ or $\deg(r) \leq n - 1$. To prove "spanning", consider any element $f(x) + (m(x)) \in K[x]/(m(x))$. Dividing the polynomial $f(x)$ by the monic polynomial $m(x)$ gives $f(x) = q(x)m(x) + r(x)$ where $r(x) = 0$ or $\deg(r) \leq n - 1$. The result now follows from the fact that $f(x) + (m(x)) = r(x) + (m(x))$. To prove "independence", assume for contradiction that there exists a nonzero polynomial $r(x) \in K[x]$ such that $\deg(r) \leq n - 1$ and $r(x) + (m(x)) = 0 + (m(x))$. The fact that $r(x) \in (m(x))$ means that we have $r(x) = f(x)m(x)$ for some nonzero $f(x) \in K[x]$ and the fact that K is a domain implies that

$$n = \deg(m) \leq \deg(m) + \deg(f) = \deg(r) \leq n - 1,$$

which is the desired contradiction. □

(c) "Multiplication by x" defines a K-linear endomorphism $K[x]/(m(x)) \to K[x]/(m(x))$. Find the matrix of this endomorphism in terms of the basis from part (b). We will call this matrix $C_m \in \mathrm{Mat}_n(K)$.

Proof. To find the matrix we just need to know what "multiplication by x" does to the basis elements. Note that we have

$$\begin{aligned} x \cdot 1 &= x, \\ x \cdot x &= x^2, \\ &\vdots \\ x \cdot x^{n-2} &= x^{n-1}, \\ x \cdot x^{n-1} &= x^n = -a_0 1 - a_1 x - \cdots - a_{n-1} x^{n-1}, \end{aligned}$$

where each polynomial is interpreted as the coset in $K[x]/(m(x))$ that it generates. The corresponding matrix is

$$C_m = \begin{pmatrix} & & & & -a_0 \\ 1 & & & & -a_1 \\ & 1 & & & -a_2 \\ & & \ddots & & \vdots \\ & & & 1 & -a_{n-1} \end{pmatrix},$$

where we interpret the blank entries as zeroes. [This is called the companion matrix of the monic polynomial $m(x)$.] ☐

(d) You may assume without proof that $m(x)$ is both the minimal polynomial **and** the characteristic polynomial of the matrix C_m. In this case, prove that $m(x)$ is both the minimal **and** the characteristic polynomial of the **transpose** matrix $(C_m)^T$.

Proof. Consider **any** matrix $A \in \mathrm{Mat}_n(K)$ and recall that the minimal polynomial is the unique monic polynomial $f(x) \in K[x]$ of minimum degree satisfying $f(A) = 0 \in \mathrm{Mat}_n(K)$. First note that $f(A)^T = f(A^T)$ for all polynomials $f(x) \in K[x]$ and hence we have $f(A) = 0 \iff f(A^T) = 0$. It follows that A and A^T have the same minimal polynomial. Then recall that the characteristic polynomial of A is defined as $\det(xI_n - A) \in K[x]$. Since

$$\det(xI_n - A) = \det((xI_n - A)^T) = \det(xI_n - A^T),$$

we conclude that A and A^T have the same characteristic polynomial. In particular, both of these statements are true when $A = C_m$. ☐

(e) Define a (finitely-generated and torsion) $K[x]$-module structure on $M = K^n$ by letting x act as the matrix $(C_m)^T$. Since $K[x]$ is a PID we know (from the FTFGMPID) that there exist unique, monic, nonconstant polynomials $f_1(x)|f_2(x)|\cdots|f_d(x)$ such that $M \cong \oplus_{i=1}^d K[x]/(f_i(x))$ as $K[x]$-modules. Use part (d) to **compute these polynomials**. [Hint: You can quote results from class.]

Proof. From class we know that $f_d(x)$ is the minimal polynomial of $(C_m)^T$ and that $\prod_{i=1}^d f_i(x)$ is the characteristic polynomial of $(C_m)^T$. And from part (d) we know that $m(x)$ is the minimal and the characteristic polynomial of $(C_m)^T$. Since the polynomials $f_i(x)$ are nonconstant this implies that $d = 1$ and $f_d(x) = m(x)$. ☐

(f) Finally, prove that there exists an invertible matrix $P \in \mathrm{GL}_n(K)$ such that

$$PC_mP^{-1} = (C_m)^T.$$

Proof. Recall that a $K[x]$-module is the same as a pair (V, φ) where V is a K-vector space and x acts on V by the K-linear endomorphism $\varphi \in \mathrm{End}_K(V)$. Furthermore, recall that a morphism of $K[x]$-modules $(V_1, \varphi_1) \to (V_2, \varphi_2)$ is the same as a K-linear function $\Phi : V_1 \to V_2$ satisfying $\Phi \circ \varphi_1 = \varphi_2 \circ \Phi$. Thus an **isomorphism** of $K[x]$-modules is the same as an isomorphism of K-vector spaces $\Phi : V_1 \to V_2$ satisfying $\Phi \circ \varphi_1 \circ \Phi^{-1} = \varphi_2$. After choosing bases for V_1 and V_2 this becomes a matrix equation:

$$P[\varphi_1]P^{-1} = [\varphi_2].$$

Finally, consider the $K[x]$-modules corresponding to pairs (K^n, C_m) and $(K^n, (C_m)^T)$. From part (e) we know that each of these is isomorphic to $K[x]/(m(x))$ as a $K[x]$-module, hence they are isomorphic to each other. It follows from the above observations that there exists an invertible matrix $P \in \mathrm{GL}_n(K)$ such that

$$PC_mP^{-1} = (C_m)^T.$$

□

[Remark: This strange result would be quite difficult to prove directly. Indeed, I have no idea how to find such a matrix P for a specific companion matrix C_m. (The situation is easier for a Jordan block $J_\lambda \in \mathrm{Mat}_n(K)$: if P is the anti-identity matrix (with 1s on the anti-diagonal) then we have $PJ_\lambda P^{-1} = (J_\lambda)^T$.) Now let K be any field and consider any matrix $A \in \mathrm{Mat}_n(K)$. From part (f) and the existence of Rational Canonical Form we conclude that there exists a matrix $P \in \mathrm{GL}_n(K)$ such that

$$PAP^{-1} = A^T.$$

Strange but true!]