Math 761 **Fall 2015**
Homework 1 **Drew Armstrong**

On this homework you will further explore the idea of Galois connections. We will begin by defining a notion of Galois connection for general posets. Let $(P, \leq)$ and $(Q, \leq)$ be posets. A pair of maps $* : P \rightleftarrows Q : *$ is called a Galois connection if it satisfies the following property:

$$\boxed{\text{for all } p \in P \text{ and } q \in Q \text{ we have } p \leq q^* \iff q \leq p^*}$$

Problem 1. Equivalent Definition. Prove that a pair of maps $* : P \rightleftarrows Q : *$ is a Galois connection (as defined above) if and only if the following two statements hold:

- For all $p \in P$ and $q \in Q$ we have
$$p \leq p^{**} \quad \text{and} \quad q \leq q^{**}.$$
- For all $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ we have
$$p_1 \leq p_2 \Longrightarrow p_2^* \leq p_1^* \quad \text{and} \quad q_1 \leq q_2 \Longrightarrow q_2^* \leq q_1^*.$$

[Hint: Since the statements come in dual pairs, you only have to prove half of them.]

Proof. Since the definition of Galois connection is symmetric with respect to P and Q, we never have to say which poset a given element comes from.

First, assume that for all elements x and y we have $x \leq y^* \iff y \leq x^*$. Substituting $y = x^*$ tells us that $x \leq x^{**} \iff x^* \leq x^*$. Since $x^* \leq x^*$ is always true (by definition of partial order), we conclude that $x \leq x^{**}$ for all elements x. Now consider any elements x_1, x_2 such that $x_1 \leq x_2$. From the previous remark we know that $x_2 \leq x_2^{**}$, and then by transitivity of partial order we have $x_1 \leq x_2^{**} = (x_2^*)^*$. Finally, our original assumption (with $x = x_1$ and $y = x_2^*$) implies that $x_2^* \leq x_1^*$.

Conversely, assume that for all elements x we have $x \leq x^{**}$ and for all elements x_1, x_2 we have $x_1 \leq x_2 \Longrightarrow x_2^* \leq x_1^*$. Now let x and y be any elements, and suppose that $x \leq y^*$. Applying $*$ to both sides gives $y^{**} \leq x^*$. Then since $y \leq y^{**}$, the transitivity of partial order implies that $y \leq x^*$. The implication $y \leq x^* \Longrightarrow x \leq y^*$ follows by switching the roles of x and y. $\square$

Recall that a lattice is a poset $(P, \leq)$ in which every pair of elements $x, y \in P$ has a (necessarily unique) join $x \vee y$ and meet $x \wedge y$. By induction, any **finite** subset $A \subseteq P$ also has a join $\bigvee A \in P$ and meet $\bigwedge A \in P$.

Problem 2. Lattice Structure. Let $* : P \rightleftarrows Q : *$ be a Galois connection. If, in addition, P and Q happen to be **lattices**, prove that for all $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ we have

- $p_1^* \vee p_2^* \leq (p_1 \wedge p_2)^*$ and $q_1^* \vee q_2^* \leq (q_1 \wedge q_2)^*$
- $p_1^* \wedge p_2^* = (p_1 \vee p_2)^*$ and $q_1^* \wedge q_2^* = (q_1 \vee q_2)^*$

Proof. Again, due to symmetry we won't worry which poset a given element comes from. We will freely use the result of Problem 1.

First note that for all elements x_1, x_2 we have $x_1 \wedge x_2 \leq x_1$ and $x_1 \wedge x_2 \leq x_2$ by definition. Applying $*$ to both inequalities gives $x_1^* \leq (x_1 \wedge x_2)^*$ and $x_2^* \leq (x_1 \wedge x_2)^*$; in other words,

$(x_1 \wedge x_2)^*$ is an upper bound of x_1^* and x_2^*. By the universal property of join (i.e., the join is the "least upper bound"), we conclude that

$$x_1^* \vee x_2^* \leq (x_1 \wedge x_2)^*.$$

Similarly, we have $x_1 \leq x_1 \vee x_2$ and $x_2 \leq x_1 \vee x_2$ by definition. Applying $*$ to both sides gives $(x_1 \vee x_2)^* \leq x_1^*$ and $(x_1 \vee x_2)^* \leq x_2^*$; in other words, $(x_1 \vee x_2)^*$ is a lower bound of x_1^* and x_2^*. By the universal property of meets (i.e., the meet is the "greatest lower bound"), we conclude that

