12/8/15

HW 4 due now.
Final Exam Thurs 2:00 – 4:30pm

The Final Exam is not cumulative. It will cover the material discussed since the Midterm. Here are the topics.

(1) Functors & Natural Transformations.

Let $\mathcal{C}$ & $\mathcal{D}$ be categories. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ consists of

- a function $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$

- for each pair $X, Y \in \mathrm{Obj}(\mathcal{C})$ a function

$$F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

satisfy the two rules

- $\forall\, X \in \mathcal{C}$, $F(id_X) = id_{F(X)}$.

- $\forall\, \alpha : X \to Y$ & $\beta : Y \to Z$

$$F(\beta \circ \alpha) = F(\beta) \circ F(\alpha).$$

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A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $F : \mathcal{C} \to \mathcal{D}^{op}$ or $F : \mathcal{C}^{op} \to \mathcal{D}$. That is for all $\alpha : X \to Y$ & $\beta : Y \to Z$ we have

$$F(\beta \circ \alpha) = F(\alpha) \circ F(\beta).$$

Now let $F, G : \mathcal{C} \to \mathcal{D}$ be covariant functors. A natural transformation

$$\Phi : F \to G$$

assigns to each object $X \in F$ a morphism $\Phi(X) : F(X) \to G(Y)$ such that for all objects $X, Y \in \mathcal{C}$ and morphisms $\alpha : X \to Y$, the following square commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\Phi(X)"] \arrow[d, "F(\alpha)"'] & G(X) \arrow[d, "G(\alpha)"] \\ F(Y) \arrow[r, "\Phi(Y)"] & G(Y) \end{tikzcd}$$

We say that Φ is a natural isomorphism "$F \approx G$" if $\Phi(X)$ is an isomorphism $\forall X \in \mathcal{C}$.

(2) The Category of G-sets.

Let $\mathcal{C}$ & $\mathcal{D}$ be categories with $\mathcal{C}$ small. Then we can define the functor category

$$\mathcal{D}^{\mathcal{C}}$$

whose objects are functors $F : \mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations.

For example, let G be a group thought of as a category with one object. Then a functor $F : G \to \text{Set}$ is called a "G-set".

A G-set $F : G \to \text{Set}$ consists of a set

$$X := F(*)$$

and a function

$$F : G = \text{Aut}_G(*) \to \text{End}_{\text{Set}}(X).$$

The axioms of a functor imply that this is actually a group homomorphism

$$F : G \longrightarrow \text{Aut}_{\text{Set}}(X).$$

To be more concrete, a G-set consists of a set X and a function $G \times X \to X$ written as $(g, x) \mapsto g(x)$ satisfying the two axioms

- $\forall x \in X$, $1_G(x) = x$
- $\forall x \in X$ & $g, h \in G$, $g(h(x)) = (gh)(x)$.

This is equivalent to the previous definition via the identification

$$g(x) = F(g)(x).$$

Now let X & Y be two G-sets with the actions $G \times X \to X$ & $G \times Y \to Y$ written implicitly. Then a morphism of G-sets is just a function $\Phi : X \to Y$ such that for all $x \in X$ & $g \in G$ we have

$$\Phi(g(x)) = g(\Phi(x)).$$

Check that is the same as a natural transformation of functors $G \to \text{Set}$.

③ Fundamental Theorem of G-sets.

Given a G-set X and an element $x \in X$ we define

$$\mathrm{Orb}_G(x) := \{ y \in X : \exists g \in G,\ y = g(x) \}$$

$$\mathrm{Stab}_G(x) := \{ g \in G : g(x) = x \}.$$

☆ Theorem (FTGS):

(i) For each $x \in X$, $\mathrm{Stab}_G(x) \subseteq G$ is a subgroup and we have an isomorphism of G-sets

$$\mathrm{Orb}_G(x) \approx_G G/\mathrm{Stab}_G(x)$$

$$g(x) \longleftrightarrow g\,\mathrm{Stab}_G(x).$$

(ii) Given two subgroups $H, K \subseteq G$ we have

$$G/H \approx_G G/K$$

if and only if $\exists\, g \in G,\ gHg^{-1} = K$.

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Know how to prove the FTGS. Part (i) is straightforward. The key to Part (ii) is the identity

$$\mathrm{Stab}_G(g(x)) = g\,\mathrm{Stab}_G(x)\,g^{-1}.$$

Two Examples from HW4:

- Let $Gr_1(r,n)$ be the set of r-element subsets of $\{1,2,\ldots,3\}$. Then we have an isomorphism of S_n-sets

$$Gr_1(r,n) \approx S_n / (S_r \times S_{n-r}).$$

- Let $Gr_K(r,n)$ be the set of r-dimensional subspaces of K^n. Then we have an isomorphism of $GL_n(K)$-sets

$$Gr_K(r,n) \approx \frac{GL_n(K)}{\mathrm{Mat}_{r,n-r}(K) \rtimes (GL_r(K) \times GL_{n-r}(K))}$$

General Example:

Let $H, K \subseteq G$ be subgroups. Then we define an action of $H \times K$ on G by

$$(H \times K) \times G \longrightarrow G$$

$$((h,k), g) \longmapsto hgk^{-1}$$

The orbits are called double cosets $\mathrm{Orb}_{H\times K}(g) =: HgK$ and we denote the set of orbits by

$$H \backslash G / K := \{ HgK : g \in G \}.$$

If H & K are finite, be able to prove that

$$|HgK| = |H| \cdot |K| \,/\, |H \cap gKg^{-1}|.$$

Hint: Show that

$$HgK \longleftrightarrow \mathrm{Orb}_H(gK) \times K$$

and $\mathrm{Stab}_H(gK) = H \cap gKg^{-1}$.

