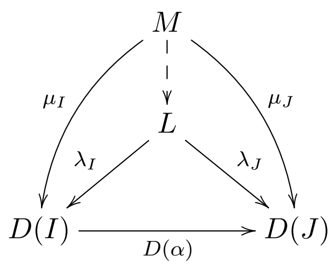
Math 762 **Spring 2016**
Homework 2 **Drew Armstrong**

Problem 1. Limits and Colimits. Let $\mathcal{C}$ be any category and let $\mathcal{I}$ be a small category. A diagram of shape $\mathcal{I}$ in $\mathcal{C}$ is just a covariant functor $D : \mathcal{I} \to \mathcal{C}$. The limit of the diagram (if it exists) is a structure $\varprojlim D = (L, \{\lambda_I\}_{I \in \mathcal{I}})$ where

- $L \in \mathcal{C}$ is an object and $\lambda_I : L \to D(\mathcal{I})$ are morphisms such that for all objects $I, J \in \mathcal{I}$ and morphisms $\alpha \in \mathrm{Hom}_{\mathcal{I}}(I, J)$ we have $D(\alpha) \circ \lambda_I = \lambda_J$.
- If $M \in \mathcal{C}$ is another object with morphisms $\mu_I : M \to D(\mathcal{I})$ satisfying the first property, then **there exists a unique morphism** $M \to L$ such that:



In other words, the limit $\varprojlim D$ is a final object in a certain category (of "cones"). We define the colimit $\varinjlim D$ by reversing all arrows in the definition.

(a) Given objects $A, B \in \mathcal{C}$, express the categorical product $A \times B$ as a limit.
(b) Suppose there exists a zero object $0 \in \mathcal{C}$. Given a morphism $\varphi : X \to Y$ in $\mathcal{C}$, express the categorical kernel of φ as a limit.
(c) (Optional) Let R be a ring and think of $\mathcal{I} = (\mathbb{N}, \leq)$ as a category with a unique arrow $i \to j$ for each $i \leq j$ in $\mathbb{N}$. Now let $D : \mathcal{I} \to R$-Mod be a diagram in which each morphism $D(\alpha)$ is injective. Prove that the colimit $\varinjlim D$ exists.

Problem 2. Length Equals Dimension For Vector Spaces. Consider a field K and a vector space $V \in K$-Vec. Recall that a composition series of length n is a chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

in which each quotient V_{i+1}/V_i is simple (i.e. has no nontrivial subspaces). Prove that V has a composition series of length n if and only if it has a basis of size n. Hence the length of a vector space is the same as its dimension.

Problem 3. Localization of a Module. Let M be a module over a commutative ring R and let $S \subseteq R$ be a submonoid. Then we define the set of "fractions"

$$S^{-1}M := \left\{ \left[\frac{m}{s}\right] : m \in M, s \in S \right\}$$

with the equivalence relation

$$\left[\frac{m}{s}\right] = \left[\frac{n}{t}\right] \quad \Longleftrightarrow \quad \exists\, u \in S,\, u(sn - tm) = 0.$$

(a) Prove that this is indeed an equivalence relation.
(b) Prove that the operations

$$\left[\frac{m}{s}\right] + \left[\frac{n}{t}\right] = \left[\frac{tm + sn}{st}\right] \quad \text{and} \quad r\left[\frac{m}{s}\right] = \left[\frac{rm}{s}\right]$$

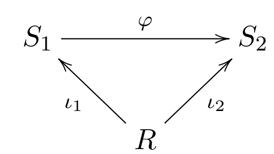
are well-defined.

(c) Prove that the operations from part (b) make $S^{-1}M$ into an R-module and that the map $M \to S^{-1}M$ defined by $m \mapsto [m/1]$ is an R-module homomorphism.

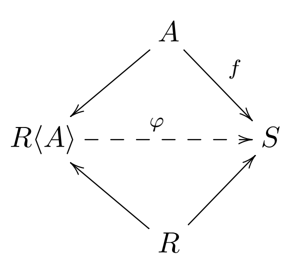
Problem 4. Rank Exists Over a Domain. Let R be an integral domain and let $S = R\backslash\{0\}$. Then we can identify the field of fractions $K = \text{Frac}(R)$ with the localization $S^{-1}R$. Let M be any R-module and let $A \subseteq M$ be any subset.

(a) Show that we can regard $S^{-1}M$ as a K-module.
(b) If A is R-linearly independent in M prove that the image of A under $M \to S^{-1}M$ (let's call it $S^{-1}A$) is K-linearly independent in $S^{-1}M$.
(c) If $A \subseteq M$ is a **maximal** R-linearly independent subset prove that $S^{-1}A \subseteq S^{-1}M$ is a **maximal** K-linearly independent subset. Conclude that any two such sets have the same cardinality. [Hint: Let $A \subseteq M$ be R-linearly independent. Then we know from part (a) that $S^{-1}A \subseteq S^{-1}M$ is K-linearly independent. Suppose there exists $n \in S^{-1}M \setminus S^{-1}A$ such that $S^{-1}A \cup \{n\}$ is K-linearly independent. Then we can write $n = [m/s]$ for some $m \in M \setminus A$ and $s \in S$. Show that the set $A \cup \{m\} \subseteq M$ is R-linearly independent.]

Problem 5. The Category R-Alg. Let R be a ring. We define an R-algebra as a pair (ι, S) where S is a ring and $\iota : R \to S$ is a ring homomorphism such that $\text{im}\,\iota \subseteq Z(S)$. A morphism of R-algebras $\varphi : (\iota_1, S_1) \to (\iota_2, S_2)$ is defined as a ring homomorphism $\varphi : S_1 \to S_2$ such that



(a) Explain why $\mathbb{Z}$-Alg = Rng.
(b) Prove that an R-algebra (ι, S) is also an R-module in a natural way (i.e. by forgetting the monoid structure on S).
(c) Conversely, given any set A there exists an R-algebra $R\langle A\rangle$ (called the free R-algebra generated by A) with the following universal property: For all set functions $f : A \to S$ there exists a unique R-algebra homomorphism $\varphi : R\langle A\rangle \to S$ such that



If $|A| = n$ then we can identify $R\langle A\rangle$ with the R-algebra $R\langle x_1, \dots, x_n\rangle$ of polynomials in the n **noncommuting indeterminates** $x_1, \dots, x_n$. Use this idea to find the initial object in the category R-Alg.
(d) Given a subset $A \subseteq S$ of an R-algebra, let $i_A : R\langle A\rangle \to S$ be the unique R-algebra morphism defined in part (c). If i_A is injective we say that the set A is R-algebraically independent in S and if i_A is surjective we say A is an R-algebraic generating set for S. If i_A is bijective we say that A is an R-algebra basis for S. Prove that an algebra basis is necessarily a **maximal** algebraically independent set and a **minimal** algebraic generating set. [To make notation easier you can assume that the basis is finite.]