$$(x_1 \vee x_2)^* \leq x_1^* \wedge x_2^*.$$

Finally, note that we have $x_1^* \wedge x_2^* \leq x_1^*$ and $x_1^* \wedge x_2^* \leq x_2^*$ by definition. By the definition of Galois connection this implies that $x_1 \leq (x_1^* \wedge x_2^*)^*$ and $x_2 \leq (x_1^* \wedge x_2^*)^*$; in other words, $(x_1^* \wedge x_2^*)^*$ is an upper bound of x_1 and x_2. By the universal property of join this implies that $x_1 \vee x_2 \leq (x_1^* \wedge x_2^*)^*$. Applying the definition of Galois connection once more gives

$$x_1^* \wedge x_2^* \leq (x_1 \vee x_2)^*,$$

and putting together the previous two results gives

$$x_1^* \wedge x_2^* = (x_1 \vee x_2)^*.$$

□

In the next problem you will show that the first inequalities are sometimes strict.

Problem 3. Counterexample. Consider the usual topology on the set of real numbers $\mathbb{R}$. Let $\mathscr{O} \subseteq 2^{\mathbb{R}}$ be the collection of open sets and let $\mathscr{C} \subseteq 2^{\mathbb{R}}$ be the collection of closed sets. Let $- : 2^{\mathbb{R}} \to 2^{\mathbb{R}}$ be the "topological closure" and let $\circ : 2^{\mathbb{R}} \to 2^{\mathbb{R}}$ be the "topological interior". One can check (you don't need to) that for all $O \in \mathscr{O}$ and $C \in \mathscr{C}$ we have

$$O \subseteq C^\circ \iff O^- \subseteq C.$$

In other words, we have a Galois connection $- : \mathscr{O} \rightleftarrows \mathscr{C} : \circ$ where $\mathscr{O}$ is partially ordered by inclusion (" $\leq$ " = " $\subseteq$ ") and $\mathscr{C}$ is partially ordered by **reverse-inclusion** ("$\leq$ " = " $\supseteq$ "). Note that $\mathscr{O}$ is a lattice with $\wedge = \cap$ and $\vee = \cup$, whereas $\mathscr{C}$ is a lattice with $\wedge = \cup$ and $\vee = \cap$.

In this case, find **specific elements** $O_1, O_2 \in \mathscr{O}$ and $C_1, C_2 \in \mathscr{C}$ such that

$$O_1^- \vee O_2^- \lneq (O_1 \wedge O_2)^- \quad \text{and} \quad C_1^\circ \vee C_2^\circ \lneq (C_1 \wedge C_2)^\circ.$$

Proof. First I'll verify that that this is a Galois connection (even though I didn't ask you to do so). Consider $O \in \mathscr{O}$ and $C \in \mathscr{C}$, and suppose that $O \subseteq C^\circ$. Since $C^\circ \subseteq C$ (property of interior) transitivity implies $O \subseteq C$. Then applying $-$ gives $O^- \subseteq C^-$ (property of closure). Since $C^- = C$ (definition of closed) we get $O^- \subseteq C$ as desired. The other direction is similar.

Recall that we are regarding $\mathscr{C}$ as a poset under reverse-inclusion, so that $\wedge = \cup$ and $\vee = \cap$. Thus we are looking for two open sets O_1, O_2 such that

$$(O_1 \cap O_2)^- \subsetneq O_1^- \cap O_2^-.$$

I will take the open intervals $O_1 = (0,1)$ and $O_2 = (1,2)$. Then we have $O_1 \cap O_2 = \emptyset$ so that $(O_1 \cap O_2)^- = \emptyset^- = \emptyset$. On the other hand, the closures are the closed intervals $O_1^- = [0,1]$ and $O_2^- = [0,2]$ so that $O_1^- \cap O_2^- = \{1\}$, which is strictly bigger than $\emptyset$.

