Math 761 **Fall 2015**
Final Exam **Drew Armstrong**

Problem 1. Functors vs. G-Sets.

(a) Define what it means for $F : \mathcal{C} \to \mathcal{D}$ to be a (covariant) functor.

A functor $F : \mathcal{C} \to \mathcal{D}$ assigns to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$ and to each morphism $\alpha : X \to Y$ in $\mathcal{C}$ a morphism $F(\alpha) : F(X) \to F(Y)$ in $\mathcal{D}$ such that the following two axioms hold:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X in $\mathcal{C}$,
- $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$ for all morphisms α, β in $\mathcal{C}$ such that $\alpha \circ \beta$ is defined.

(b) Given functors $F, G : \mathcal{C} \to \mathcal{D}$, define what it means for $\Phi : F \to G$ to be a natural transformation.

A natural transformation $\Phi : F \to G$ assigns to each object X in $\mathcal{C}$ a morphism $\Phi(X) : F(X) \to G(X)$ in $\mathcal{D}$ such that for all morphisms $\alpha : X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi(X)} & G(X) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ F(Y) & \xrightarrow[\Phi(Y)]{} & G(Y) \end{array}$$

(c) Let G be a group thought of as a category with one object "$*$", so that $G = \mathrm{Aut}(*)$. Prove that a functor $F : G \to \mathsf{Set}$ defines a set X and a group homomorphism $F : G \to \mathrm{Aut}_{\mathsf{Set}}(X)$.

A functor $F : G \to \mathsf{Set}$ assigns to the unique object $*$ of G a set $X := F(*)$ and assigns to each group element $g \in G = \mathrm{Aut}(*)$ a function $F(g) \in \mathrm{End}_{\mathsf{Set}}(X)$ satisfying the following two properties:

- $F(1_G) = \mathrm{id}_X$,
- $F(gh) = F(g) \circ F(h)$ for all $g, h \in G$.

It remains only to show that for each $g \in G$ the function $F(g) : X \to X$ is invertible. Indeed, note that $F(g) \circ F(g^{-1}) = F(gg^{-1}) = F(1_G) = \mathrm{id}_X$ and $F(g^{-1}) \circ F(g) = F(g^{-1}g) = F(1_G) = \mathrm{id}_X$ so that $F(g)$ is invertible with inverse $F(g^{-1})$.

(d) Consider two functors $F_1, F_2 : G \to \mathsf{Set}$ that define group homomorphisms $F_1 : G \to \mathrm{Aut}_{\mathsf{Set}}(X_1)$ and $F_2 : G \to \mathrm{Aut}_{\mathsf{Set}}(X_2)$. Prove that a natural transformation $\Phi : F_1 \to F_2$ defines a G-equivariant set function $\Phi : X_1 \to X_2$ (and say what this means).

A natural transformation $\Phi : F_1 \to F_2$ assigns to the unique object a function $\Phi(*) : F_1(*) \to F_2(*)$. Since $X_1 := F_1(*)$ and $X_2 := F_2(*)$ we might as well just call this function $\Phi : X_1 \to X_2$. Then the defining property of natural transformation says that for all group elements $g : * \to *$ the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi} & X_2 \\ {\scriptstyle F_1(g)}\downarrow & & \downarrow{\scriptstyle F_2(g)} \\ X_1 & \xrightarrow[\Phi]{} & X_2 \end{array}$$

In other words, for all elements $x \in X_1$ we have

$$\Phi(F_1(g)(x)) = F_2(g)(\Phi(x)).$$

If the functors F_1 and F_2 are implicitly understood then we will often write this as

$$\Phi(g(x)) = g(\Phi(x)).$$

Problem 2. The Fundamental Theorem of G-Sets. Let G be a group acting on a set X. Denote the action $G \times X \to X$ implicitly by $(g, x) \mapsto g(x)$.

(a) For each $x \in X$, prove that the prescription $g(x) \leftrightarrow g\mathsf{Stab}(x)$ defines a bijection

$$\mathsf{Orb}(x) \leftrightarrow G/\mathsf{Stab}(x).$$

Proof. For all $x \in X$ and $g, h \in G$ we have

$$\begin{aligned} g(x) = h(x) &\Leftrightarrow (h^{-1}g)(x) = x \\ &\Leftrightarrow h^{-1}g \in \mathsf{Stab}(x) \\ &\Leftrightarrow g\mathsf{Stab}(x) = h\mathsf{Stab}(x). \end{aligned}$$

□

(b) Prove that the bijection from part (a) is G-equivariant.

Proof. Let $\Phi : \mathsf{Orb}(x) \to G/\mathsf{Stab}(x)$ denote the bijection $\Phi(g(x)) = g\mathsf{Stab}(x)$ from part (a). Then for all $g(x) \in \mathsf{Orb}(x)$ and $h \in G$ we have

$$\begin{aligned} \Phi(h(g(x))) &= \Phi((hg)(x)) \\ &= (hg)\mathsf{Stab}(x) \\ &= h(g\mathsf{Stab}(x)) \\ &= h(\Phi(g(x))). \end{aligned}$$

□

Problem 3. Double Cosets. Let G be a group and consider two subgroups $H, K \subseteq G$.