④ The Class Equation.

Let G act on itself by $g(h) := ghg^{-1}$. The orbits are called conjugacy classes

$$K_G(a) := \{ b \in G : \exists g \in G,\ b = gag^{-1} \}$$

and the stabilizers are called centralizers.

$$Z_G(a) := \{ g \in G : gag^{-1} = a \}.$$

The intersection of all centralizers is called the center of G

$$Z(G) := \bigcap_{a \in G} Z_G(a).$$

If $K_1, K_2, \ldots, K_n$ are the classes of G and $Z_1, Z_2, \ldots, Z_n$ are the corresponding centralizers (up to isomorphism), use the FTGS to prove that we have an isomorphism of G-sets

$$G \approx Z(G) \sqcup \Big(\bigsqcup_{Z_i \neq G} G/Z_i \Big).$$

Hint: $K_G(a) = \{a\} \iff a \in Z(G)$. ///

If G is finite, we obtain

$$|G| = |Z(G)| + \sum_{Z_i \neq G} \frac{|G|}{|Z_i|},$$

which is called the class equation.

⑤ Application: Sylow Theory.

Let $|G| = p^\alpha m$ with p prime and $p \nmid m$.

☆ Theorem (Sylow):

(i) For all $0 \leq \beta \leq \alpha$, $\exists$ subgro $H \leq G$ with $|H| = p^\beta$.

(ii) If $H, K \leq G$ are subgroups with $|K| = p^\alpha$ & $|H| = p^\beta$ ($\beta \leq \alpha$), then $\exists\, g \in G$ such that $gHg^{-1} \subseteq K$.

(iii) If $n_p := \#\{K \leq G : |K| = p^\alpha\}$ then

- $n_p \mid m$
- $n_p = 1 \mod p$.

You do <u>not</u> need to memorize the proof but you should know what goes into it.

The hardest part is the following Lemma.

Lemma: Let A be a finite abelian group and let p be prime. If $p \mid |A|$ then A has an element of order p.

[We proved this with a tricky argument. We'll see the correct proof next semester when we prove the Fundamental Theorem of Finitely Generated $\mathbb{Z}$-modules.]

The rest of the proof is more straightforward.

$Z_i \neq G$

Proof of (i): If $p^\alpha \mid |Z_i|$ then we're done by induction. Otherwise, the class equation says that $p \mid |Z(G)|$, hence $Z(G)$ has an element $z \in Z(G)$ of order p (by the lemma). By induction $G/\langle z \rangle$ has p-subgroups of all orders and we can lift these up to G.

Proof of (ii) : Let $H, K \leq G$ with $|K| = p^\alpha$ and $|H| = p^\beta$ ($\beta \leq \alpha$). Decompose G into double cosets to get

$$|G| = \sum_i \frac{|H| \cdot |K|}{|K \cap g_i H g_i^{-1}|}.$$

Show by contradiction that one of the denominators has size p^β, hence

$$g_i H g_i^{-1} \leq K.$$

Proof of (iii) : Let $G \curvearrowright Syl_p(G)$ by conjugation and fix $K \in Syl_p(G)$.

- By (ii) and FTGS we know that

$$|Syl_p(G)| = |Orb_G(K)| = |G| / |N_G(K)|.$$

Since $K \leq N_G(K)$ we have $p^\alpha \mid |N_G(K)|$ and it follows that $|Syl_p(G)| \mid m$.

- Now let $K \curvearrowright Syl_p(G)$ by conjugation, so

$$|Syl_p(G)| = \sum_i \frac{|K|}{|Stab_K(H_i)|}$$

for some $H_i \in Syl_p(K)$. Show that $Stab_K(H_i) = K \iff H_i = K$ and hence

$$|Syl_p(G)| = 1 + \sum_{H_i \neq K} \frac{|K|}{|Stab_K(H_i)|}$$

$$= 1 \mod p.$$

You should know how to apply Sylow to study groups of small order.

Example: Prove that there is no simple group of order 12.

Proof: Let $|G| = 12 = 2^2 \cdot 3$.

Let $n_2 = |Syl_2(G)|$ & $n_3 = |Syl_3(G)|$. By Sylow (i) we have $n_3 \geq 1$. If $n_3 = 1$ then by Sylow (ii) we obtain a <u>normal</u> subgroup of size 3. So assume that $n_3 \geq 2$. Then by Sylow (iii) we have

$$n_3 = 1 \mod 3, \text{ hence } n_3 \geq 4.$$

Since these subgroups intersect trivially (they are cyclic), G must contain at least 8 elements of order 3.

But then there is room for <u>only one</u> Sylow 2-subgroup (of size 4), which must therefore be normal. //

(6) Finite matrix groups.

You should also remember that

$$|GL_n(q)| = q^{\binom{n}{2}} (q-1)^n [n]_q!$$

$$|SL_n(q)| = q^{\binom{n}{2}} (q-1)^{n-1} [n]_q!$$

$$|PSL_n(q)| = \frac{q^{\binom{n}{2}} (q-1)^{n-1} [n]_q!}{\gcd(n, q-1)}$$

In case I want to use these as an example somewhere.