We are also looking for two closed sets C_1, C_2 such that

$$C_1^\circ \cup C_2^\circ \subsetneq (C_1 \cup C_1)^\circ.$$

I will take the closed intervals $C_1 = [0,1]$ and $C_2 = [1,2]$. The interiors are the open intervals $C_1^\circ = (0,1)$ and $C_2^\circ = (1,2)$ so that $C_1^\circ \cup C_2^\circ = (0,1) \cup (1,2)$. On the other hand we have $C_1 \cup C_2 = [0,2]$ so that $(C_1 \cup C_2)^\circ = (0,2)$, which is strictly bigger than $(0,1) \cup (1,2)$. □

[Remark: The result of Problem 5 below will imply that there is an isomorphism between the subposet $\mathscr{C}^\circ \subseteq \mathscr{O}$ of "$\circ-$ closed sets" and the subposet $\mathscr{O}^- \subseteq \mathscr{C}$ of "$-\circ$ closed" sets. You might wonder (as I did) what kind of sets these are. I found out that the elements of $\mathscr{C}^\circ$ are called "regular open sets" and the elements of $\mathscr{O}^-$ are called "regular closed sets". I wasn't able to learn much about them except for the following facts: (1) $\mathscr{O}^-$ and $\mathscr{C}^\circ$ are Boolean lattices, (2) convex sets and their complements are regular.]

Now you will investigate under what conditions the first inequalities in Problem 2 become equalities.

Problem 4. Closed Elements. Let $* : P \rightleftarrows Q : *$ be a Galois connection between lattices P and Q. We will say that $p \in P$ (resp. $q \in Q$) is $**$-closed if $p^{**} = p$ (resp. $q^{**} = q$).

(a) Prove that the meet of any two $**$-closed elements is $**$-closed.
(b) Prove that the following two conditions are equivalent:
- The join of any two $**$-closed elements is $**$-closed.
- For all $**$-closed elements $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ we have

$$p_1^* \vee p_2^* = (p_1 \wedge p_2)^* \quad \text{and} \quad q_1^* \vee q_2^* = (q_1 \wedge q_2)^*.$$

Proof. For part (a) assume that x_1 and x_2 are $**$-closed, i.e., that $x_1^{**} = x_1$ and $x_2^{**} = x_2$. By definition of meet we have $x_1 \wedge x_2 \leq x_1$ and $x_1 \wedge x_2 \leq x_2$ and since $**$ preserves order [because $*$ reverses order; see Problem 1] this implies that $(x_1 \wedge x_2)^{**} \leq x_1^{**} = x_1$ and $(x_1 \wedge x_2)^{**} \leq x_2^{**} = x_2$. In other words, $(x_1 \wedge x_2)^{**}$ is a lower bound of x_1 and x_2. Since $x_1 \wedge x_2$ is the greatest lower bound this implies that $(x_1 \wedge x_2)^{**} \leq x_1 \wedge x_2$. Combining this with the fact that $x_1 \wedge x_2 \leq (x_1 \wedge x_2)^{**}$ [see Problem 1] gives

$$(x_1 \wedge x_2)^{**} = x_1 \wedge x_2.$$

In other words, $x_1 \wedge x_2$ is $**$-closed.

For part (b) first assume that for all x_1, x_2 we have $x_1^* \vee x_2^* = (x_1 \wedge x_2)^*$. We will show that the join of any two $**$-closed elements is $**$-closed. So let y_1, y_2 be any two $**$-closed elements, i.e., let $y_1^{**} = y_1$ and $y_2^{**} = y_2$. Then we have $y_1 = x_1^*$ and $y_2 = x_2^*$ where $x_1 = y_1^*$ and $x_2 = y_2^*$, so that

$$y_1 \vee y_2 = x_1^* \vee x_2^* = (x_1 \wedge x_2)^*,$$

and this is $**$-closed because $(x_1 \wedge x_2)^{***} = (x_1 \wedge x_2)^*$ [see Problem 5(a)].

Conversely, assume the join of any two $**$-closed elements is $**$-closed and consider any $**$-closed elements x_1, x_2. We will show that $x_1^* \vee x_2^* = (x_1 \wedge x_2)^*$. To do this, first note that by definition of join we have $x_1^* \leq x_1^* \vee x_2^*$ and $x_2^* \leq x_1^* \vee x_2^*$. Applying $*$ to both inequalities gives $(x_1^* \vee x_2^*)^* \leq x_1^{**} = x_1$ and $(x_1^* \vee x_2^*)^* \leq x_2^{**} = x_2$. In other words, $(x_1^* \vee x_2^*)^*$ is a lower bound of x_1 and x_2. Since $x_1 \wedge x_2$ is the greatest lower bound, this implies that

$$(x_1^* \vee x_2^*)^* \leq x_1 \wedge x_2 \tag{1}$$

Since x_1^* and x_2^* are $**$-closed [see Problem 5(a)] we have by assumption that $x_1^* \vee x_2^*$ is also $**$-closed. Finally, apply $*$ to both sides of (1) to get

$$(x_1 \wedge x_2)^* \leq (x_1^* \vee x_2^*)^{**} = x_1^* \vee x_2^*.$$

Combining this with the inequality $x_1^* \vee x_2^* \leq (x_1 \wedge x_2)^*$ [see Problem 2] gives the result. $\square$

Finally, let's put everything together. Basically, if we have a Galois connection between lattices in which joins of closed elements are closed, then this restricts to an **isomorphism** on their sublattices of closed elements. If $(P, \leq)$ is a poset we'll use the notation P^{op} for the same set of elements with the **opposite** partial order (and hence with meets and joins switched).