(a) Prove that the rule $((h, k), g) \mapsto hgk^{-1}$ defines an action of the direct product $H \times K$ on G. The orbits are called double cosets; we denote them by $HgK := \mathsf{Orb}_{H\times K}(g)$.

Proof. For all $(h, k) \in H \times K$ and $g \in G$ we will write $(h, k) \bullet g := ghk^{-1}$. To prove that this is an action, first note that for all $g \in G$ we have

$$(1, 1) \bullet g = 1g1^{-1} = g.$$

Then note that for all $(h_1, k_1), (h_2, k_2) \in H \times K$ and $g \in G$ we have

$$\begin{aligned} (h_1, k_1) \bullet [(h_2, k_2) \bullet g] &= (h_1, k_1) \bullet h_2gk_2^{-1} \\ &= h_1(h_2gk_2^{-1})k_1^{-1} \\ &= (h_1h_2)g(k_1k_2)^{-1} \\ &= (h_1h_2, k_1k_2) \bullet g \\ &= [(h_1, k_1) \cdot (h_2, k_2)] \bullet g. \end{aligned}$$

□

(b) Now consider the action of H on the set G/K by $(h, gK) \mapsto (hg)K$. For all $g \in G$, prove that we have a bijection $HgK \leftrightarrow \mathsf{Orb}_H(gK) \times K$. [Hint: Show that HgK is the union of the elements of $\mathsf{Orb}_H(gK)$.]

Proof. Consider any element $hgk \in HgK$. Since $hgk \in (hg)K = h(gK) \in \mathsf{Orb}_H(gK)$ we conclude that HgK is contained in the union $\cup_{C \in \mathsf{Orb}_H(gK)} C$. Conversely, given any $x \in \cup_{C \in \mathsf{Orb}_H(gK)} C$ there exists $h \in H$ such that $x \in h(gK) \subseteq HgK$. We conclude that

$$HgK = \bigcup_{C \in \mathsf{Orb}_H(gK)} C.$$

Since any coset of K is in bijection with K, and since any two distinct cosets are disjoint, we obtain a bijection

$$HgK \leftrightarrow \left(\bigsqcup_{C \in \mathsf{Orb}_H(gK)} K \right) \leftrightarrow \mathsf{Orb}_H(gK) \times K.$$

□

(c) Use part (b) and Problem 2(a) to prove that we have a bijection

$$HgK \leftrightarrow H/(H \cap gKg^{-1}) \times K.$$

Proof. Note that for all $g \in G$ we have

$$\begin{aligned} \mathsf{Stab}_H(gK) &= \{h \in H : h(gK) = gK\} \\ &= \{h \in H : (g^{-1}hg)K = K\} \\ &= \{h \in H : g^{-1}hg \in K\} \\ &= \{h \in H : h \in gKg^{-1}\} \\ &= H \cap gKg^{-1}, \end{aligned}$$

so from Problem 2(a) we obtain a bijection

$$\mathsf{Orb}_H(gK) \leftrightarrow H/\mathsf{Stab}_H(gK) = H/(H \cap gKg^{-1}).$$

Combining this with part (b) gives the desired result. □

(d) Now let G be finite and consider an element $g \in G$. Use part (c) and Problem 2(a) to compute the size of the stabilizer $\mathsf{Stab}_{H \times K}(g) := \{(h, k) \in H \times K : hgk^{-1} = g\}$.

Proof. From part (c) we have

$$|\mathsf{Orb}_{H \times K}(g)| = |HgK| = |H/(H \cap gKg^{-1})| \cdot |K| = \frac{|H| \cdot |K|}{|H \cap gKg^{-1}|}$$

and from Problem 2(a) we have

$$|\mathsf{Orb}_{H \times K}(g)| = \frac{|H \times K|}{|\mathsf{Stab}_{H \times K}(g)|} = \frac{|H| \cdot |K|}{|\mathsf{Stab}_{H \times K}(g)|}.$$

Combining these gives $|\mathsf{Stab}_{H \times K}(g)| = |H \cap gKg^{-1}|$. □

Problem 4. The Sylow Theorem.

(a) Accurately state all three parts of the Sylow Theorem.

Let G be a finite group and let p be a prime number. If $|G| = p^\alpha m$ with $p \nmid m$ then

(S1) For each $0 \le \beta \le \alpha$ there exists a subgroup $H \subseteq G$ of size $|H| = p^\beta$.
(S2) Given subgroups $H, K \in G$ with $|H| = p^\alpha$ and $|K| = p^\beta$ for some $0 \le \beta \le \alpha$, there exists a group element $g \in G$ such that $gKg^{-1} \subseteq H$.
(S3) Let n_p be the number of subgroups of G with size p^α. Then we have $n_p | m$ and $n_p = 1 \pmod p$.

(b) Use the Sylow Theorem to prove that no group of size 30 is simple. [Hint: Count elements of orders 2, 3, and 5.]

