10/27/15

HW3 due now

Midterm Exam on Thursday

I didn't find any really appropriate previous exams to study from, so today I'll give a thorough list of the topics for the midterm.

(1) Categories.

A category $\mathcal{C}$ consists of

- a "collection" $\mathrm{Obj}(\mathcal{C})$ of objects.
- for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a <u>set</u> of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$
- for all $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$ a function

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

called <u>composition</u>,

satisfying two axioms:

$\downarrow$

- for all $\alpha : A \to B$, $\beta : B \to C$, $\gamma : C \to D$ we have

$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$$

- for all $X \in Obj(\mathcal{C})$ there exists $id_X \in Hom_{\mathcal{C}}(X, X) =: End_{\mathcal{C}}(X)$ such that for all $\alpha : X \to Y$ we have

$$\alpha \circ id_X = \alpha .$$

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You should know the definitions of

- isomorphism / automorphism
- initial / final object
- zero object / morphism
- product / coproduct
- kernel / cokernel

and some examples in specific categories.

(2) Posets & Lattices.

A poset is a pair $(P, \leq)$ where P is a set and $\leq$ is a partial order:

- $x \le x$
- $x \le y \ \& \ y \le z \implies y \le z$
- $x \le y \ \& \ y \le x \implies x = y$.

Alternatively, a poset P is a (small) category in which

$$|\mathrm{Hom}_P(x,y)| \in \{0,1\} \quad \forall \, x,y \in \mathrm{Obj}(P).$$

We will write

$$x \le y \iff |\mathrm{Hom}_P(x,y)| = 1$$

and $x = y$ if x, y are isomorphic in P.

The product / coproduct if they exist are called meet / join. The initial / final objects if they exist are called 0 / 1.

(3) Galois Connections.

Let P, Q be posets and consider a pair of maps

$$L : P \rightleftarrows Q : R.$$

We say (L, R) is a covariant Galois connection if for all $x \in P$, $y \in Q$ we have a bijection

$$\mathrm{Hom}_P(x, R(y)) \longleftrightarrow \mathrm{Hom}_Q(L(x), y)$$

$$(\text{i.e., } x \leq R(y) \iff L(x) \leq y).$$

[We use the word covariant because it follows that L, R are covariant functors.]

We say $L : P \rightleftarrows Q : R$ is a contravariant Galois connection if and only if $L : P \rightleftarrows Q^{op} : R$ is a covariant Galois connection. The notation becomes easier if we write

$$* : P \rightleftarrows Q : *$$

for a contravariant connection, i.e.,

$$x \leq y^* \iff y \leq x^*.$$

You should know the results of HW1
and be able to prove the more basic facts.

You should know the following examples of
Galois connections.

- fields & automorphisms
- image / preimage of a group hom
- join / meet fixed elements of a lattice.

(4) Equivalence & Quotient.

Let $\sim$ be an equivalence relation on
a set S:

- $x \sim x$
- $x \sim y \implies y \sim x$
- $x \sim y$ & $y \sim z \implies x \sim z$.

We say $f : S \to T$ is a class function
if for all $x, y \in S$ we have

$$x \sim y \implies f(x) = f(y).$$

We say $\pi : S \to Q$ is a quotient if

- $\pi : S \to Q$ is a class function
- for all class functions $f : S \to T$ there exists a unique function $\bar{f} : Q \to T$ such that

$$\begin{tikzcd} & S \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ Q \arrow[rr, dashed, "\exists! \bar{f}"'] & & T \end{tikzcd}$$

Prove that the quotient exists and say in what sense it is unique. We call "the" quotient

$$\pi : S \to S/\sim .$$

Application: If $f : S \to T$ is any function, then we define an equivalence on S by

$$x \sim y \iff f(x) = f(y),$$

and we call the quotient $\pi : S \to S/f$.

Prove that this leads to a canonical factorization:

$$\begin{tikzcd} S \arrow[r, two heads, "\pi"'] \arrow[rrr, bend left, "f"] & S/f \arrow[r, "\bar{f}"', "\sim"] & \operatorname{im} f \arrow[r, hook, "i"'] & T \end{tikzcd}$$

Show how to lift this factorization to the category of groups:

- $\pi : G \to G/\sim$ is a group hom if and only if $\sim$ is G-invariant, in which case $N := [1]_\sim$ is a normal subgroup.

- For all $g \in G$ we have

$$[g]_\sim = gN = Ng$$

- If $\varphi : G \to H$ is a group hom, then the equivalence

$$x \sim y \iff \varphi(x) = \varphi(y)$$

is G-invariant with $[1]_\sim = \ker\varphi$.

- Universal Property of Group Quotient:

Let $N \trianglelefteq G$. If $\varphi : G \to H$ is a group hom with $N \subseteq \ker\varphi$, then there exists a unique group hom $\bar{\varphi} : G/N \to H$ such that

$$\begin{tikzcd} & G \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ G/N \arrow[rr, dashed, "\exists! \bar{\varphi}"'] & & H \end{tikzcd}$$

(5) Isomorphism Theorems.

Apply the above universal property to prove the following.

- Given $\varphi : G \to G'$ we have an isomorphism of groups

$$\bar{\varphi} : G/\ker\varphi \xrightarrow{\sim} \operatorname{im}\varphi.$$

• Let $N \trianglelefteq G$ and $N \trianglelefteq H \trianglelefteq G$. Then we have $H/N \trianglelefteq G/N$ and an isomorphism

$$\frac{G}{H} \approx \frac{G/N}{H/N}$$

• Let $H, K \leq G$ with $K \trianglelefteq G$. Then we have $H \cap K \trianglelefteq H$, $K \trianglelefteq HK$, and an isomorphism

$$\frac{H}{H \cap K} \approx \frac{HK}{K}.$$

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Be able to state and use (but not prove) the Jordan-Hölder Theorem.

(6) Products of Groups.

Let $H, K \leq G$ be arbitrary subgroups. If H, K are finite then we have

$$|H| \cdot |K| = |HK| \cdot |H \cap K|,$$

even though HK might not be a group.

Prove that HK is a group if and only if $HK = KH$, in which case we have

$$H \vee K = HK.$$

If $H \wedge K = H \cap K = 1$ and $H \vee K = HK = G$ we say that G is a product of H & K. There are three cases:

(i) In general we write

$$G = H \bowtie K$$

and call this a Zappa-Szép product.

(ii) If $H \trianglelefteq G$ or $K \trianglelefteq G$ we write

$$G = H \rtimes K \quad \text{or} \quad G = H \ltimes K,$$

respectively, and call this a semi-direct product. Know the "external" characterization and the theorem on right-split short exact sequences.

(iii) If $H \trianglelefteq G$ and $K \trianglelefteq G$ we write

$$G = H \times K$$

and call this the direct product. In this case we have $hk = kh \ \forall h \in H, k \in K$. Know the external characterization as the product object in the category of groups.

⑦ Specific Examples of Groups

- cyclic
- dihedral
- symmetric / alternating
- general / special linear.