Problem 5. Galois Correspondence. Let $* : P \rightleftarrows Q : *$ be a Galois connection between lattices P and Q. Denote the image of $* : P \to Q$ by $P^* \subseteq Q$ and denote the image of $* : Q \to P$ by $Q^* \subseteq P$. We will think of these as subposets with the induced partial order.

(a) Prove that $Q^* \subseteq P$ and $P^* \subseteq Q$ are precisely the subposets of $**$-closed elements.
(b) Prove that the restricted maps $* : Q^* \rightleftarrows P^* : *$ are an **isomorphism of posets**:

$$Q^* \approx (P^*)^{\mathrm{op}}.$$

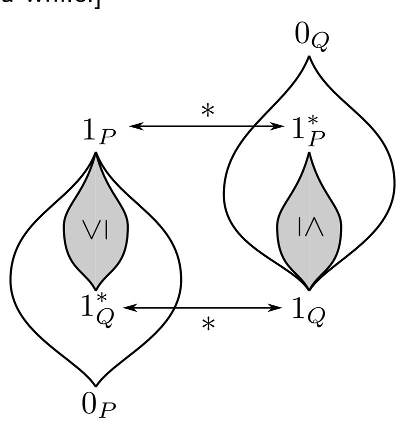
(c) If, in addition, the join of any two $**$-closed elements is $**$-closed, prove that $Q^* \subseteq P$ and $P^* \subseteq Q$ are **sublattices**, and that the isomorphism from (b) is an **isomorphism of lattices**.

Proof. For part (a), consider an element x^* in the image of $*$. From Problem 2 we have $x^* \leq (x^*)^{**}$. On the other hand, applying $*$ to both sides of the inequality $x \leq x^{**}$ gives $(x^*)^{**} = (x^{**})^* \leq x^*$. We conclude that $(x^*)^{**} = x^*$, hence x^* is $**$-closed. Conversely, let y be $**$-closed. Since $y = y^{**} = (y^*)^*$ we conclude that y is in the image of $*$.

For part (b) we first note that $* : Q^* \rightleftarrows P^* : *$ are inverse functions (and hence bijections). Indeed, given an element x^* in the image of $*$ then we know from part (a) that $(x^*)^{**} = x^*$. Since $*$ reverses order [see Problem 1], we obtain a poset isomorphism $Q^* \approx (P^*)^{\mathrm{op}}$.

For part (c) assume that the join of $**$-closed elements is $**$-closed. By part (a) and Problem 4(a) this implies that $Q^* \subseteq P$ and $P^* \subseteq Q$ are sublattices. Finally, Problems 2 and 4(b) imply that the poset isomorphism from part (b) is an isomorphism of lattices. □

[Remark: For the purpose of this problem I defined a sublattice to be a subposet of a lattice closed under finite meets and joins. If the lattice has a 0 and 1, I don't require that a sublattice contains these. For example, if P and Q have top elements 1_P and 1_Q, respectively, then it will follow that Q^* and P^* have the same top elements. However, the bottom elements of Q^* and P^* will be 1_P^* and 1_Q^*, respectively, which might not equal 0_P and 0_Q (see the picture below). An isomorphism of *complete lattices* would necessarily preserve 0 and 1. Don't you hate all this terminology? Yeah, I'm done with lattice theory for a while.]



Epilogue: You might ask whether the definition of Galois connection given above is more general than the one discussed in class. The answer is: "yes and no". The answer is "yes" in the sense that this definition applies to more general posets. However, if P and Q happen to be Boolean lattices then the answer is "no". I will define a Boolean lattice as the collection of subsets of a set U, partially ordered by inclusion. Note that the lattice operations are $\wedge = \cap$ and $\vee = \cup$.