Proof. Let G be a group and suppose that $|G| = 30 = 2 \cdot 3 \cdot 5$. Let n_2, n_3, n_5 be the numbers of subgroups of sizes 2, 3 and 5, respectively. From (S3) we know that

- $n_5 | 6$ and $n_5 = 1 \pmod 5$,
- $n_3 | 10$ and $n_3 = 1 \pmod 3$,
- $n_2 | 15$ and $n_2 = 1 \pmod 2$,

which implies that $n_5 \in \{1, 6\}$, $n_3 \in \{1, 10\}$ and $n_2 \in \{1, 3, 5, 15\}$. If any of n_2, n_3 or n_5 equals 1 then by (S2) we obtain a nontrivial normal subgroup, so assume for contradiction that $n_5 \ge 6$, $n_3 \ge 10$ and $n_2 \ge 3$. By Lagrange's Theorem, any two subgroups of coprime order intersect trivially and any two distinct subgroups of the same prime order intersect trivially. This implies that we have

$$\begin{aligned} |G| &\ge n_5(5-1) + n_3(3-1) + n_2(2-1) + 1 \\ 30 &\ge 6(4) + 10(2) + 3(1) + 1 \\ 30 &\ge 48, \end{aligned}$$

which is a contradiction. □

Problem 5. Sylow vs. Kolchin. Let p be prime and consider the group $G := \mathrm{GL}_n(p)$ of invertible $n \times n$ matrices over the field $\mathbb{Z}/p\mathbb{Z}$. Recall that we have $|G| = p^{\binom{n}{2}}(p-1)^n [n]_p!$.

(a) Let $U \subseteq G$ be the subgroup consisting of upper triangular matrices with 1's on the diagonal. Prove that U is a Sylow p-subgroup of G.

Proof. A matrix in U is determined by its entries above the diagonal. Since the number of entries above the diagonal is

$$1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$$

and since the entries can take any value in $\mathbb{Z}/p\mathbb{Z}$ we conclude that

$$|U| = |\mathbb{Z}/p\mathbb{Z}|^{\binom{n}{2}} = p^{\binom{n}{2}}.$$

Now suppose for contradiction that p divides $(p-1)^n [n]_p!$. Since p is prime Euclid's Lemma tells us that p divides $p-1$ or p divides $[n]_p!$. The first is impossible because $p - 1 < p$ and the second is impossible because

$$[n]_p! = (1)(1+p)(1+p+p^2)\cdots(1+p+p^2+\cdots+p^{n-1}) = 1 \pmod p.$$

We conclude that U is a Sylow p-subgroup. □

(b) Let $N_G(U) := \{g \in G : gUg^{-1} = U\}$. Prove that the number of Sylow p-subgroups of G equals $|G|/|N_G(U)|$. [Hint: Use the Sylow Theorem.]

Proof. Let $\mathsf{Syl}_p(G)$ be the set of Sylow p-subgroups and consider the action of G on $\mathsf{Syl}_p(G)$ by conjugation. Part (S2) of the Sylow Theorem says that any two Sylow p-subgroups are conjugate, hence $\mathsf{Syl}_p(G) = \mathsf{Orb}_G(U)$. Then Problem 2(a) tells us that

$$|\mathsf{Syl}_p(G)| = |\mathsf{Orb}_G(U)| = \frac{|G|}{|\mathsf{Stab}_G(U)|} = \frac{|G|}{|N_G(U)|}.$$

□

(c) Now you can assume that $B := N_G(U)$ is the group of **all** invertible upper triangular matrices. Use this information to compute the number of Sylow p-subgroups of G.

Proof. Consider an upper triangular matrix $g \in B$. Since $\det(g)$ is the product of the diagonal entries, the fact that $\det(g) \neq 0$ implies that none of the diagonal entries is zero. Thus the number of ways to choose the diagonal entries is $(p-1)^n$. The number of ways to choose the entries above the diagonal is again $p^{\binom{n}{2}}$ because the entries are unrestricted. We conclude that $|B| = p^{\binom{n}{2}}(p-1)^n$ and hence

$$|\mathsf{Syl}_p(G)| = \frac{|G|}{|N_G(U)|} = \frac{|G|}{|B|} = \frac{p^{\binom{n}{2}}(p-1)^n[n]_p!}{p^{\binom{n}{2}}(p-1)^n} = [n]_p!.$$

□

[Remark: If q is a power of p then similar reasoning shows that the number of Sylow p-subgroups of $\mathrm{GL}_n(q)$ is $[n]_q!$. This suggests that Sylow p-subgroups of $\mathrm{GL}_n(q)$ are some kind of "q-analogue" of permutations of the set $\{1, 2, \ldots, n\}$. Here's a possible explanation: When the field K is infinite there is no such thing as a Sylow subgroup of $\mathrm{GL}_n(K)$ so we just look at the set of cosets G/B. We call this the "complete flag variety" because it is in bijection with the set of maximal chains of subspaces of K^n (i.e., complete flags). Finally, note that there is a bijection between permutations of the set $\{1, 2, \ldots, n\}$ and maximal chains of subsets of this set.]