Math 761 **Fall 2015**
Homework 4 **Drew Armstrong**

Problem 1. Burnside's Lemma. Let X be a G-set and for all $g \in G$ define the set

$$\mathsf{Fix}(g) := \{x \in X : g(x) = x\} \subseteq X.$$

(a) If G and X are finite, prove that

$$\sum_{g\in G} |\mathsf{Fix}(g)| = \sum_{x \in X} |\mathsf{Stab}(x)|.$$

(b) Let X/G be the set of orbits. Use part (a) to prove that

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |\mathsf{Fix}(g)|.$$

Problem 2. The Dodecahedron. Let D be the group of rotational symmetries of a regular dodecahedron.

(a) Describe the conjugacy classes of D and use this to prove that D is simple. [Hint: Any normal subgroup is a union of conjugacy classes.]

(b) Compute the number of distinguishable ways to color the faces of a dodecahedron with k colors. [Hint: Let X be the set of all colorings, so that $|X| = k^{12}$. Many of these colorings are indistinguishable after rotation so we really want to know the number of orbits $|X/D|$. Use part (a) and Burnside's Lemma.]

(c) Prove that D is isomorphic to the alternating group A_5. [Hint: There are five cubes that can be inscribed in a dodecahedron. The action of D defines a nontrivial homomorphism $\varphi : D \to S_5$. Composing this with the "sign" homomorphism $\sigma : S_5 \to \{\pm 1\}$ gives a homomorphism $\sigma\varphi : D \to \{\pm 1\}$. Since D is simple the first homomorphism must be injective and the second must be trivial.]

Problem 3. Affine Space. What is space? In general it is possible to "subtract points" to obtain a vector, but it is not possible to "add points" unless we fix an arbitrary origin. Let V be a vector space. We say that A is an affine space over V if there exists a "subtraction function" $(p, q) \mapsto [p, q]$ from $A \times A$ to V such that

- $[p, -] : A \to V$ is a bijection for all $p \in A$,
- $[p, q] + [q, r] = [p, r]$ for all $p, q, r \in A$.

(a) We say that a group action is free if all stabilizers are trivial and we say it is transitive if every orbit is the full set. We say that an action is regular if it is free and transitive. Prove that an affine space over a vector space V is the same thing as a regular V-set (thinking of V as an abelian group).

(b) Let A be an affine space over V and denote the induced regular action of V on A by $v(p) =$ "$p + v$". We say that a function $f : A \to A$ is affine if there exists a **linear** function $df : V \to V$ such that for all points $p \in A$ and vectors $v \in V$ we have

$$f(p + v) = f(p) + df(v).$$

In this case show that $df([p, q]) = [f(p), f(q)]$ for all $p, q \in A$, so that df is uniquely determined by f (we call it the differential of f). Prove that f is invertible if and only if df is invertible, in which case we have $d(f^{-1}) = (df)^{-1}$.

(c) Let $\mathsf{GA}(V)$ be the group of invertible affine functions $A \to A$ (called the **general affine group** of V). Prove that we have an isomorphism

$$\mathsf{GA}(V) \approx V \rtimes \mathsf{GL}(V)$$

where $\mathsf{GL}(V)$ acts on V in the obvious way. [Hint: Show that the differential map $d : \mathsf{GA}(V) \to \mathsf{GL}(V)$ is a group homomorphism with kernel isomorphic to V. Show that "choosing an origin" $o \in A$ defines a section $s : \mathsf{GL}(V) \to \mathsf{GA}(V)$.]

Problem 4. Grassmannians.

(a) Let $\mathsf{Gr}_1(r,n)$ denote the set of r-element subsets of $\{1,2,\ldots,n\}$. Show that the obvious action of the symmetric group S_n on $\mathsf{Gr}_1(r,n)$ is transitive with stabilizer isomorphic to $S_r \times S_{n-r}$. Then use orbit-stabilizer to compute $|\mathsf{Gr}_1(r,n)|$.

(b) Let K be a field and let $\mathsf{Gr}_K(r,n)$ denote the set of r-dimensional subspaces of K^n. Show that the obvious action of $\mathsf{GL}_n(K)$ on $\mathsf{Gr}_K(r,n)$ is transitive with stabilizer isomorphic to

$$\mathsf{Mat}_{r,n-r}(K) \rtimes (\mathsf{GL}_r(K) \times \mathsf{GL}_{n-r}(K)),$$

where $\mathsf{Mat}_{r,n-r}(K)$ is the additive group of $r \times (n-r)$ matrices. [Hint: Show that the stabilizer is isomorphic to the group of block matrices

$$\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)$$

with $A \in \mathsf{GL}_r(K)$, $B \in \mathsf{GL}_{n-r}(K)$, and $C \in \mathsf{Mat}_{r,n-r}(K)$.]

(c) When K is the finite field of size q we will write $\mathsf{Gr}_q(r,n) := \mathsf{Gr}_K(r,n)$. Use orbit-stabilizer and part (b) to compute $|\mathsf{Gr}_q(r,n)|$. [Hint: Define $\mathsf{GL}_n(q) := \mathsf{GL}_n(K)$. You can assume the formula

$$|\mathsf{GL}_n(q)| = q^{\binom{n}{2}}(q-1)^n [n]_q!,$$

where $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$.] Now compare your answers from parts (a) and (c).