Problem 6. Boolean Galois Connections. Let S and T be sets and consider the corresponding Boolean lattices $P = 2^S$ and $Q = 2^T$. For any relation $R \subseteq S \times T$ and for any subsets $A \subseteq S$ and $B \subseteq T$ we will define the sets $A^R \subseteq T$ and $B^R \subseteq S$ as follows:

- $A^R = \{t \in T : \forall a \in A, aRt\}$
- $B^R = \{s \in S : \forall b \in B, sRb\}$

In class we called this an "abstract Galois connection" and we showed that it has many nice properties. Now let $* : P \rightleftarrows Q : *$ be a Galois connection of posets in the sense defined above. Prove that **there exists a unique relation** $R \subseteq S \times T$ such that for all $A \subseteq S$ and $B \subseteq T$ we have

$$A^* = A^R \quad \text{and} \quad B^* = B^R.$$

[Hint: Consider the singleton subsets of S and T. You will need to use the fact that the power set 2^U is a complete lattice, i.e., it is possible to take the intersection and union of arbitrary collections of subsets.]

Proof. Let S and T be sets and let $* : 2^S \rightleftarrows 2^T : *$ be a Galois connection of posets. That is, for all subsets $A \subseteq T$ and $B \subseteq T$ we have $A \subseteq B^* \iff B \subseteq A^*$. In particular, for all elements $s \in S$ and $t \in T$ we have

$$\{s\} \subseteq \{t\}^* \iff \{t\} \subseteq \{s\}^*.$$

Define the relation $R \subseteq S \times T$ by setting "sRt" (i.e., "$(s,t) \in R$") whenever either of these equivalent conditions is true.

I claim that for all $A \subseteq S$ and $B \subseteq T$ we have $A^* = A^R$ and $B^* = B^R$. To see this, first note that $R : 2^S \rightleftarrows 2^T : R$ is a Galois connection and so it satisfies all of the properties proved in this homework. Indeed, for all subsets $A \subseteq S$ and $B \subseteq T$ we have

$$\begin{aligned} A \subseteq B^R &\iff \forall a \in A, a \in B^R \\ &\iff \forall a \in A, \forall b \in B, aRb \\ &\iff \forall b \in B, \forall a \in A, aRb \\ &\iff \forall b \in B, b \in A^R \\ &\iff B \subseteq A^R. \end{aligned}$$

Now we observe that the result is true for singleton subsets. Indeed, we have

$$\begin{aligned} \{a\}^R &= \{t \in T : \forall s \in \{a\}, sRt\} \\ &= \{t \in T : aRt\} \\ &= \{t \in T : \{t\} \subseteq \{a\}^*\} \\ &= \{t \in T : t \in \{a\}^*\} \\ &= \{a\}^*. \end{aligned}$$

To finish the proof we will use the fact (details omitted) that the proof from Problem 2 can be generalized to show that for **arbitrary** collections of sets $\{X_i\}_{i \in I}$ we have

$$\cap_{i \in I} X_i^* = \left(\cup_{i \in I} X_i\right)^*.$$

Finally, for all subsets $A \subseteq S$ we have

$$\begin{aligned} A^* &= (\cup_{a\in A}\{a\})^* \\ &= \cap_{a\in A}\{a\}^* \\ &= \cap_{a\in A}\{a\}^R \\ &= (\cup_{a\in A}\{a\})^R \\ &= A^R. \end{aligned}$$

To see that the relation R is unique, suppose there exists another relation $R' \subseteq S \times T$ with the same properties. Then for all $t \in T$ we have $\{t\}^R = \{t\}^* = \{t\}^{R'}$, and hence for all $(s,t) \in S \times T$ we have

$$sRt \iff s \in \{t\}^R \iff s \in \{t\}^* \iff s \in \{t\}^{R'} \iff sR't.$$

□

[Remark: The theory of Galois connections between posets is a special case of the theory of adjoint functors between categories. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a pair of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is called an adjunction if there is a family of bijections $\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \approx \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ that is "natural" in X and Y. Recall that a poset is just a category in which $|\mathrm{Hom}(X,Y)| \in \{0,1\}$ for all X and Y, and we write "$X \leq Y$" to mean that $|\mathrm{Hom}(X,Y)| = 1$. Thus if $\mathcal{C}$ and $\mathcal{D}$ are posets then the condition $\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \approx \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ becomes $X \leq G(Y) \iff F(X) \leq Y$. The results we found about Galois connections preserving lattice structure can be generalized by saying: G preserves limits and F preserves colimits.]

> The slogan is "Adjoint functors arise everywhere".
>
> — Saunders Mac Lane